

REAL ANALYSIS

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Chapter I

Topology Preliminaries

LECTURE 1

1. Review of basic topology concepts

In this lecture we review some basic notions from topology, the main goal being to set up the language. Except for one result (Uryson Lemma) there will be no proofs.

DEFINITIONS. A *topology* on a (non-empty) set X is a family $\mathcal{T}$ of subsets of X , which are called *open sets*, with the following properties:

(TOP₁): *both the empty set $\emptyset$ and the total set X are open;*

(TOP₂): *an arbitrary union of open sets is open;*

(TOP₃): *a finite intersection of open sets is open.*

In this case the system $(X, \mathcal{T})$ is called a *topological space*.

If $(X, \mathcal{T})$ is a topological space and $x \in X$ is an element in X , a subset $N \subset X$ is called a *neighborhood of x* if there exists some open set D such that $x \in D \subset N$.

A collection $\mathcal{N}$ of neighborhoods of x is called a *basic system of neighborhoods of x* , if for any neighborhood M of x , there exists some neighborhood N in $\mathcal{N}$ such that $x \in N \subset M$.

A collection $\mathcal{V}$ of neighborhoods of x is called a *fundamental system of neighborhoods of x* if for any neighborhood M of x there exists a finite sequence $V_1, V_2, \dots, V_n$ of neighborhoods in $\mathcal{V}$ such that $x \in V_1 \cap V_2 \cap \dots \cap V_n \subset M$.

A topology is said to have the *Hausdorff property* if:

(H) *for any $x, y \in X$ with $x \neq y$, there exist open sets $U \ni x$ and $V \ni y$ such that $U \cap V = \emptyset$.*

If $(X, \mathcal{T})$ is a topological space, a subset $F \subset X$ will be called *closed*, if its complement $X \setminus F$ is open. The following properties are easily derived from the definition:

(C₁) *both the empty set $\emptyset$ and the total set X are closed;*

(C₂) *an arbitrary intersection of closed sets is closed;*

(C₃) *a finite union of closed sets is closed.*

Using the above properties of open/closed sets, one can perform the following constructions. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$ be an arbitrary subset. Consider the set $\text{Int}(A)$ to be the union of all open sets D with $D \subset A$ and consider the set $\overline{A}$ to be the intersection of all closed sets F with $F \supset A$. The set $\text{Int}(A)$ (sometimes denoted simply by $\overset{\circ}{A}$) is called the *interior of A* , while the set $\overline{A}$ is called the *closure of A* . The properties of these constructions are summarized in the following:

PROPOSITION 1.1. *Let $(X, \mathcal{T})$ be a topological space, and let A be an arbitrary subset of X .*

A. (Properties of the interior)

- (i) The set $\text{Int}(A)$ is open and $\text{Int}(A) \subset A$.
- (ii) If D is an open set such that $D \subset A$, then $D \subset \text{Int}(A)$.
- (iii) x belongs to $\text{Int}(A)$ if and only if A is a neighborhood of x .
- (iv) A is open if and only if $A = \text{Int}(A)$.

B. (Properties of the closure)

- (i) The set $\overline{A}$ is closed and $\overline{A} \supset A$.
- (ii) If F is a closed set with $F \supset A$, then $F \supset \overline{A}$.
- (iii) A point x belongs to $\overline{A}$, if and only if, $A \cap N \neq \emptyset$ for any neighborhood N of x .
- (iv) A is closed if and only if $A = \overline{A}$.

C. (Relationship between interior and closure) $\text{Int}(X \setminus A) = X \setminus \overline{A}$ and $\overline{X \setminus A} = X \setminus \text{Int}(A)$.

DEFINITION. Suppose $(X, \mathcal{T})$ is a topological space. Assume $A \subset X$ is a subset of X . On A we can introduce a natural topology, sometimes denoted by $\mathcal{T}|_A$ which consists of all subsets of A of the form $A \cap U$ with U open set in X . This topology is called the *relative* (or *induced*) *topology*.

REMARK 1.1. If A is already open in the topology $\mathcal{T}$, then a subset $V \subset A$ is open in the induced topology if and only if V is open in the topology $\mathcal{T}$ (this follows from the fact that the intersection of any two open sets in $\mathcal{T}$ is again an open set in $\mathcal{T}$).

DEFINITION. Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are topological spaces and x is an element in X . A map $f : X \rightarrow Y$ is said to be *continuous at x* , if for any neighborhood N of $f(x)$ in the topology $\mathcal{S}$ (on Y), the set

$$f^{-1}(N) = \{x \in X \mid f(x) \in N\}$$

is a neighborhood of x in the topology $\mathcal{T}$ (on X).

If f is continuous at every point in X , then f is said to be *continuous*.

Continuity is “well behaved” with respect to compositions:

PROPOSITION 1.2. Suppose $(X, \mathcal{T})$, $(Y, \mathcal{S})$, and $(Z, \mathcal{Z})$ are topological spaces, and $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two functions.

- (i) If f is continuous at a point $x \in X$, and if g is continuous at $f(x)$, then $g \circ f$ is continuous at x .
- (ii) If f and g are (globally) continuous, then so is $g \circ f$.

The identity map on a topological space is always continuous.

In terms of open/closed sets, the characterization of continuity is given by the following.

PROPOSITION 1.3. If $(X, \mathcal{T})$, $(Y, \mathcal{S})$ are topological spaces and $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{S})$ is a map, then the following are equivalent:

- (i) f is continuous.
- (ii) Whenever $U \subset Y$ is an open set, it follows that $f^{-1}(U)$ is also an open set (in X).
- (iii) Whenever $F \subset Y$ is a closed set, it follows that $f^{-1}(F)$ is also a closed set (in X).

We conclude this section with a useful technical result.

THEOREM 1.1 (Urysohn's Lemma). *Let (X, T) be a topological Hausdorff space with the following property:*

- (N) *For any two disjoint closed sets $A, B \subset X$, there exist two disjoint open sets $U, V \subset X$, such that $U \supset A$ and $V \supset B$.*

Then for any two disjoint closed sets $A, B \subset X$, there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f|_A = 0$ and $f|_B = 1$.

PROOF. We begin with a refinement of property (N):

- (N') *For any disjoint closed sets $A, B \subset X$, there exist two open sets $U, W \subset X$, such that $A \subset U$, $\overline{U} \subset W$, and $\overline{W} \cap B = \emptyset$.*

To prove (N'), we first apply (N) to find two disjoint open sets $W, Z \subset X$ such that

$$(1) \quad W \supset A \text{ and } Z \supset B.$$

Next we apply again (N) to the pair of closed sets A and $X \setminus W$, and find two disjoint open sets $U, V \subset X$ such that

$$(2) \quad U \supset A \text{ and } V \supset X \setminus W.$$

On the one hand, using the fact that $U \cap V = \emptyset$ and the fact that V is open, we get the inclusion $\overline{U} \subset X \setminus V$. Using (2) this gives

$$\overline{U} \subset X \setminus V \subset W.$$

On the other hand, using the fact that $W \cap Z = \emptyset$ and the fact that Z is open, we get $\overline{W} \subset X \setminus Z$. But using (1) this will give

$$\overline{W} \subset X \setminus Z \subset X \setminus B,$$

and we are done.

To prove the Theorem, start with two disjoint closed sets $A, B \subset X$. For every integer $n \geq 0$ we define the set $\mathcal{D}_n = \{\frac{k}{2^n} : k \in \mathbb{Z}, 0 \leq k \leq 2^n\}$, and we consider

$$\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n.$$

(Notice that $\mathcal{D}_n \subset \mathcal{D}_{n+1}$, for all $n \geq 0$.)

We are going to construct a family $(V_t)_{t \in \mathcal{D}}$ of open sets in X with the following properties

- (i) $V_0 \supset A$ and $\overline{V_1} \cap B = \emptyset$;
- (ii) $\overline{V_t} \subset V_s$, for all $t, s \in \mathcal{D}$ with $t < s$.

Let us start by constructing V_0 and V_1 . We use property (N') to find open sets $U, W \subset X$, with

$$A \subset U \subset \overline{U} \subset W \text{ and } \overline{W} \cap B = \emptyset,$$

and we simply take $V_0 = U$ and $V_1 = W$.

The construction of the family $(V_t)_{t \in \mathcal{D}}$ is carried on recursively. Assume, for some integer $n \geq 0$, we have constructed the sets $(V_t)_{t \in \mathcal{D}_n}$ with property (i) and (ii) (satisfied for $t, s \in \mathcal{D}_n$), and let us construct the next block of sets $(V_t)_{t \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n}$. We start off by observing that for every $t \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n$, then the numbers

$$t^{\pm} = t \pm \frac{1}{2^{n+1}}$$

belong to $\mathcal{D}_n$. Apply (N') to the pair of disjoint closed sets $\overline{V}_{t-}$ and $X \setminus V_{t+}$ to find two open sets $U, W \subset X$ such that

$$\overline{V}_{t-} \subset U \subset \overline{U} \subset W \text{ and } \overline{W} \cap X \setminus V_{t+} = \emptyset.$$

Notice that the equality $\overline{W} \cap (X \setminus V_{t+}) = \emptyset$, coupled with the inclusion $\overline{U} \subset W$, gives $\overline{U} \cap (X \setminus V_{t+}) = \emptyset$, so we get $\overline{U} \subset V_{t+}$. We can then define $V_t = U$, and we will obviously have the inclusions

$$(3) \quad \overline{V}_{t-} \subset V_t \subset \overline{V}_t \subset V_{t+}.$$

Now the extended family $(V_t)_{t \in \mathcal{D}_{n+1}}$ will also satisfy property (ii), since for $t, s \in \mathcal{D}_{n+1}$ with $t < s$, one of the following will hold:

- either $t, s \in \mathcal{D}_n$, or
- $t \in \mathcal{D}_n, s \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n$, and $t \leq s^-$, or
- $t \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n, s \in \mathcal{D}_n$, and $t^+ \leq s$, or
- $t, s \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n$, and $t^+ \leq s^-$.

(In either case, one uses (3) combined with the inductive hypothesis.)

Having constructed the family $(V_t)_{t \in \mathcal{D}}$, with properties (i) and (ii), we define the functions $f : X \rightarrow [0, 1]$ by

$$f(x) = \begin{cases} \inf\{t \in \mathcal{D} : x \in V_t\}, & \text{if } x \in V_1 \\ 1, & \text{if } x \notin V_1 \end{cases}$$

Claim 1: The function f is equivalently defined by

$$(4) \quad f(x) = \begin{cases} 0, & \text{if } x \in \overline{V}_0 \\ \sup\{t \in \mathcal{D} : x \notin \overline{V}_t\}, & \text{if } x \notin \overline{V}_0 \end{cases}$$

Let us denote by $g : X \rightarrow [0, 1]$ be the function defined by formula (4). Fix some point $x \in X$. We break the proof in several cases

CASE I: $x \in \overline{V}_0$.

In particular, using (ii) we get $x \in V_t$, for all $t \in \mathcal{D}$, with $t > 0$, and since $x \in V_1$, we have

$$f(x) = \inf\{t \in \mathcal{D} : x \in V_t\} = \inf\{t \in \mathcal{D} : t > 0\} = 0 = g(x).$$

CASE II: $x \notin V_1$.

Using (ii) we have $x \notin \overline{V}_t$, for all $t \in \mathcal{D}$, with $t < 1$, and since $x \notin \overline{V}_0$, we have

$$g(x) = \sup\{t \in \mathcal{D} : x \notin \overline{V}_t\} = \sup\{t \in \mathcal{D} : t < 1\} = 1 = f(x).$$

CASE III: $x \in V_1 \setminus \overline{V}_0$.

By the definition of $f(x)$ we know:

$$(5) \quad x \notin V_t, \quad \forall t \in \mathcal{D}, \text{ with } t < f(x).$$

$$(6) \quad \forall \varepsilon > 0, \exists s_\varepsilon \in \mathcal{D}, \text{ with } f(x) \leq s_\varepsilon < f(x) + \varepsilon, \text{ such that } x \in V_{s_\varepsilon}.$$

By the definition of $g(x)$ we know:

$$(7) \quad x \in \overline{V}_t, \quad \forall t \in \mathcal{D}, \text{ with } t > g(x);$$

$$(8) \quad \forall \varepsilon > 0, \exists r_\varepsilon \in \mathcal{D}, \text{ with } g(x) \geq r_\varepsilon > g(x) - \varepsilon, \text{ such that } x \notin \overline{V}_{r_\varepsilon}.$$

Using (6) and (8) we see that we must have

$$(9) \quad s_\varepsilon \geq r_\varepsilon, \quad \forall \varepsilon > 0.$$

Indeed, if there exists some $\varepsilon > 0$ for which we have $s_\varepsilon < r_\varepsilon$, then using (6) we would have

$$x \in V_{s_\varepsilon} \subset \overline{V_{s_\varepsilon}} \subset V_{r_\varepsilon} \subset \overline{V_{r_\varepsilon}},$$

which contradicts (8).

Now the inequality (9) gives

$$f(x) + \varepsilon > g(x) - \varepsilon, \quad \forall \varepsilon > 0,$$

so we have in fact the inequality

$$f(x) \geq g(x).$$

Suppose now this inequality is strict. Using (5) and (7) we will get

$$(10) \quad x \in \overline{V_t} \text{ and } x \notin V_t, \text{ for all } t \in \mathcal{D}, \text{ with } f(x) > t > g(x).$$

Using the fact that $\mathcal{D}$ is dense in $[0, 1]$, we could then find at least two elements $t_1, t_2 \in \mathcal{D}$ such that

$$f(x) > t_1 > t_2 > g(x).$$

In this case (10) immediately creates a contradiction, since

$$x \in \overline{V_{t_2}} \subset V_{t_1}.$$

Claim 2: The function f is continuous.

Since any open set in $\mathbb{R}$ is a union of open intervals, it suffices to prove the following two properties¹

(USC): $f^{-1}((\infty, t))$ is open for all $t \in \mathbb{R}$;

(LSC): $f^{-1}((t, \infty))$ is open for all $t \in \mathbb{R}$.

In order to prove property (USC) it suffices to prove the equality

$$(11) \quad f^{-1}((\infty, t)) = \bigcup_{\substack{s \in \mathcal{D} \\ s < t}} V_s.$$

Start with a point $x \in f^{-1}((t, \infty))$, which means that $f(x) < t$. Using (6), there exists some $s \in \mathcal{D}$ with $f(x) < s < t$, such that $x \in V_s$, so x indeed belongs to the right hand side of (11). Conversely, if x belongs to the right hand side of (11), there exists some $s < t$ such that $x \in V_s$. By the definition of $f(x)$, it follows that $f(x) \leq s < t$, so $x \in f^{-1}((\infty, t))$.

In order to prove property (LSC) it suffices to prove the equality

$$(12) \quad f^{-1}((t, \infty)) = \bigcup_{\substack{r \in \mathcal{D} \\ r > t}} (X \setminus \overline{V_r}).$$

Start with a point $x \in f^{-1}((t, \infty))$, which means that $f(x) > t$. Using (8), there exists some $r \in \mathcal{D}$ with $f(x) > r > t$, such that $x \notin \overline{V_r}$, that is, $x \in X \setminus \overline{V_r}$, so x indeed belongs to the right hand side of (12). Conversely, if x belongs to the right hand side of (12), there exists some $r > t$ such that $x \in X \setminus \overline{V_r}$, i.e. $x \notin \overline{V_r}$. By the equivalent definition of $f(x)$ given by Claim 1, it follows that $f(x) \geq r > t$, so $x \in f^{-1}((t, \infty))$.

¹ The condition (USC) means that f is *upper semi-continuous*, while the condition (LSC) means that f is *lower semi-continuous*.

Having proven that f is continuous, let us finish the proof. Since $A \subset V_0$, by the definition of f , we get $f|_A = 0$. Since $B \subset X \setminus V_1$, again by the definition of f , we get $f|_B = 1$. $\square$

DEFINITION. A Hausdorff space $(X, \mathcal{T})$ with property (N) is called *normal*.

LECTURE 2

2. Ultrafilters

In this lecture we discuss a set theoretical concept, which turns out to be technically useful in topology.

DEFINITION. Suppose X is a fixed (non-empty) set. A *filter in X* is a (non-empty) family $\mathcal{F}$ of non-empty subsets of X which has the property²:

(F) *Whenever F and G belong to $\mathcal{F}$, it follows that $F \cap G$ also belongs to $\mathcal{F}$.*

What is important here is that all the sets in the filter are assumed to be *non-empty*. The set of all filters in X can be ordered by inclusion. A simple application of Zorn's Lemma yields:

- *For each filter $\mathcal{F}$ there exists at least one **maximal** filter $\mathcal{U}$ with $\mathcal{U} \supset \mathcal{F}$.*

Maximal filters will be called *ultrafilters*.

An interesting feature of ultrafilters is given by the following:

LEMMA 2.1. *Let X be a non-empty set, and let $\mathcal{U}$ be a filter on X . The following are equivalent:*

- (i) *$\mathcal{U}$ is an ultrafilter.*
- (ii) *For any subsets $A \subset X$, it follows that either A or $X \setminus A$ belongs to $\mathcal{U}$, but not both!*

PROOF. (i) $\Rightarrow$ (ii). Assume $\mathcal{U}$ is an ultrafilter. First remark that X always belongs to $\mathcal{U}$. (Otherwise, if X does not belong to $\mathcal{U}$, the family $\mathcal{U} \cup \{X\}$ will be obviously a new filter which will contradict the maximality of $\mathcal{U}$).

Let us assume that A is non-empty and it does not belong to $\mathcal{U}$. This means that the family

$$\mathcal{M} = \mathcal{U} \cup \{A \cap U \mid U \in \mathcal{U}\}$$

is no longer a filter (otherwise, the maximality of $\mathcal{U}$ will be contradicted). Note that if F and G belong to $\mathcal{M}$, then automatically $F \cap G$ belongs to $\mathcal{M}$. This means that the only thing that can prevent $\mathcal{M}$ from being a filter, must be the fact that one of the sets in $\mathcal{M}$ is empty. That is, there is some set $V \in \mathcal{U}$ such that $A \cap V = \emptyset$. In other words, $V \subset X \setminus A$. But then, it follows that for any $U \in \mathcal{U}$ we have $U \cap (X \setminus A) \supset U \cap V \neq \emptyset$ and then the set

$$\mathcal{N} = \mathcal{U} \cup \{U \cap (X \setminus A) \mid U \in \mathcal{U}\}$$

will be a filter. By maximality, it follows that $\mathcal{N} = \mathcal{U}$, in particular, $X \setminus A$ belongs to $\mathcal{U}$. It is obvious that A and $X \setminus A$ cannot simultaneously belong to $\mathcal{U}$, because this will force $\emptyset = A \cap (X \setminus A)$ to belong to $\mathcal{U}$.

² Some textbooks may use a slightly different definition.

(ii) $\Rightarrow$ (i). Assume property (ii) holds, but $\mathcal{U}$ is not maximal, which means that there exists some ultrafilter $\mathcal{V}$ with $\mathcal{V} \supsetneq \mathcal{U}$. Pick then some set $A \in \mathcal{V} \setminus \mathcal{U}$. Since $A \notin \mathcal{U}$, by (ii) we must have $X \setminus A \in \mathcal{U}$. This would force both A and $X \setminus A$ to belong to $\mathcal{V}$, which is impossible. $\square$

Exercise 1. Let $\mathcal{U}$ be an ultrafilter on X , and let $A \in \mathcal{U}$. Prove that the collection

$$\mathcal{U}|_A = \{U \cap A : U \in \mathcal{U}\}$$

is an ultrafilter on A .

REMARK 2.1. If $\mathcal{U}$ is an ultrafilter on X , and $A \in \mathcal{U}$, then $\mathcal{U}$ contains all sets B with $A \subset B \subset X$. Indeed, if we start with such a B , then by the above result, either $B \in \mathcal{U}$ or $X \setminus B \in \mathcal{U}$. Notice however that in the case $X \setminus B \in \mathcal{U}$ we would get

$$\mathcal{U} \ni (X \setminus B) \cap A = \emptyset,$$

which is impossible. Therefore B must belong to $\mathcal{U}$.

We are in position now to define the notion of convergence for ultrafilters, by means of the following.

PROPOSITION 2.1. *Let (X, T) be a topological space, let $\mathcal{U}$ be an ultrafilter in X , and let x be a point in X . The following are equivalent:*

- (i) *Every neighborhood of x belongs to $\mathcal{U}$.*
- (ii) *There exists $\mathcal{N}$ a basic system of neighborhoods of x , with $\mathcal{N} \subset \mathcal{U}$.*
- (iii) *There exists $\mathcal{V}$ a fundamental system of neighborhoods of x , with $\mathcal{V} \subset \mathcal{U}$.*

If the ultrafilter $\mathcal{U}$ satisfies one of the equivalent conditions above, we say that $\mathcal{U}$ is *convergent to x* , and we write $\mathcal{U} \rightarrow x$.

PROOF. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (i). Let $\mathcal{V}$ be a fundamental system of neighborhoods of x , with $\mathcal{V} \subset \mathcal{U}$. Start with an arbitrary neighborhood M of x . By the properties of $\mathcal{V}$, there exists a finite sequence $V_1, \dots, V_n \in \mathcal{V}$, with

$$x \in V_1 \cap \dots \cap V_n \subset M.$$

Since $\mathcal{V} \subset \mathcal{U}$, and $\mathcal{U}$ is a filter, it follows that the intersection $W = V_1 \cap \dots \cap V_n$ belongs to $\mathcal{U}$. By Remark 2.1 it follows that M itself belongs to $\mathcal{U}$. Since M was arbitrary, it follows that $\mathcal{U}$ indeed satisfies condition (i). $\square$

The Hausdorff property has a nice ultrafilter characterization:

PROPOSITION 2.2. *For a topological space (X, T) , the following are equivalent:*

- (i) *The topology T is Hausdorff.*
- (ii) *Every convergent ultrafilter in X has a unique limit.*

PROOF. (i) $\Rightarrow$ (ii). Assume the topology is Hausdorff. Let $\mathcal{U}$ be an ultrafilter in X which is convergent to both x and y . If $x \neq y$, then by the Hausdorff property, there exist two open sets $U, V \subset X$, with $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Since U is a neighborhood of x , we must have $U \in \mathcal{U}$. Likewise, we must have $V \in \mathcal{U}$. But this is impossible, since it will force $\mathcal{U} \ni U \cap V = \emptyset$.

(ii) $\Rightarrow$ (i). Assume X satisfies condition (ii), but the topology is not Hausdorff. This means that there exist two points $x, y \in X$, with $x \neq y$, such that

$$(*) \text{ for any open sets } U, V \subset X, \text{ with } U \ni x \text{ and } V \ni y, \text{ we have } U \cap V \neq \emptyset.$$

Let $\mathcal{N}_x$ denote the collection of all neighborhoods of x , and $\mathcal{N}_y$ denote the collection of all neighborhoods of y . By condition (*) we have

$$M \cap N \neq \emptyset, \quad \forall M \in \mathcal{N}_x, N \in \mathcal{N}_y.$$

This proves that the collection

$$\mathcal{F} = \{M \cap N : M \in \mathcal{N}_x, N \in \mathcal{N}_y\}$$

is a filter in X . Notice that, since X is a neighborhood for both x and y , we have the inclusion $\mathcal{F} \supset \mathcal{N}_x \cup \mathcal{N}_y$. So if we take $\mathcal{U}$ to be an ultrafilter, with $\mathcal{U} \supset \mathcal{F}$, it follows that $\mathcal{U} \supset \mathcal{N}_x$, hence $\mathcal{U}$ converges to x , but also $\mathcal{U} \supset \mathcal{N}_y$, hence $\mathcal{U}$ is also convergent to y . By condition (ii) this is impossible. $\square$

EXAMPLES 2.1. A. Let x be a point in X . We can consider the collection $\mathcal{U}_x = \{U \subset X \mid U \ni x\}$. Clearly $\mathcal{U}_x$ is an ultrafilter in X . This is called a *constant* ultrafilter at x . If $(X, \mathcal{T})$ is a topological space, then it is obvious that $\mathcal{U}_x$ is convergent to x .

B. (Example of a convergent non-constant ultrafilter.) Suppose $(X, \mathcal{T})$ is a topological space and x is a point in X such that for any neighborhood N of x , we have $N \setminus \{x\} \neq \emptyset$. Consider the collection

$$\mathcal{F} = \{N \setminus \{x\} \mid N \text{ neighborhood of } x\}.$$

Then $\mathcal{F}$ is a filter. If we take $\mathcal{U}$ any ultrafilter which contains $\mathcal{F}$, we get a non-constant (sometimes called *free*) ultrafilter. It is again clear that $\mathcal{U}$ is again convergent to x .

C. (Example of a non-convergent ultrafilter.) Let $\mathbb{N}$ be the set of non-negative integers. Equip $\mathbb{N}$ with the *discrete topology* (in which *every* subset is open). Consider the collection $\mathcal{F}$ consisting of all subsets $F \subset \mathbb{N}$ which have *finite* complement $\mathbb{N} \setminus F$. It is easy to check that $\mathcal{F}$ is a filter. Pick then $\mathcal{U}$ to be any ultrafilter with $\mathcal{U} \supset \mathcal{F}$. Since on $\mathbb{N}$ we use the discrete topology, it follows that the only convergent ultrafilters are the constant ones. Note however, that if $n \in \mathbb{N}$, then the set $\mathbb{N} \setminus \{n\}$ belongs to $\mathcal{F}$, hence to $\mathcal{U}$. This means that the singleton set $\{n\}$ cannot belong to $\mathcal{U}$. Therefore $\mathcal{U}$ cannot be constant.

REMARK 2.2. Maps between sets can be put to act on ultrafilters. More explicitly one has the following construction. Suppose $f : X \rightarrow Y$ is a map and $\mathcal{U}$ is an ultrafilter in X . Consider the collection

$$f_*(\mathcal{U}) = \{V \subset Y \mid f^{-1}(V) \in \mathcal{U}\}.$$

Then $f_*(\mathcal{U})$ is an ultrafilter on Y . Indeed, it is easy to show that $f_*(\mathcal{U})$ is a filter. To prove that it is maximal, let us take $\mathcal{F}$ a filter on Y with $\mathcal{F} \supset f_*(\mathcal{U})$ and let us consider an arbitrary set F which belongs to $\mathcal{F}$. Since $\mathcal{U}$ is an ultrafilter on X it follows that either $f^{-1}(F)$ or $X \setminus f^{-1}(F)$ belongs to $\mathcal{U}$. If $X \setminus f^{-1}(F)$ belongs to $\mathcal{U}$, using the equality $X \setminus f^{-1}(F) = f^{-1}(Y \setminus F)$ it follows that $Y \setminus F$ belongs to $f_*(\mathcal{U})$, hence to $\mathcal{F}$. But this is impossible, since F also belongs to $\mathcal{F}$ and this will force the empty set $F \cap (Y \setminus F)$ to belong to the filter $\mathcal{F}$. This contradiction shows that the set $f^{-1}(F)$ belongs to $\mathcal{U}$, which means precisely that F belongs to $f_*(\mathcal{U})$. This argument proves the inclusion $\mathcal{F} \subset f_*(\mathcal{U})$, so $f_*(\mathcal{U})$ is indeed a maximal filter.

REMARK 2.3. With the above notations, one has

$$f(U) \in f_*(\mathcal{U}), \quad \forall U \in \mathcal{U}.$$

One can prove this property by contradiction. Assume $f(U)$ does not belong to $f_*(\mathcal{U})$, for some $U \in \mathcal{U}$. Then $Y \setminus f(U)$ belongs to $f_*(\mathcal{U})$, which means that the set

$$M = f^{-1}(Y \setminus f(U)) = X \setminus f^{-1}(f(U))$$

belongs to $\mathcal{U}$. But using the obvious inclusion $U \subset f^{-1}(f(U))$, this gives $M \cap U = \emptyset$, which is impossible.

Continuity can be nicely characterized using ultrafilters:

PROPOSITION 2.3. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces, and let x be element in X . For a function $f : X \rightarrow Y$, the following are equivalent:*

- (i) *f is continuous at x .*
- (ii) *Whenever $\mathcal{U}$ is an ultrafilter on X convergent to x , it follows that the ultrafilter $f_*(\mathcal{U})$ in Y , convergent to $f(x)$.*

PROOF. (i) $\Rightarrow$ (ii). Assume that f is continuous at x . Start with an ultrafilter $\mathcal{U}$ on X , with $\mathcal{U} \rightarrow x$. Let N be an arbitrary neighborhood of $f(x)$. Since f is continuous at x , it follows that $f^{-1}(N)$ is a neighborhood of x . In particular we get $f^{-1}(N) \in \mathcal{U}$, which proves that $N \in f_*(\mathcal{U})$. Since the ultrafilter $f_*(\mathcal{U})$ contains all neighborhoods of $f(x)$, it means that indeed $f_*(\mathcal{U})$ is convergent to $f(x)$.

(ii) $\Rightarrow$ (i). Assume f satisfies condition (ii), but f is not continuous at x . This means that there exists some neighborhood V of $f(x)$ such that $f^{-1}(V)$ is not a neighborhood of x . Consider the collection

$$\mathcal{F} = \{N \setminus f^{-1}(V) : N \text{ neighborhood of } x\}.$$

Our assumption on V shows that all the sets in $\mathcal{F}$ are non-empty. (Otherwise $f^{-1}(V)$ would contain some neighborhood of x , which would force $f^{-1}(V)$ itself to be a neighborhood of x .) It is also clear that $\mathcal{F}$ is a filter. Let $\mathcal{U}$ be an ultrafilter with $\mathcal{U} \supset \mathcal{F}$.

Claim: The ultrafilter $\mathcal{U}$ is convergent to x .

To prove this, start with some arbitrary neighborhood N of x . If N does not belong to $\mathcal{U}$, then $X \setminus N$ belongs to $\mathcal{U}$. But then $(X \setminus N) \cap (N \setminus f^{-1}(V)) = \emptyset$ belongs to $\mathcal{U}$, which is impossible. So $\mathcal{U}$ contains all neighborhoods of x , which means that indeed $\mathcal{U}$ is convergent to x .

Using our assumption on V , plus condition (ii), it follows that $V \in f_*(\mathcal{U})$, which means that $f^{-1}(V) \in \mathcal{U}$. But this leads to a contradiction, since $X \setminus f^{-1}(V)$ clearly belongs to $\mathcal{F} \subset \mathcal{U}$. $\square$

LECTURE 3

3. Constructing topologies

In this section we discuss several methods for constructing topologies on a given set.

DEFINITION. If $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on the same space X , such that $\mathcal{T}' \subset \mathcal{T}$ (as sets), then $\mathcal{T}$ is said to be *stronger than* $\mathcal{T}'$. Equivalently, we will say that $\mathcal{T}'$ is *weaker than* $\mathcal{T}$.

Remark that this condition is equivalent to the continuity of the map

$$\text{Id} : (X, \mathcal{T}) \rightarrow (X, \mathcal{T}').$$

COMMENT. Given a (non-empty) set X , and a collection $\mathcal{S}$ of subsets of X , one can ask the following:

Question 1: *Is there a topology on X with respect to which all the sets in $\mathcal{S}$ are open?*

Of course, this question has an affirmative answer, since we can take as the topology the collection of *all* subsets of X . Therefore the above question is more meaningful if stated as:

Question 2: *Is there the weakest topology on X with respect to which all the sets in $\mathcal{S}$ are open?*

The answer to this question is again affirmative, and it is based on the following:

REMARK 3.1. If X is a non-empty set, and $(\mathcal{T}_i)_{i \in I}$ is a family of topologies on X , then the intersection

$$\bigcap_{i \in I} \mathcal{T}_i$$

is again a topology on X .

In particular, if one starts with an arbitrary family $\mathcal{S}$ of subsets of X , and if we take

$$\Theta(\mathcal{S}) = \{\mathcal{T} : \mathcal{T} \text{ topology on } X \text{ with } \mathcal{T} \supset \mathcal{S}\},$$

then the intersection

$$\text{TOP}(\mathcal{S}) = \bigcap_{\mathcal{T} \in \Theta(\mathcal{S})} \mathcal{T}$$

is the weakest (i.e. smallest) among all topologies with respect to which all sets in $\mathcal{S}$ are open.

The topology $\text{TOP}(\mathcal{S})$ defined above can also be described constructively as follows.

PROPOSITION 3.1. *Let $\mathcal{S}$ be a collection of subsets of X . Then the sets in $\text{TOP}(\mathcal{S})$, which are a proper subsets of X , are those which can be written a (arbitrary) unions of finite intersections of sets in $\mathcal{S}$.*

PROOF. It is useful to introduce the following notations. First we define $\mathcal{V}(\mathcal{S})$ to be the collection of all sets which are finite intersections of sets in $\mathcal{S}$. In other words,

$$B \in \mathcal{V}(\mathcal{S}) \iff \exists D_1, \dots, D_n \in \mathcal{S} \text{ such that } D_1 \cap \dots \cap D_n = B.$$

With the above notation, what we need to prove is that for a set $A \subseteq X$, we have

$$A \in \text{TOP}(\mathcal{S}) \iff \exists \mathcal{V}_A \subset \mathcal{V}(\mathcal{S}) \text{ such that } A = \bigcup_{B \in \mathcal{V}_A} B.$$

The implication “ $\Leftarrow$ ” is pretty obvious. Since $\text{TOP}(\mathcal{S})$ is a topology, and every set in $\mathcal{S}$ is open with respect to $\text{TOP}(\mathcal{S})$, it follows that every finite intersection of sets in $\mathcal{S}$ is again in $\text{TOP}(\mathcal{S})$, which means that every set in $\mathcal{V}(\mathcal{S})$ is again open with respect to $\text{TOP}(\mathcal{S})$. But then arbitrary unions of sets in $\mathcal{V}(\mathcal{S})$ are again open with respect to $\text{TOP}(\mathcal{S})$.

To prove the implication “ $\Rightarrow$ ” we define

$$\mathcal{T}_0 = \{A \subset X : \exists \mathcal{V}_A \subset \mathcal{V}(\mathcal{S}) \text{ such that } A = \bigcup_{B \in \mathcal{V}_A} B\},$$

and we will show that

$$(1) \quad \text{TOP}(\mathcal{S}) \subset \{X\} \cup \mathcal{T}_0.$$

By the definition of $\text{TOP}(\mathcal{S})$ it suffices to prove the following

Claim: The collection $\mathcal{T}_1 = \{X\} \cup \mathcal{T}_0$ is a topology on X , which contains all the sets in $\mathcal{S}$.

The fact that $\mathcal{T}_1 \supset \mathcal{S}$ is trivial.

The fact that $\emptyset, X \in \mathcal{T}_1$ is also clear.

The fact that arbitrary unions of sets in $\mathcal{T}_1$ again belong to $\mathcal{T}_1$ is again clear, by construction.

Finally, we need to show that if $A_1, A_2 \in \mathcal{T}_1$, then $A_1 \cap A_2 \in \mathcal{T}_1$. If either $A_1 = X$ or $A_2 = X$, there is nothing to prove. Assume that both A_1 and A_2 are proper subsets of X , so there are subsets $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{V}(\mathcal{S})$, such that

$$A_1 = \bigcup_{B \in \mathcal{V}_1} B \text{ and } A_2 = \bigcup_{E \in \mathcal{V}_2} E.$$

Then it is clear that

$$A_1 \cap A_2 = \bigcup_{\substack{B \in \mathcal{V}_1 \\ E \in \mathcal{V}_2}} (B \cap E),$$

with all the sets $B \cap E$ in $\mathcal{V}(\mathcal{S})$, so $A_1 \cap A_2$ indeed belongs to $\mathcal{T}_1$. $\square$

DEFINITION. Let X be a (non-empty) set, let $\mathcal{T}$ be a topology on X . A collection $\mathcal{S}$ of subsets of X , with the property that

$$\mathcal{T} = \text{TOP}(\mathcal{S}),$$

is called a *sub-base for $\mathcal{T}$* . According to the above remark, the above condition is equivalent to the fact that every open set $D \subseteq X$ can be written as a *union of finite intersections of sets in $\mathcal{S}$* .

Convergence of ultrafilters is characterized using sub-bases as follows;

PROPOSITION 3.2. *Let $(X, \mathcal{T})$ be a topological space, let $\mathcal{S}$ be a sub-base for $\mathcal{T}$, and let x be some point in X . For an ultrafilter $\mathcal{U}$ on X , the following are equivalent:*

- (i) $\mathcal{U}$ is convergent to x ;
- (ii) $\mathcal{U}$ contains all the sets $S \in \mathcal{S}$ with $S \ni x$.

PROOF. The implication (i) $\Rightarrow$ (ii) is trivial.

To prove the implication (ii) $\Rightarrow$ (i), we assume $\mathcal{U}$ has property (ii), we consider some neighborhood N of x , and let us prove that N belongs to $\mathcal{U}$. Since N is a neighborhood of x , there exists some open set D , with $x \in D \subset N$. Furthermore, by Proposition 3.1, either

- (a) $D = X$, or
- (b) there exist sets $S_1, S_2, \dots, S_n \in \mathcal{S}$ with

$$x \in S_1 \cap S_2 \cap \dots \cap S_n \subset D \subset N.$$

In case (a) we immediately have $N = X$, and we obviously get $N \in \mathcal{U}$. In case (b) it follows that $S_1, \dots, S_n \in \mathcal{U}$, so the intersection $S_1 \cap S_2 \cap \dots \cap S_n$ also belongs to $\mathcal{U}$. By Remark 2.1 it then follows that N itself belongs to $\mathcal{U}$. $\square$

There are instances when sub-bases have a particular feature, which enables one to describe all open sets in an easier fashion.

PROPOSITION 3.3. *Let $(X, \mathcal{T})$ be a topological space. Suppose $\mathcal{V}$ is a collection of subsets of X . The following are equivalent:*

- (i) $\mathcal{V}$ is a sub-base for $\mathcal{T}$, and
- (2) $\forall U, V \in \mathcal{V}$ and $x \in U \cap V$, $\exists W \in \mathcal{V}$ with $x \in W \subset U \cap V$.
- (ii) Every open set $A \subsetneq X$ is a union of sets in $\mathcal{V}$.

PROOF. (i) $\Rightarrow$ (ii). From property (i), it follows that every finite intersection of sets in $\mathcal{V}$ is a union of sets in $\mathcal{V}$. Then the desired implication is immediate from the previous result.

(ii) $\Rightarrow$ (i). Assume (ii) and start with two sets $U, V \in \mathcal{V}$, and an element $x \in U \cap V$. Since $U \cap V$ is open, by (ii) either we have $U \cap V = X$, in which case we get $U = V = X$, and we take $W = X$, or $U \cap V \subsetneq X$, in which case $U \cap V$ is a union of sets in $\mathcal{V}$, so in particular there exists $W \in \mathcal{V}$ with $x \in W \subset U \cap V$. $\square$

DEFINITION. If $(X, \mathcal{T})$ is a topological space, a collection $\mathcal{V}$ which satisfies the above equivalent conditions, is called a *base for $\mathcal{T}$* .

The following is a useful technical result.

LEMMA 3.1. *Let $(Y, \mathcal{T})$ be a topological space, let X be some (non-empty) set, and let $f : X \rightarrow Y$ be a function. Then the collection*

$$f^*(\mathcal{T}) = \{f^{-1}(D) : D \in \mathcal{T}\}$$

is a topology on X . Moreover, $f^(\mathcal{T})$ is the weakest topology on X , with respect to which the map f is continuous.*

PROOF. Clearly $\emptyset = f^{-1}(\emptyset)$ and $X = f^{-1}(Y)$ both belong to $f^*(\mathcal{T})$. If $(A_i)_{i \in I}$ is a family of sets in $f^*(\mathcal{T})$, say $A_i = f^{-1}(D_i)$, for some $D_i \in \mathcal{T}$, for all

$i \in I$, then the equality

$$\bigcup_{i \in I} A_i = \bigcup_{i \in I} f^{-1}(D_i) = f\left(\bigcup_{i \in I} D_i\right)$$

clearly shows that $\bigcup_{i \in I} A_i$ again belongs to $f^*(\mathcal{T})$. Likewise, if $A_1, A_2 \in f^*(\mathcal{T})$, say $A_1 = f^{-1}(D_1)$ and $A_2 = f^{-1}(D_2)$ for some $D_1, D_2 \in \mathcal{T}$, then the equality

$$A_1 \cap A_2 = f^{-1}(D_1) \cap f^{-1}(D_2) = f^{-1}(D_1 \cap D_2)$$

proves that $A_1 \cap A_2$ again belongs to $f^*(\mathcal{T})$.

Having proven that $f^*(\mathcal{T})$ is a topology on X , let us prove now the second statement. The fact that f is continuous with respect to $f^*(\mathcal{T})$ is clear by construction. If $\mathcal{T}'$ is another topology which still makes f continuous, then this will force all the sets of the form $f^{-1}(D)$, $D \in \mathcal{T}$ to belong to $\mathcal{T}'$, which means that $f^*(\mathcal{T}) \subset \mathcal{T}'$. $\square$

REMARK 3.2. Using the above notations, if $\mathcal{V}$ is a (sub)base for $\mathcal{T}$, then

$$f^*(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\}$$

is a (sub)base for $f^*(\mathcal{T})$. This is pretty obvious since the correspondence

$$\{\text{subsets of } Y\} \ni D \longmapsto f^{-1}(D)$$

is compatible with the operation of intersection and union (of arbitrary families).

REMARK 3.3. As a consequence of the above remark, we see that (sub)bases can be useful for verifying continuity. More specifically, if $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ are topological spaces, and $\mathcal{V}$ is a sub-base for $\mathcal{T}'$, then a function $f : X \rightarrow X'$ is continuous, if and only if $f^{-1}(V)$ is open, for all $V \in \mathcal{V}$.

The construction outlined in Lemma 3.1 can be generalized as follows.

PROPOSITION 3.4. *Let X be a set, and let $\Phi = (f_i, Y_i)_{i \in I}$ be a family consisting of maps $f_i : X \rightarrow Y_i$, where Y_i is a topological space, for all $i \in I$. Then there is a unique topology $\mathcal{T}^\Phi$ on X , with the following properties*

- (i) *Each of the maps $f_i : X \rightarrow Y_i$, $i \in I$ is continuous with respect to $\mathcal{T}^\Phi$.*
- (ii) *Given a topological space $(Z, \mathcal{S})$, and a map $g : Z \rightarrow X$, such that the composition $f_i \circ g : Z \rightarrow Y_i$ is continuous, for every $i \in I$, it follows that g is continuous as a map $(Z, \mathcal{S}) \rightarrow (X, \mathcal{T}^\Phi)$.*

PROOF. For every $i \in I$ we define

$$\mathcal{D}_i = \{f_i^{-1}(D) : D \text{ open subset of } Y_i\},$$

and we form the collection

$$\mathcal{D} = \bigcup_{i \in I} \mathcal{D}_i.$$

Take $\mathcal{T}^\Phi = \text{TOP}(\mathcal{D})$. Property (i) follows from the simple observation that, by construction, every set in $\mathcal{D}$ is open.

To prove property (ii) start with a topological space $(Z, \mathcal{S})$, and a map $g : Z \rightarrow X$, such that the composition $f_i \circ g : Z \rightarrow Y_i$ is continuous, for every $i \in I$, and let us prove that g is continuous. By Remark 3.3 it suffices to prove that $g^{-1}(A)$ is open (in Z) for every $A \in \mathcal{D}$. By the definition of $\mathcal{D}$ this is equivalent to proving the fact that, for each $i \in I$, and each open set $D \subset Y_i$, the set $g^{-1}(f_i^{-1}(D))$ is open. But this is obvious, since we have

$$g^{-1}(f_i^{-1}(D)) = (f_i \circ g)^{-1}(D),$$

and $f_i \circ g : Z \rightarrow Y_i$ is continuous.

To prove the uniqueness, let $\mathcal{T}$ be another topology on X with properties (i) and (ii). Consider the map $h = \text{Id} : (X, \mathcal{T}) \rightarrow (X, \mathcal{T}^\Phi)$. Using property (i) for $\mathcal{T}$, combined with property (ii) for $\mathcal{T}^\Phi$, it follows that h is continuous, which means that $\mathcal{T}^\Phi \subset \mathcal{T}$. Reversing the roles, and arguing exactly the same way, we also get the other inclusion $\mathcal{T} \subset \mathcal{T}^\Phi$. $\square$

REMARK 3.4. Using the above setting, assume that for each $i \in I$ a sub-base $\mathcal{S}_i$ for the topology of Y_i is given. Consider the sets $f_i^* \mathcal{S}_i = \{f_i^{-1}(S) : S \in \mathcal{S}_i\}$. Then the collection

$$\mathcal{S} = \bigcup_{i \in I} f_i^* \mathcal{S}_i$$

is a sub-base for the topology $\mathcal{T}^\Phi$.

To prove this, we take $\mathcal{T} = \text{TOP}(\mathcal{S})$, so that we obviously have the inclusion $\mathcal{T} \subset \mathcal{T}^\Phi$. In order to prove the equality $\mathcal{T} = \mathcal{T}^\Phi$, all we have to prove are (use the notations from the proof of the above Proposition) the inclusions

$$\mathcal{D}_i \subset \mathcal{T}, \quad \forall i \in I.$$

By construction however, we have $\mathcal{D}_i = f_i^* \mathcal{T}_i$, and since $\mathcal{S}_i$ is a sub-base for $\mathcal{T}_i$, it follows that $f_i^* \mathcal{S}_i$ is a sub-base for $\mathcal{D}_i$, which means that we have

$$\mathcal{D}_i = \text{TOP}(f_i^* \mathcal{S}_i) \subset \text{TOP}(\mathcal{S}) = \mathcal{T}.$$

COMMENT. Using the notations above, it is immediate that the topology $\mathcal{T}^\Phi$ can also be described as *the weakest topology on X , with respect to which all the maps $f_i : X \rightarrow Y_i$, $i \in I$, are continuous*. In the light of this remark, we will call the topology $\mathcal{T}^\Phi$ the *weak topology defined by Φ* .

Convergence for ultrafilters can be nicely characterized:

PROPOSITION 3.5. *Let X be a set, and let $\Phi = (f_i, Y_i)_{i \in I}$ be a family consisting of maps $f_i : X \rightarrow Y_i$, where Y_i is a topological space, for all $i \in I$. For an ultrafilter $\mathcal{U}$ on X and a point $x \in X$, the following are equivalent:*

- (i) $\mathcal{U}$ is convergent to x , with respect to the topology $\mathcal{T}^\Phi$;
- (ii) for every $i \in I$, the ultrafilter $f_{i*}(\mathcal{U})$ is convergent to $f_i(x)$.

PROOF. (i) $\Rightarrow$ (ii). This implication is clear, since all maps $f_i : (X, \mathcal{T}^\Phi) \rightarrow (Y_i, \mathcal{T}_i)$, $i \in I$, are continuous.

(ii) $\Rightarrow$ (i). Suppose $\mathcal{U}$ satisfies (ii). Then for every $i \in I$, the ultrafilter $f_{i*}(\mathcal{U})$ contains all the open sets $D \subset Y_i$ with $D \ni f_i(x)$. This means that $f_i^{-1}(D) \in \mathcal{U}$. But by construction, the topology $\mathcal{T}^\Phi$ has

$$\mathcal{S} = \bigcup_{i \in I} \{f_i^{-1}(D) : D \subset Y_i \text{ open}\},$$

as a sub-base, so if we define

$$\mathcal{S}_x = \{S \in \mathcal{S} : S \ni x\},$$

we clearly have $\mathcal{U} \supset \mathcal{S}_x$. Then the fact that $\mathcal{U}$ converges to x follows from Proposition 3.2. $\square$

EXAMPLE 3.1. (The product topology) Suppose we have a family $(X_i, \mathcal{T}_i)$, $i \in I$ of topological spaces. Consider the Cartesian product $X = \prod_{i \in I} X_i$. For each $j \in I$

we consider the projection $\pi_j : X \rightarrow X_j$. The weakest topology on X , defined by the family $\Phi = \{\pi_j\}_{j \in I}$, is called the *product topology*.

A sub-base for the product topology can be defined as follows. For each $i \in I$, we choose a sub-base $\mathcal{S}_i$ for $\mathcal{T}_i$ (for instance we can take $\mathcal{S}_i = \mathcal{T}_i$), and we take

$$\mathcal{S} = \bigcup_{i \in I} \pi_i^* \mathcal{S}_i = \bigcup_{i \in I} \{\pi_i^{-1}(D) : D \in \mathcal{S}_i\}.$$

Then $\mathcal{S}$ is a sub-base for the product topology.

For a point $x = (x_i)_{i \in I} \in X$, and an ultrafilter $\mathcal{U}$ on X , the condition $\mathcal{U} \rightarrow x$ is equivalent to the fact that $\pi_{i*}(\mathcal{U}) \rightarrow x_i$, $\forall i \in I$.

Another method of constructing topologies is based on the following “dual” version of Lemma 3.1.

LEMMA 3.2. *Let $(Y, \mathcal{T})$ be a topological space, let X be some (non-empty) set, and let $f : Y \rightarrow X$ be a function. Then the collection*

$$f_*(\mathcal{T}) = \{D \subset X : f^{-1}(D) \in \mathcal{T}\}$$

is a topology on X . Moreover, $f_(\mathcal{T})$ is the strongest topology on X , with respect to which the map f is continuous.*

PROOF. Since $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(X) = Y$, it follows that $\emptyset$ and X both belong to $f_*(\mathcal{T})$. If $(A_i)_{i \in I}$ is a family of sets in $f_*(\mathcal{T})$, then the sets $f^{-1}(A_i)$, $i \in I$ belong to $\mathcal{T}$. In particular the set

$$f^{-1}\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f^{-1}(A_i)$$

will again belong to $\mathcal{T}$, which means that $\bigcup_{i \in I} A_i$ belongs to $f_*(\mathcal{T})$. Likewise, if $A_1, A_2 \in f_*(\mathcal{T})$, then the sets $f^{-1}(A_1)$ and $f^{-1}(A_2)$ both belong to $\mathcal{T}$. The intersection

$$f^{-1}(A_1 \cap A_2) = f^{-1}(A_1) \cap f^{-1}(A_2)$$

will then belong to $\mathcal{T}$, which means that $A_1 \cap A_2$ again belongs to $f_*(\mathcal{T})$.

Having proven that $f_*(\mathcal{T})$ is a topology on X , let us prove now the second statement. The fact that f is continuous with respect to $f_*(\mathcal{T})$ is clear by construction. If $\mathcal{T}'$ is another topology which still makes f continuous, then this will force all the sets of the form $f^{-1}(A)$, $A \in \mathcal{T}'$ to belong to $\mathcal{T}$, which means that A will in fact belong to $f_*(\mathcal{T})$. In other words, we have the inclusion $\mathcal{T}' \subset f_*(\mathcal{T})$. $\square$

A generalization of the above construction is given in the following.

PROPOSITION 3.6. *Let X be a set, and let $\Phi = (f_i, Y_i)_{i \in I}$ be a family consisting of maps $f_i : Y_i \rightarrow X$, where Y_i is a topological space, for all $i \in I$. Then there is a unique topology $\mathcal{T}_\Phi$ on X , with the following properties*

- (i) *Each of the maps $f_i : Y_i \rightarrow X$, $i \in I$ is continuous with respect to $\mathcal{T}_\Phi$.*
- (ii) *Given a topological space $(Z, \mathcal{S})$, and a map $g : X \rightarrow Z$, such that the composition $g \circ f_i : Y_i \rightarrow Z$ is continuous, for every $i \in I$, it follows that g is continuous as a map $(X, \mathcal{T}_\Phi) \rightarrow (Z, \mathcal{S})$.*

PROOF. For each $i \in I$, let $\mathcal{T}_i$ denote the topology on Y_i . We define

$$\mathcal{T}_\Phi = \bigcap_{i \in I} f_{i*}(\mathcal{T}_i).$$

Property (i) is obvious by construction.

To prove property (ii), start with some topological space $(Z, \mathcal{S})$ and a map $g : X \rightarrow Z$ such that $g \circ f_i : Y_i \rightarrow Z$ is continuous, for all $i \in I$. Start with some open set $D \subset Z$, and let us prove that the set $A = g^{-1}(D)$ is open in X , i.e. $A \in \mathcal{T}_\Phi$. Notice that, for each $i \in I$, one has

$$f_i^{-1}(A) = f_i^{-1}(g^{-1}(D)) = (g \circ f_i)^{-1}(D),$$

so using the continuity of $g \circ f_i$ we get the fact that $f_i^{-1}(A)$ is open in Y_i , which means that $A \in f_{i*}(\mathcal{T}_i)$. Since this is true for all $i \in I$, we then get $A \in \mathcal{T}_\Phi$.

To prove uniqueness, let $\mathcal{T}$ be another topology on X with properties (i) and (ii). Consider the map $h = \text{Id} : (X, \mathcal{T}) \rightarrow (X, \mathcal{T}^\Phi)$. Using property (i) for $\mathcal{T}_\Phi$, combined with property (ii) for $\mathcal{T}$, it follows that h is continuous, which means that $\mathcal{T}^\Phi \subset \mathcal{T}$. Reversing the roles, and arguing exactly the same way, we also get the other inclusion $\mathcal{T} \subset \mathcal{T}^\Phi$. $\square$

COMMENT. Using the notations above, it is immediate that the topology $\mathcal{T}_\Phi$ can also be described as *the strongest topology on X , with respect to which all the maps $f_i : Y_i \rightarrow X$, $i \in I$, are continuous*. In the light of this remark, we will call the topology $\mathcal{T}_\Phi$ the *strong topology defined by Φ* .

EXAMPLE 3.2. (The disjoint union topology) Suppose we have a family $(X_i, \mathcal{T}_i)$, $i \in I$ of topological spaces. Consider the disjoint union³ $X = \bigsqcup_{i \in I} X_i$. For each $i \in I$

we consider the inclusion $\epsilon_i : X_i \rightarrow X$. The strongest topology on X , defined by the family $\Phi = \{\epsilon_i\}_{i \in I}$, is called the *disjoint union topology*.

If we think each X_i as a subset of X , then X_i is open in X , for all $i \in I$. Moreover, a set $D \subset X$ is open, if and only if $D \cap X_i$ is open (in X_i), for all $i \in I$. For a point $x \in X$, there exists a unique $i(x) \in I$, with $x \in X_{i(x)}$. With this notation, an ultrafilter $\mathcal{U}$ on X is convergent to x , if and only if $X_{i(x)} \in \mathcal{U}$, and the collection

$$\mathcal{U}|_{X_{i(x)}} = \{U \cap X_{i(x)} : U \in \mathcal{U}\}$$

is an ultrafilter on $X_{i(x)}$, which converges to x .

³ Formally one uses the sets $Z = \bigcup_{i \in I} X_i$, and $Y = I \times Z$, and one realizes the disjoint union as $X = \bigcup_{i \in I} \{i\} \times X_i$.

LECTURE 4

4. Compactness

DEFINITION. Let X be a topological space X . A subset $K \subset X$ is said to be *compact set in X* , if it has the *finite open cover* property:

(F.O.C) *Whenever $\{D_i\}_{i \in I}$ is a collection of open sets such that $K \subset \bigcup_{i \in I} D_i$, there exists a finite sub-collection $D_{i_1}, \dots, D_{i_n}$ such that*

$$K \subset D_{i_1} \cup \dots \cup D_{i_n}.$$

An equivalent description is the *finite intersection* property:

(F.I.P.) *If $\{F_i\}_{i \in I}$ is a collection of closed sets such that for any finite sub-collection $F_{i_1}, \dots, F_{i_n}$ we have $K \cap F_{i_1} \cap \dots \cap F_{i_n} \neq \emptyset$, it follows that*

$$K \cap \left(\bigcap_{i \in I} F_i \right) \neq \emptyset.$$

A topological space $(X, \mathcal{T})$ is called compact if X itself is a compact set.

REMARK 4.1. Suppose $(X, \mathcal{T})$ is a topological space, and K is a subset of X . Equip K with the induced topology $\mathcal{T}|_K$. Then it is straightforward from the definition that the following are equivalent:

- K is compact, as a subset in $(X, \mathcal{T})$;
- $(K, \mathcal{T}|_K)$ is a compact space, that is, K is compact as a subset in $(K, \mathcal{T}|_K)$.

The following three results give methods of constructing compact sets.

PROPOSITION 4.1. *A finite union of compact sets is compact.*

PROOF. Immediate from the definition. □

PROPOSITION 4.2. *Suppose $(X, \mathcal{T})$ is a topological space and $K \subset X$ is a compact set. Then for every closed set $F \subset X$, the intersection $F \cap K$ is again compact.*

PROOF. Immediate, using the finite intersection property. □

PROPOSITION 4.3. *Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are topological spaces, $f : X \rightarrow Y$ is a continuous map, and $K \subset X$ is a compact set. Then $f(K)$ is compact.*

PROOF. Immediate from the definition. □

Besides the two equivalent conditions (F.O.C) and (F.I.P.), there are some other useful characterizations of compactness, listed in the following.

THEOREM 4.1. *Let $(X, \mathcal{T})$ be a topological space. The following are equivalent:*

- (i) X is compact.

(ii) (Alexander sub-base Theorem) *There exists a sub-base $\mathcal{S}$ with the finite open cover property:*

(s) *For any collection $\{S_i \mid i \in I\} \subset \mathcal{S}$ with $X = \bigcup_{i \in I} S_i$, there exists a finite sub-collection $\{S_{i_1}, S_{i_2}, \dots, S_{i_n}\}$ (for some finite sequence of indices $i_1, i_2, \dots, i_n \in I$) such that $X = S_{i_1} \cup S_{i_2} \cup \dots \cup S_{i_n}$.*

(iii) *Every ultrafilter in X is convergent.*

PROOF. (i) $\Rightarrow$ (ii). This is obvious. (In fact *any* sub-base has the open cover property.)

(ii) $\Rightarrow$ (iii). Let $\mathcal{U}$ be an ultrafilter on X . Assume $\mathcal{U}$ is not convergent to any point $x \in X$. By Proposition 3.2 it follows that, for each $x \in X$, one can find a set $S_x \in \mathcal{S}$ with $S_x \ni x$, but such that $S_x \notin \mathcal{U}$. Using property (s), one can find a finite collection of points $x_1, \dots, x_n \in X$, such that

$$(1) \quad S_{x_1} \cup \dots \cup S_{x_n} = X.$$

Since $S_{x_p} \notin \mathcal{U}$, it means that $X \setminus S_{x_p}$ belongs to $\mathcal{U}$, for every $p = 1, \dots, n$. Then, using (1), we get

$$\mathcal{U} \ni (X \setminus S_{x_1}) \cap \dots \cap (X \setminus S_{x_n}) = \emptyset,$$

which is impossible.

(iii) $\Rightarrow$ (i). Assuming property (iii), we will show that X has the finite intersection property. Start with a family of closed sets $\{F_i\}_{i \in I}$, with the property that

$$(2) \quad \bigcap_{i \in J} F_i \neq \emptyset, \text{ for every finite subset } J \subset I.$$

We want to prove that $\bigcap_{i \in I} F_i \neq \emptyset$. For every finite subset $J \subset I$ we define the non-empty closed set $F_J = \bigcap_{i \in J} F_i$. It is clear that

$$\mathcal{F} = \{F_J : J \text{ finite subset of } I\}$$

is a *filter*. Let then $\mathcal{U}$ be an ultrafilter with $\mathcal{U} \supset \mathcal{F}$. By (iii) there exists some $x \in X$ such that $\mathcal{U} \rightarrow x$, which means that $\mathcal{U}$ contains all neighborhoods of x . Start now with some arbitrary index $i \in I$. Since we clearly have $F_i \in \mathcal{F} \subset \mathcal{U}$, it follows that $X \setminus F_i$ cannot belong to $\mathcal{U}$, which means that $X \setminus F_i$ is *not* a neighborhood of x . Since $X \setminus F_i$ is already open, this forces $x \notin X \setminus F_i$, which means that $x \in F_i$. Since this is true for all $i \in I$, it proves that the intersection $\bigcap_{i \in I} F_i$ contains x , so it is non-empty. $\square$

An interesting application of the above result is the following:

THEOREM 4.2 (Tihonov). *Suppose one has a family $(X_i, \mathcal{T}_i)_{i \in I}$ of compact topological spaces. Then the product space $\prod_{i \in I} X_i$ is compact in the product topology.*

PROOF. We are going to use the ultrafilter characterization (iii) from the preceding Theorem. Let $\mathcal{U}$ be an ultrafilter on $X = \prod_{i \in I} X_i$. Denote by $\pi_i : X \rightarrow X_i$, $i \in I$ the coordinate maps. Since each X_i is compact, it follows that, for every $i \in I$, the ultrafilter $\pi_{i*}(\mathcal{U})$ (in X_i) is convergent to some point $x_i \in X_i$. If we form the element $x = (x_i)_{i \in I} \in X$, this means that $\pi_{i*}(\mathcal{U})$ is convergent to $\pi_i(x)$, for every $i \in I$. Then, by the ultrafilter characterization of the product topology (see section 3) it follows that $\mathcal{U}$ is convergent to x . $\square$

COMMENT. Another interesting application of Theorem 4.1 is the following construction. Suppose $(X, \mathcal{T})$ is a compact Hausdorff space, and $(x_i)_{i \in I} \subset X$ is an arbitrary family of elements. (Here I is an arbitrary set.) Suppose $\mathcal{U}$ is an ultrafilter on I . If we regard the family $(x_i)_{i \in I}$ simply as a function $f : I \rightarrow X$, then we can construct the ultrafilter $f_*(\mathcal{U})$ on X . More explicitly

$$f_*(\mathcal{U}) = \{U \subset X : \text{the set } \{i \in I : x_i \in U\} \text{ belongs to } \mathcal{U}\}.$$

Since X is compact Hausdorff, the ultrafilter $f_*(\mathcal{U})$ is convergent to some unique point $x \in X$. This point is denoted by $\lim_{\mathcal{U}} x_i$.

We conclude this section with some results on compactness in Hausdorff spaces.

PROPOSITION 4.4. *Suppose $(X, \mathcal{T})$ is topological Hausdorff space.*

- (i) *Any compact set $K \subset X$ is closed.*
- (ii) *If K is a compact set, then a subset $F \subset K$ is compact, if and only if F is closed (in X).*

PROOF. (i) The key step is contained in the following

Claim: For every $x \in X \setminus K$, there exists some open set D_x with $x \in D_x \subset X \setminus K$.

Fix $x \in X \setminus K$. For every $y \in K$, using the Hausdorff property, we can find two open sets U_y and V_y with $U_y \ni x$, $V_y \ni y$, and $U_y \cap V_y = \emptyset$. Since we obviously have $K \subset \bigcup_{y \in K} V_y$, by compactness, there exist points $y_1, \dots, y_n \in K$, such that $K \subset V_{y_1} \cup \dots \cup V_{y_n}$. The claim immediately follows if we then define $D_x = U_{y_1} \cap \dots \cap U_{y_n}$.

Using the Claim we now see that we can write the complement of K as a union of open sets:

$$X \setminus K = \bigcup_{x \in X \setminus K} D_x,$$

so $X \setminus K$ is open, which means that K is indeed closed. (ii). If F is closed, then F is compact by Proposition 4.2. Conversely, if F is compact, then by (i) F is closed. $\square$

PROPOSITION 4.5. *Every compact Hausdorff space is normal.*

PROOF. Let X be a compact Hausdorff space. Let $A, B \subset X$ be two closed sets with $A \cap B = \emptyset$. We need to find two open sets $U, V \subset X$, with $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$. We start with the following

Particular case: Assume B is a singleton, $B = \{b\}$.

The proof follows line by line the first part of the proof of part (i) from Proposition 4.4. For every $a \in A$ we find open sets U_a and V_a , such that $U_a \ni a$, $V_a \ni b$, and $U_a \cap V_a = \emptyset$. Using Proposition 4.4 we know that A is compact, and since we clearly have $A \subset \bigcup_{a \in A} U_a$, there exist $a_1, \dots, a_n \in A$, such that $U_{a_1} \cup \dots \cup U_{a_n} \supset A$. Then we are done by taking $U = U_{a_1} \cup \dots \cup U_{a_n}$ and $V = V_{a_1} \cap \dots \cap V_{a_n}$.

Having proven the above particular case, we proceed now with the general case. For every $b \in B$, we use the particular case to find two open sets U_b and V_b , with $U_b \supset A$, $V_b \ni b$, and $U_b \cap V_b = \emptyset$. Arguing as above, the set B is compact, and we have $B \subset \bigcup_{b \in B} V_b$, so there exist $b_1, \dots, b_n \in B$, such that $V_{b_1} \cup \dots \cup V_{b_n} \supset B$. Then we are done by taking $U = U_{b_1} \cap \dots \cap U_{b_n}$ and $V = V_{b_1} \cup \dots \cup V_{b_n}$. $\square$

LECTURE 5

5. Topology preliminaries V: Locally compact spaces

DEFINITION. A *locally compact space* is a Hausdorff topological space with the property

(LC) *Every point has a compact neighborhood.*

One key feature of locally compact spaces is contained in the following:

LEMMA 5.1. *Let X be a locally compact space, let K be a compact set in X , and let D be an open subset, with $K \subset D$. Then there exists an open set E with:*

- (i) $\overline{E}$ compact;
- (ii) $K \subset E \subset \overline{E} \subset D$.

PROOF. Let us start with the following

Particular case: Assume K is a singleton $K = \{x\}$.

Start off by choosing a compact neighborhood N of x . Using the results from section 4, when equipped with the induced topology, the set N is normal. In particular, if we consider the closed sets $A = \{x\}$ and $B = N \setminus D$ (which are also closed in the induced topology), it follows that there exist sets $U, V \subset N$, such that

- $U \supset \{x\}$, $V \supset B$, $U \cap V = \emptyset$;
- U and V are open in the induced topology on N .

The second property means that there exist open sets $U_0, V_0 \subset X$, such that $U = N \cap U_0$ and $V = N \cap V_0$. Let $E = \text{Int}(U)$. By construction E is open, and $E \ni x$. Also, since $E \subset U \subset N$, it follows that

$$(1) \quad \overline{E} \subset \overline{N} = N.$$

In particular this gives the compactness of E . Finally, since we obviously have

$$E \cap V_0 \subset U \cap V_0 = N \cap U_0 \cap V_0 = U \cap V = \emptyset,$$

we get $E \subset X \setminus V_0$, so using the fact that $X \setminus V_0$ is closed, we also get the inclusion $\overline{E} \subset X \setminus V_0$. Finally, combining this with (1) and with the inclusion $N \setminus D \subset V \subset V_0$, we will get

$$\overline{E} \subset N \cap (X \setminus V_0) \subset N \cap (N \setminus D) \subset D,$$

and we are done.

Having proven the particular case, we proceed now with the general case. For every $x \in K$ we use the particular case to find an open set $E(x)$, with $\overline{E(x)}$ compact, and such that $x \in E(x) \subset \overline{E(x)} \subset D$. Since we clearly have $K \subset \bigcup_{x \in K} E(x)$, by compactness, there exist $x_1, \dots, x_n \in K$, such that $K \subset E(x_1) \cup \dots \cup E(x_n)$. Notice that if we take $E = E(x_1) \cup \dots \cup E(x_n)$, then we clearly have

$$K \subset E \subset \overline{E} \subset \overline{E(x_1)} \cup \dots \cup \overline{E(x_n)} \subset D,$$

and we are done. $\square$

One of the most useful result in the analysis on locally compact spaces is the following.

THEOREM 5.1 (Urysohn's Lemma for locally compact spaces). *Let X be a locally compact space, and let $K, F \subset X$ be two disjoint sets, with K compact, F closed, and $K \cap F = \emptyset$. Then there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f|_K = 1$ and $f|_F = 0$.*

PROOF. Apply Lemma 5.1 for the pair $K \subset X \setminus F$ and find an open set E , with $\overline{E}$ compact, such that $K \subset E \subset \overline{E} \subset X \setminus F$. Apply again Lemma 5.1 for the pair $K \subset E$ and find another open set G with $\overline{G}$ compact, such that $K \subset G \subset \overline{G} \subset E$.

Let us work for the moment in the space $\overline{E}$ (equipped with the induced topology). This is a compact Hausdorff space, hence it is normal. In particular, using Urysohn Lemma (see section 1) there exists a continuous function $g : \overline{E} \rightarrow [0, 1]$ such that $g|_K = 1$ and $g|_{\overline{E} \setminus G} = 0$. Let us now define the function $f : X \rightarrow [0, 1]$ by

$$f(x) = \begin{cases} g(x) & \text{if } x \in \overline{E} \\ 0 & \text{if } x \in X \setminus \overline{E} \end{cases}$$

Notice that $f|_E = g|_E$, so $f|_E$ is continuous. If we take the open set $A = X \setminus \overline{G}$, then it is also clear that $f|_A = 0$. So now we have two open sets E and A , with $A \cup E = X$, and $f|_A$ and $f|_E$ both continuous. Then it is clear that f is continuous. The other two properties $f|_K = 1$ and $f|_F = 0$ are obvious. $\square$

We now discuss an important notion which makes the linkage between locally compact spaces and compact spaces

DEFINITION. Let X be a locally compact space. By a *compactification of X* one means a pair (θ, T) consisting of a compact Hausdorff space T , and of a continuous map $\theta : X \rightarrow T$, with the following properties

- (i) $\theta(X)$ is a dense open subset of T ;
- (ii) when we equip $\theta(X)$ with the induced topology, the map $\theta : X \rightarrow \theta(X)$ is a homeomorphism.

Notice that, when X is already compact, any compactification (θ, T) of X is necessarily made up of a compact space T , and a homeomorphism $\theta : X \rightarrow T$.

EXAMPLE 5.1 (Alexandrov compactification). Suppose X is a locally compact space, which is not compact. We form a disjoint union with a singleton $X^\alpha = X \sqcup \{\infty\}$, and we equip the space X^α with the topology in which a subset $D \subset X^\alpha$ is declared to be open, if either D is an open subset of X , or there exists some compact subset $K \subset X$, such that $D = (X \setminus K) \sqcup \{\infty\}$. Define the inclusion map $\iota : X \hookrightarrow X^\alpha$. Then (ι, X^α) is a compactification of X , which is called the Alexandrov compactification. The fact that $\iota(X)$ is open in X^α , and $\iota : X \rightarrow \iota(X)$ is a homeomorphism, is clear. The density of $\iota(X)$ in X^α is also clear, since every open set $D \subset X^\alpha$, with $D \ni \infty$, is of the form $(X \setminus K) \sqcup \{\infty\}$, for some compact set $K \subset X$, and then we have $D \cap \iota(X) = \iota(X \setminus K)$, which is non-empty, because X is not compact.

Remark that, if X is already compact, we can still define the topological space $X^\alpha = X \sqcup \{\infty\}$, but this time the singleton set $\{\infty\}$ will be also be open. Although $\iota(X)$ will still be open in X^α , it will not be dense in X^α .

One should regard the Alexandrov compactification as a minimal one. It turns out that there exists another compactification which is described below, which can be regarded as the largest.

THEOREM 5.2 (Stone-Čech). *Let X be a locally compact space. Consider the set*

$$F = \{f : X \rightarrow [0, 1] : f \text{ continuous}\},$$

and consider the product space

$$T = \prod_{f \in F} [0, 1],$$

equipped with the product topology, and define the map $\theta : X \rightarrow T$ by

$$\theta(x) = (f(x))_{f \in F}, \quad \forall x \in X.$$

Equip the closure $\overline{\theta(X)}$ with the topology induced from T . Then the pair $(\theta, \overline{\theta(X)})$ is a compactification of X .

PROOF. For every $f \in F$, let us denote by $\pi_f : T \rightarrow [0, 1]$ the coordinate map. Remark that $\theta : X \rightarrow T$ is continuous. This is immediate from the definition of the product topology, since the continuity of θ is equivalent to the continuity of all compositions $\pi_f \circ \theta$, $f \in F$. The fact that these compositions are continuous is however trivial, since we have $\pi_f \circ \theta = f$, $\forall f \in F$.

Denote for simplicity $\overline{\theta(X)}$ by B . By Tihonov's Theorem, the space T is compact (and obviously Hausdorff), so the set B is compact as well, being a closed subset of T . By construction, $\theta(X)$ is dense in B , and θ is continuous.

At this point, it is interesting to point out the following property

Claim 1: For every $f \in F$, there exists a unique continuous map $\tilde{f} : B \rightarrow [0, 1]$, such that $\tilde{f} \circ \theta = f$.

The uniqueness is trivial, since $\theta(X)$ is dense in B . The existence is also trivial, because we can take $\tilde{f} = \pi_f|_B$.

We can show now that θ is injective. If $x, y \in X$ are such that $x \neq y$, then using Urysohn Lemma we can find $f \in F$, such that $f(x) \neq f(y)$. The function $\tilde{f}$ given by Claim 1, clearly satisfies

$$\tilde{f}(\theta(x)) = f(x) \neq f(y) = \tilde{f}(\theta(y)),$$

which forces $\theta(x) \neq \theta(y)$.

In order to show that $\theta(X)$ is open in B , we need some preparations. For every compact subset $K \subset X$, we define

$$F_K = \{f : X \rightarrow [0, 1] : f \text{ continuous}, f|_{X \setminus K} = 0\}.$$

On key observation is the following.

Claim 2: If $K \subset X$ is compact, and if $f \in F_K$, then the continuous function $\tilde{f} : B \rightarrow [0, 1]$, given by Claim 1, has the property $\tilde{f}|_{B \setminus \theta(K)} = 0$.

We start with some $\alpha \in B \setminus \theta(K)$, and we use Urysohn Lemma to find some continuous function $\phi : B \rightarrow [0, 1]$ such that $\phi(\alpha) = 1$ and $\phi|_{\theta(K)} = 0$. Consider the function $\psi = \phi \cdot \tilde{f}$. Notice that $(\phi \circ \theta)|_K = 0$, which combined with the fact that $f|_{X \setminus K} = 0$, gives

$$\psi \circ \theta = (\phi \circ \theta) \cdot (\tilde{f} \circ \theta) = (\phi \circ \theta) \cdot f = 0,$$

so using Claim 1 (the uniqueness part), we have $\psi = 0$. In particular, since $\phi(\alpha) = 1$, this forces $\tilde{f}(\alpha) = 0$, thus proving the Claim.

We define now the collection

$$F_c = \bigcup_{\substack{K \subset X \\ K \text{ compact}}} F_K.$$

Define the set

$$S = \bigcap_{f \in F_c} \pi_f^{-1}(\{0\}).$$

By the definition of the product topology, it follows that S is closed in T . The fact that $\theta(X)$ is open in B , is then a consequence of the following fact.

Claim 3: One has the equality $\theta(X) = B \setminus S$.

Start first with some point $x \in X$, and let us show that $\theta(x) \notin S$. Choose some open set $D \subset X$, with $\bar{D}$ compact, such that $D \ni x$, and apply Urysohn Lemma to find some continuous map $f : X \rightarrow [0, 1]$ such that $f(x) = 1$ and $f|_{X \setminus D} = 0$. It is clear that $f \in F_{\bar{D}} \subset F_c$, but $\pi_f(\theta(x)) = f(x) = 1 \neq 0$, which means that $\theta(x) \notin \pi_f^{-1}(\{0\})$, hence $\theta(x) \notin S$. Conversely, let us start with some point $\alpha = (\alpha_f)_{f \in F} \in B \setminus S$, and let us prove that $\alpha \in \theta(X)$. Since $\alpha \notin S$, there exists some $f \in F_c$, such that $\pi_f(\alpha) > 0$. Since $f \in F_c$, there exists some compact subset $K \subset X$, such that $f|_{X \setminus K} = 0$. Using Claim 2, we know that $\tilde{f}|_{B \setminus \theta(K)} = 0$. Since $\tilde{f}(\alpha) = \pi_f(\alpha) \neq 0$, this forces $\alpha \in \theta(K) \subset \theta(X)$.

To finish the proof of the Theorem, all we need to prove now is the fact that $\theta : X \rightarrow \theta(X)$ is a homeomorphism, which amounts to proving that, whenever $D \subset X$ is open, it follows that $\theta(D)$ is open in B . Fix an open subset $D \subset X$. In order to show that $\theta(D)$ is open in B , we need to show that $\theta(D)$ is a neighborhood for each of its points. Fix some point $\alpha \in \theta(D)$, i.e. $\alpha = \theta(x)$, for some $x \in D$. Choose some compact subset $K \subset D$, such that $x \in \text{Int}(K)$, and apply Urysohn Lemma to find a function $f \in F_K$, with $f(x) = 1$. Consider the continuous function $\tilde{f} : B \rightarrow [0, 1]$ given by Claim 1, and apply Claim 2 to conclude that $\tilde{f}|_{B \setminus \theta(K)} = 0$. In particular the open set

$$N = \tilde{f}^{-1}((1/2, \infty)) \subset B$$

is contained in $\theta(K) \subset \theta(D)$. Since $\tilde{f}(\alpha) = f(x) = 1$, we clearly have $\alpha \in N$. $\square$

DEFINITION. The compactification $(\theta, \overline{\theta(X)})$, constructed in the above Theorem, is called the *Stone-Cech compactification of X* . The space $\overline{\theta(X)}$ will be denoted by X^β . Using the map θ , we shall identify from now on X with a dense open subset of X^β . Remark that if X is compact, then $X^\beta = X$.

COMMENT. The Stone-Cech compactification is an inherent “Zorn Lemma type” construction. For example, if X is a non-compact locally compact space, and if $\mathcal{U}$ is an ultrafilter on X , then neither $\mathcal{U}$ is convergent to a point in X (this happens when $\mathcal{U}$ contains at least one compact subset of X), or $\mathcal{U}$ produces a point in $X^\beta \setminus X$. If $\theta : X \rightarrow X^\beta$ denotes the inclusion map, then one considers the ultrafilter $\theta_*\mathcal{U}$ on X^β , and by compactness this ultrafilter converges to some (unique) point in X^β . This way one gets a correspondence

$$\mathbf{lim}_X : \{\mathcal{U} \subset \mathcal{P}(X) : \mathcal{U} \text{ ultrafilter on } X\} \rightarrow X^\beta.$$

This correspondence is surjective. The injectivity obstruction is characterized as follows. For two ultrafilters $\mathcal{U}_1, \mathcal{U}_2$, the condition $\mathbf{lim}_X(\mathcal{U}_1) \neq \mathbf{lim}_X(\mathcal{U}_2)$ is equivalent to the existence of two disjoint open sets $D_1 \in \mathcal{U}_1$ and $D_2 \in \mathcal{U}_2$.

Exercise 1. Suppose a set X is equipped with the discrete topology. Prove that the correspondence $\mathbf{lim}_X$ is bijective.

The Stone-Cech compactification is functorial, in the following sense.

PROPOSITION 5.1. *If X and Y are locally compact spaces, and if $\Phi : X \rightarrow Y$ is a continuous map, then there exists a unique continuous map $\Phi^\beta : X^\beta \rightarrow Y^\beta$, such that $\Phi^\beta|_X = \Phi$.*

PROOF. We use the notations from Theorem 5.2. Define

$$F = \{f : X \rightarrow [0, 1] : f \text{ continuous}\} \text{ and } G = \{g : Y \rightarrow [0, 1] : g \text{ continuous}\},$$

the product spaces

$$T_X = \prod_{f \in F} [0, 1] \text{ and } T_Y = \prod_{g \in G} [0, 1],$$

as well as the maps $\theta_X : X \rightarrow T_X$ and $\theta_Y : Y \rightarrow T_Y$, defined by

$$\begin{aligned} \theta_X(x) &= (f(x))_{f \in F}, \quad \forall x \in X; \\ \theta_Y(y) &= (g(y))_{g \in G}, \quad \forall y \in Y. \end{aligned}$$

With these notations, we have $X^\beta = \overline{\theta_X(X)} \subset T_X$ and $Y^\beta = \overline{\theta_Y(Y)} \subset T_Y$. Using the fact that we have a correspondence $G \ni g \mapsto g \circ \Phi \in F$, we define the map

$$\Psi : T_X \ni (\alpha_f)_{f \in F} \mapsto (\alpha_{g \circ \Phi})_{g \in G} \in T_Y.$$

Remark that Ψ is continuous. This fact is pretty obvious, because when we compose with coordinate projections $\pi_g : T_Y \rightarrow [0, 1]$, $g \in G$, we have $\pi_g \circ \Psi = \pi_{g \circ \Phi}$ where $\pi_{g \circ \Phi} : T_X \rightarrow [0, 1]$ is the coordinate projection, which is automatically continuous. Remark that if we start with some point $x \in X$, then

$$(2) \quad \Psi(\theta_X(x)) = ((g \circ \Phi)(x))_{g \in G} = \theta_Y(\Phi(x)),$$

which means that we have the equality $\Psi \circ \theta_X = \theta_Y \circ \Phi$. Remark first that, since Y^β is closed, it follows that $\Psi^{-1}(Y^\beta)$ is closed in T_X . Second, using (2), we clearly have the inclusion $\theta_X(X) \subset \Psi^{-1}(\theta_Y(Y)) \subset \Psi^{-1}(Y^\beta)$, so using the fact that $\Psi^{-1}(Y^\beta)$ is closed, we get the inclusion

$$X^\beta = \overline{\theta_X(X)} \subset \Psi^{-1}(Y^\beta).$$

In other words, we get now a continuous map $\Phi^\beta = \Psi|_{X^\beta} : X^\beta \rightarrow Y^\beta$, which clearly satisfies $\Phi^\beta \circ \theta_X = \theta_Y \circ \Phi$, which using our conventions means that $\Phi^\beta|_X = \Phi$. The uniqueness is obvious, by the density of X in X^β . $\square$

Exercise 2. The Alexandrov compactification is not functorial. In other words, given locally compact spaces X and Y , and a continuous map $f : X \rightarrow Y$, in general there does not exist a continuous map $f^\alpha : X^\alpha \rightarrow Y^\alpha$, with $f^\alpha|_X = f$. Give an example of such a situation.

HINT: Consider $X = Y = \mathbb{N}$, equipped with the discrete topology, and define $f : \mathbb{N} \rightarrow \mathbb{N}$ by

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$$

It turns out that one can define a certain type of continuous maps, with respect to which the Alexandrov compactification is functorial.

DEFINITION. Let X, Y be locally compact spaces, and let $\Phi : X \rightarrow Y$ be a continuous map. We say that Φ is *proper*, if it satisfies the condition

$$K \subset Y, \text{ compact} \Rightarrow \Phi^{-1}(K) \text{ compact in } X.$$

The following is an interesting property of proper maps, which will be exploited later, is the following.

PROPOSITION 5.2. *Let X, Y be locally compact spaces, let $\Phi : X \rightarrow Y$ be a proper continuous map, and let $T \subset X$ be a closed subset. Then the set $\Phi(T)$ is closed in Y .*

PROOF. Start with some point $y \in \overline{\Phi(T)}$. This means that

$$(3) \quad D \cap \Phi(T) \neq \emptyset, \text{ for every open set } D \subset Y, \text{ with } D \ni y.$$

Denote by $\mathcal{V}$ the collection of all compact neighborhoods of y . In other words, $V \in \mathcal{V}$, if and only if $V \subset Y$ is compact, and $y \in \text{Int}(V)$. For each $V \in \mathcal{V}$ we define the set $\tilde{V} = \Phi^{-1}(V) \cap T$. Since Φ is proper, all sets $\tilde{V}$, $V \in \mathcal{V}$, are compact. Notice also that, for every finite number of sets $V_1, \dots, V_n \in \mathcal{V}$, if we form the intersection $V = V_1 \cap \dots \cap V_n$, then $V \in \mathcal{V}$, and $\tilde{V} \subset \tilde{V}_j$, $\forall j = 1, \dots, n$. Remark now that, by (3), we have $\tilde{V} \neq \emptyset$, $\forall V \in \mathcal{V}$. Indeed, if we start with some $V \in \mathcal{V}$ and we choose some point $x \in T$, such that $\Phi(x) \in V$, then $x \in \tilde{V}$. Use now the finite intersection property, to get the fact that $\bigcap_{V \in \mathcal{V}} \tilde{V} \neq \emptyset$. Pick now a point $x \in \bigcap_{V \in \mathcal{V}} \tilde{V}$. This means that $x \in T$, and

$$(4) \quad \Phi(x) \in V, \quad \forall V \in \mathcal{V}.$$

But now we are done, because this forces $\Phi(x) = y$. Indeed, if $\Phi(x) \neq y$, using the Hausdorff property, one could find some $V \in \mathcal{V}$ with $\Phi(x) \notin V$, thus contradicting (4). $\square$

Exercise 3. Let X be a locally compact space, which is non-compact, let Y be another a locally compact space, and $\Phi : X \rightarrow Y$ is a proper continuous map.

- (i) If Y is non-compact, prove that there exists a unique continuous map $\Phi^\alpha : X^\alpha \rightarrow Y^\alpha$, with $\Phi^\alpha|_X = \Phi$.
- (ii) If Y is compact, prove that there exists a unique continuous map $\Psi : X^\alpha \rightarrow Y$, with $\Psi|_X = \Phi$.

HINT: In case (i) define $\Phi^\alpha(\infty) = \infty$. In case (ii) consider the collection

$$\mathcal{W} = \{ \Phi(T) : T \subset X \text{ closed, with } \overline{X \setminus T} \text{ compact} \}.$$

Use the above result, combined with the finite intersection property, to pick a point $y \in \bigcap_{W \in \mathcal{W}} W$. Define $\Psi(\infty) = y$.

LECTURE 6

6. Metric spaces

In this section we review the basic facts about metric spaces.

DEFINITIONS. A *metric* on a non-empty set X is a map

$$d : X \times X \rightarrow [0, \infty)$$

with the following properties:

- (i) If $x, y \in X$ are points with $d(x, y) = 0$, then $x = y$;
- (ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;
- (iii) $d(x, y) \leq d(x, z) + d(y, z)$, for all $x, y, z \in X$.

A *metric space* is a pair (X, d) , where X is a set, and d is a metric on X .

NOTATIONS. If (X, d) is a metric space, then for any point $x \in X$ and any $r > 0$, we define the open and closed balls:

$$\begin{aligned}\mathcal{B}_r(x) &= \{y \in X : d(x, y) < r\}, \\ \overline{\mathcal{B}}_r(x) &= \{y \in X : d(x, y) \leq r\}.\end{aligned}$$

DEFINITION. Suppose (X, d) is a metric space. Then X carries a natural topology constructed as follows. We say that a set $D \subset X$ is *open*, if it has the property:

- for every $x \in D$, there exists some $r_x > 0$, such that $\mathcal{B}_{r_x}(x) \subset D$.

One can prove that the collection

$$\mathcal{T}_d = \{D \subset X : D \text{ open}\}$$

is indeed a *topology*, i.e. we have

- $\emptyset$ and X are open;
- if $(D_i)_{i \in I}$ is a family of open sets, then $\bigcup_{i \in I} D_i$ is again open;
- if D_1 and D_2 are open, then $D_1 \cap D_2$ is again open.

The topology thus constructed is called the *metric topology*.

REMARK 6.1. Let (X, d) be a metric space. Then for every $p \in X$, and for every $r > 0$, the set $\mathcal{B}_r(p)$ is open, and the set $\overline{\mathcal{B}}_r(p)$ is closed.

If we start with some $x \in \mathcal{B}_r(p)$, and if we define $r_x = r - d(x, p)$, then for every $y \in \mathcal{B}_{r_x}(x)$ we will have

$$d(y, p) \leq d(y, x) + d(x, p) < r_x + d(x, p) = r,$$

so y belongs to $\mathcal{B}_r(p)$. This means that $\mathcal{B}_{r_x}(x) \subset \mathcal{B}_r(p)$. Since this is true for all $x \in \mathcal{B}_r(p)$, it follows that $\mathcal{B}_r(p)$ is indeed open.

To prove that $\overline{\mathcal{B}}_r(p)$ is closed, we need to show that its complement

$$X \setminus \overline{\mathcal{B}}_r(p) = \{x \in X : d(x, p) > r\}$$

is open. If we start with some $x \in X \setminus \overline{\mathcal{B}_r(p)}$, and if we define $\rho_x = d(p, x) - r$, then for every $y \in \mathcal{B}_{\rho_x}(x)$ we will have

$$d(y, p) \geq d(p, x) - d(y, x) > d(p, x) - \rho_x = r,$$

so y belongs to $X \setminus \overline{\mathcal{B}_r(p)}$. This means that $\mathcal{B}_{\rho_x}(x) \subset X \setminus \overline{\mathcal{B}_r(p)}$. Since this is true for all $x \in X \setminus \overline{\mathcal{B}_r(p)}$, it follows that $X \setminus \overline{\mathcal{B}_r(p)}$ is indeed open.

REMARK 6.2. The metric topology on a metric space (X, d) is Hausdorff. Indeed, if we start with two points $x, y \in X$, with $x \neq y$, then if we choose r to be a real number, with

$$0 < r < \frac{d(x, y)}{2},$$

then we have $\mathcal{B}_r(x) \cap \mathcal{B}_r(y) = \emptyset$. (Otherwise, if we have a point $z \in \mathcal{B}_r(x) \cap \mathcal{B}_r(y)$, we would have $2r < d(x, y) \leq d(x, z) + d(y, z) < 2r$, which is impossible.)

REMARK 6.3. Let (X, d) be a metric space, and let M be a subset of X . Then $d|_{M \times M}$ is a metric on M , and the metric topology on M defined by this metric is precisely the *induced topology* from X . This means that a set $A \subset M$ is open in M if and only if there exists some open set $D \subset X$ with $A = M \cap D$.

The metric space framework is particularly convenient because one can use *convergence*.

DEFINITION. Let (X, d) be a metric space. For a point $x \in X$, we say that a sequence $(x_n)_{n \geq 1} \subset X$ is *convergent to x* , if $\lim_{n \rightarrow \infty} d(x_n, x) = 0$.

REMARK 6.4. Let (X, d) is a metric space, and if the sequence $(x_n)_{n \geq 1} \subset X$ is convergent to some point $x \in X$, then

$$(1) \quad \lim_{n \rightarrow \infty} d(x_n, y) = d(x, y), \quad \forall y \in X.$$

This is an immediate consequence of the inequalities

$$d(x, y) - d(x_n, x) \leq d(x_n, y) \leq d(x, y) + d(x_n, x).$$

Among other things, the equality (1) gives the fact that $(x_n)_{n \geq 1}$ cannot be convergent to any other point $y \neq x$. Therefore, if $(x_n)_{n \geq 1}$ is convergent to *some* x , then x is uniquely determined, and will be denoted by $\lim_{n \rightarrow \infty} x_n$.

Convergence is useful for characterizing closure.

PROPOSITION 6.1. *Let (X, d) be a metric space, and let $A \subset X$ be a non-empty subset. For a point $x \in X$, the following are equivalent:*

- (i) x belongs to the closure $\overline{A}$ of A ;
- (ii) there exists some sequence $(x_n)_{n \geq 1} \subset A$, with $\lim_{n \rightarrow \infty} x_n = x$.

PROOF. (i) $\Rightarrow$ (ii). Assume $x \in \overline{A}$. This means that

(*) For every open set $D \subset X$ with $D \ni x$, the intersection $D \cap A$ is non-empty.

We use this property for the open sets $\mathcal{B}_{1/n}(x)$, $n = 1, 2, \dots$. So, for every integer $n \geq 1$, we can find a point $x_n \in \mathcal{B}_{1/n}(x) \cap A$. This way we have built a sequence $(x_n)_{n \geq 1} \subset A$, such that

$$d(x_n, x) < \frac{1}{n}, \quad \forall n \geq 1.$$

It is clear that this gives $x = \lim_{n \rightarrow \infty} x_n$.

(ii) $\Rightarrow$ (i). Assume x satisfies property (ii). Fix $(x_n)_{n \geq 1} \subset A$ to be a sequence with $\lim_{n \rightarrow \infty} x_n = x$. We need to prove property (*). Start with some arbitrary

open set $D \subset X$, with $x \in D$. Let $\varepsilon > 0$ be chosen such that $\mathcal{B}_\varepsilon(x) \subset D$. Since $\lim_{n \rightarrow \infty} d(x_n, x) = 0$, there exists some n_ε such that $d(x_{n_\varepsilon}, x) < \varepsilon$. It is now clear that

$$x_{n_\varepsilon} \in \mathcal{B}_\varepsilon(x) \cap A \subset D \cap A,$$

so the intersection $D \cap A$ is indeed non-empty. $\square$

Continuity can be characterized using convergence, as follows.

PROPOSITION 6.2. *Let X and Y be metric spaces, and let $f : X \rightarrow Y$ be a function. For a point $p \in X$, the following are equivalent:*

- (i) *f is continuous at p ;*
- (ii) *for every $\varepsilon > 0$, there exists some $\delta_\varepsilon > 0$ such that*

$$d(f(x), f(p)) < \varepsilon, \text{ for all } x \in X \text{ with } d(x, p) < \delta_\varepsilon.$$

- (iii) *if $(x_n)_{n \geq 1} \subset X$ is a sequence with $\lim_{n \rightarrow \infty} x_n = p$, then $\lim_{n \rightarrow \infty} f(x_n) = f(p)$.*

PROOF. (i) $\Rightarrow$ (ii). The condition that f is continuous at p means

- (*) *for every open set $D \subset Y$, with $D \ni f(p)$, there exists some open set $E \subset X$, with $p \in E \subset f^{-1}(D)$.*

Assume f is continuous at p . For every $\varepsilon > 0$, we consider the open ball $\mathcal{B}_\varepsilon^Y(f(p))$. Using (*), there exists some open set $E \subset X$, with $E \ni p$, and $f(E) \subset \mathcal{B}_\varepsilon^Y(f(p))$. In particular, there exists $\delta > 0$, such that $\mathcal{B}_\delta^X(p) \subset E$, so now we have

$$f(\mathcal{B}_\delta^X(p)) \subset \mathcal{B}_\varepsilon^Y(f(p)),$$

which clearly gives (ii).

(ii) $\Rightarrow$ (iii). Assume f satisfies (ii), and start with some sequence $(x_n)_{n \geq 1} \subset X$, which converges to p . For every $\varepsilon > 0$, we choose $\delta_\varepsilon > 0$ as in (ii), and using the fact that $\lim_{n \rightarrow \infty} x_n = p$, we can also choose some N_ε such that

$$d(x_n, p) < \delta_\varepsilon, \quad \forall n \geq N_\varepsilon.$$

Using (ii) this will give

$$d(f(x_n), f(p)) < \varepsilon, \quad \forall n \geq N_\varepsilon.$$

In other words, we get the fact that

$$\lim_{n \rightarrow \infty} (f(x_n), f(p)) = 0,$$

which means that we indeed have $\lim_{n \rightarrow \infty} f(x_n) = f(p)$.

(iii) $\Rightarrow$ (i). Assume f satisfies (iii), but f is not continuous at p . By (*) this means that there exists some open set $D_0 \subset Y$ with $D_0 \ni f(p)$, such that

- (*)' *for every open set $E \subset X$ with $E \ni p$, we have $f(E) \not\subset D_0$.*

It is clear that any other open set D , with $f(p) \in D \subset D_0$, will again satisfy property (*'). Fix then some $r > 0$, such that $\mathcal{B}_r^Y(f(p)) \subset D_0$. Using condition (*)' it follows that for every integer $n \geq 1$, we have

$$f(\mathcal{B}_{1/n}^X(p)) \not\subset \mathcal{B}_r^Y(f(p)).$$

This means that, for every integer $n \geq 1$, we can find a point $x_n \in X$ such that

$$d(x_n, p) < \frac{1}{n} \text{ and } d(f(x_n), f(p)) \geq r.$$

It is then clear that the sequence $(x_n)_{n \geq 1} \subset X$ is convergent to p , but the sequence $(f(x_n))_{n \geq 1} \subset Y$ is not convergent to $f(p)$. This will contradict (iii). $\square$

Convergence can also be used for characterizing compactness.

THEOREM 6.1. *Let (X, d) be a metric space. The following are equivalent:*

- (i) *X is compact in the metric topology;*
- (ii) *every sequence has a convergent subsequence.*

PROOF. (i) $\Rightarrow$ (ii). Assume X is compact. Start with an arbitrary sequence $(x_n)_{n \geq 1} \subset X$. For every $n \geq 1$, we define the closed set

$$T_n = \overline{\{x_k : k > n\}}.$$

It is obvious that the family of closed sets $(T_n)_{n \geq 1}$ has the *finite intersection property*, i.e. for every finite set F of indices, we have

$$\bigcap_{n \in F} T_n \neq \emptyset.$$

(This follows from the fact that the T_n 's form a decreasing sequence of sets.) By compactness, it follows that

$$\bigcap_{n \geq 1} T_n \neq \emptyset.$$

Take a point $x \in \bigcap_{n \geq 1} T_n$. The key feature of x is the given by the following:

Claim 1: *For every $\varepsilon > 0$ and every integer $\ell \geq 1$, there exists some integer $N(\varepsilon, \ell) > \ell$ such that $d(x_{N(\varepsilon, \ell)}, x) < \varepsilon$.*

This is a consequence of the fact that, for every $\ell \geq 1$, the point x belongs to the closure $\overline{\{x_N : N > \ell\}}$, so for every $\varepsilon > 0$ we have

$$\mathcal{B}_\varepsilon(x) \cap \{x_N : N > \ell\} \neq \emptyset.$$

Using Claim 1, we define a sequence $(k_n)_{n \geq 0}$ of integers, recursively by

$$k_n = N(\frac{1}{n}, k_{n-1}), \quad \forall n \geq 1.$$

(The initial term k_0 is chosen arbitrarily.) We have, by construction, $k_0 < k_1 < k_2 < \dots$, and

$$d(x_{k_n}, x) < \frac{1}{n}, \quad \forall n \geq 1,$$

so $(x_{k_n})_{n \geq 1}$ is indeed a subsequence of $(x_k)_{k \geq 1}$, which is convergent (to x).

(ii) $\Rightarrow$ (i). Assume (ii). Before we start proving that X is compact, We shall need some preparations.

Claim 2: *For every $r > 0$ there exists a finite set $F \subset X$, such that*

$$X = \bigcup_{x \in F} \mathcal{B}_r(x).$$

We prove this by contradiction. Assume there exists some $r > 0$, such that

$$\bigcup_{x \in F} \mathcal{B}_r(x) \subsetneq X,$$

for every finite set $F \subset X$. In particular, there exists a sequence $(x_n)_{n \geq 1}$ such that

$$x_{n+1} \in X \setminus [\mathcal{B}_r(x_1) \cup \dots \cup \mathcal{B}_r(x_n)], \quad \forall n \geq 1.$$

This will force

$$d(x_m, x_n) \geq r, \quad \forall m > n \geq 1.$$

Notice that every subsequence $(x_{k_n})_{n \geq 1}$ will satisfy the same property

$$d(x_{k_m}, x_{k_n}) \geq r, \quad \forall m > n \geq 1.$$

This proves that *no subsequence of $(x_n)_{n \geq 1}$ is Cauchy*, so *no subsequence of $(x_n)_{n \geq 1}$ can be convergent*, thus contradicting (ii).

Having proven Claim 2, we choose, for every integer $n \geq 1$, finite set F_n such that

$$X = \bigcup_{x \in F_n} \mathcal{B}_{\frac{1}{n}}(x).$$

Claim 3: *The collection $\mathcal{W} = \{\mathcal{B}_{\frac{1}{n}}(x) : n \in \mathbb{N}, x \in F_n\}$ is a base for the metric topology.*

What we need to show is that every open set is a union of sets in $\mathcal{W}$. Fix an open set D and a point $p \in D$. Choose $r > 0$, such that $\mathcal{B}_r(p) \subset D$. Choose then some integer $n \geq 1$, such that $\frac{1}{n} < \frac{r}{2}$, and choose some point $x \in F_n$, such that $p \in \mathcal{B}_{\frac{1}{n}}(x)$. Notice that, for every $y \in \mathcal{B}_{\frac{1}{n}}(x)$, we have

$$d(y, p) \leq d(y, x) + d(x, p) < \frac{1}{n} + \frac{1}{n} \leq r,$$

which proves that $y \in \mathcal{B}_r(p)$. Therefore we have

$$p \in \mathcal{B}_{\frac{1}{n}}(x) \subset \mathcal{B}_r(p) \subset D.$$

Since $p \in D$ is arbitrary, this proves that D is a union of sets in $\mathcal{W}$.

We now begin proving that X is compact. Start with a collection $(D_i)_{i \in I}$ of open sets, with $\bigcup_{i \in I} D_i = X$. We need to find a finite set of indices $I_0 \subset I$, such that $\bigcup_{i \in I_0} D_i = X$. First we show that:

Claim 4: *There exists a countable set of indices $I_1 \subset I$, such that*

$$\bigcup_{i \in I_1} D_i = X.$$

The key fact is that the base $\mathcal{W}$ is *countable*. Let us enumerate the base $\mathcal{W}$ as a sequence

$$\mathcal{W} = \{W_m : m \in \mathbb{N}\}.$$

For each $i \in I$, we define the set

$$M_i = \{m \geq 1 : W_m \subset D_i\}.$$

By Claim 3, we know that for every $x \in D_i$ there exists some $m \in M_i$ such that $x \in W_m \subset D_i$. In particular this proves the equality

$$D_i = \bigcup_{m \in M_i} W_m, \quad \forall i \in I.$$

Consider then the union $M = \bigcup_{i \in I} M_i$, which is countable, being a subset of the integers. We clearly have

$$\bigcup_{m \in M} W_m = \bigcup_{i \in I} \left(\bigcup_{m \in M_i} W_m \right) = \bigcup_{i \in I} D_i = X.$$

For every $m \in M$ we choose an $i_m \in I$, such that $m \in M_{i_m}$. If we take

$$I_1 = \{i_m : m \in M\},$$

then I_1 is obviously countable, and since we clearly have $W_m \subset D_{i_m}$, we get

$$X = \bigcup_{m \in M} W_m \subset \bigcup_{m \in M} D_{i_m} = \bigcup_{i \in I_1} D_i,$$

so the Claim is proven.

Let us list the countable set I_1 as

$$I_1 = \{i_k : k \geq 1\}.$$

(Of course, if I_1 is already finite, there is nothing to prove. So we will assume that I_1 is infinite.) In order to finish the proof, we must find some k , such that $D_{i_1} \cup D_{i_2} \cup \dots \cup D_{i_k} = X$. Assume no such k can be found, which means that

$$D_{i_1} \cup D_{i_2} \cup \dots \cup D_{i_k} \subsetneq X, \quad \forall k \geq 1.$$

In other words, if we define for each $k \geq 1$, the close set

$$A_k = X \setminus (D_{i_1} \cup D_{i_2} \cup \dots \cup D_{i_k}),$$

we have

$$A_k \neq \emptyset, \quad \forall k \geq 1.$$

For each $k \geq 1$ we choose a point $x_k \in A_k$. This way we have constructed a sequence $(x_k)_{k \geq 1} \subset X$, so using property (i) we can find a convergent subsequence. This means that we have a sequence of integers

$$1 \leq k_1 < k_2 < \dots$$

and a point $x \in X$, such that $\lim_{n \rightarrow \infty} x_{k_n} = x$. Notice that, since

$$k_n \geq n, \quad \forall n \geq 1,$$

and since the sequence $(A_k)_{k \geq 1}$ is decreasing, we get the fact that, for each $m \geq 1$, we have

$$x_{k_n} \in A_m, \quad \forall n \geq m.$$

Since A_m is closed, this forces $x \in A_m$, for all $m \geq 1$. But this is clearly impossible, since

$$\bigcap_{m \geq 1} A_m = X \setminus \left(\bigcup_{m \geq 1} (D_{i_1} \cup \dots \cup D_{i_m}) \right) = X \setminus \left(\bigcup_{i \in I_1} D_i \right) = \emptyset.$$

□

COROLLARY 6.1 (of the proof). *Every compact metric space is second countable, which means that there exists a sequence $(W_m)_{m \geq 1}$ of open sets, with the property*

(B) *for every open set D , there exists a subset $M \subset \mathbb{N}$ such that*

$$D = \bigcup_{m \in M} W_m.$$

PROOF. Use (i) and the steps in the proof of $(i) \Rightarrow (ii)$, up to the proof of Claim 3. □

COROLLARY 6.2. *Let (X, d) be a metric space. For a subset $K \subset X$ the following are equivalent:*

- (i) *every sequence in K has a subsequence which is convergent to some point in K ;*
- (ii) *K is compact in X .*

PROOF. (i) $\Rightarrow$ (ii). By the above Theorem, we know that when we equip K with the metric $d|_{K \times K}$, then K is compact. This means that K is compact in the induced topology, which means exactly that K is compact in X .

(ii) $\Rightarrow$ (i). Argue as above. If K is compact in X , then K is compact when equipped with the induced topology, which means that $(K, d|_{K \times K})$ is compact. $\square$

COROLLARY 6.3. *Let X and Y be metric spaces, and let $f : X \rightarrow Y$ be a continuous map. If X is compact, then f is uniformly continuous, that is,*

- *for every $\varepsilon > 0$, there exists some $\delta_\varepsilon > 0$, such that*

$$d(f(x), f(x')) < \varepsilon, \text{ for all } x, x' \in X \text{ with } d(x, x') < \delta_\varepsilon.$$

PROOF. Suppose f is not uniformly continuous, so there exists some $\varepsilon_0 > 0$, with the property that for any $\delta > 0$ there exists $x, x' \in X$, with $d(x, x') < \delta$, but $d(f(x), f(x')) \geq \varepsilon_0$. In particular, one can construct two sequences $(x_n)_{n \geq 1}$ and $(x'_n)_{n \geq 1}$ with

$$(2) \quad d(x_n, x'_n) < \frac{1}{n} \text{ and } d(f(x_n), f(x'_n)) \geq \varepsilon_0, \quad \forall n \geq 1.$$

Using compactness, we can find a subsequence $(x_{n_k})_{k \geq 1}$ of $(x_n)_{n \geq 1}$ which converges to some point p . On the one hand, we have

$$d(p, x'_{n_k}) \leq d(p, x_{n_k}) + d(x_{n_k}, x'_{n_k}) < d(p, x_{n_k}) + \frac{1}{n_k}, \quad \forall k \geq 1,$$

which proves that

$$(3) \quad \lim_{k \rightarrow \infty} x'_{n_k} = p.$$

On the other hand, using (2) we also have

$$\varepsilon_0 \leq d(f(x_{n_k}), f(x'_{n_k})) \leq d(f(p), f(x_{n_k})) + d(f(p), f(x'_{n_k})),$$

which leads to a contradiction, because the equalities

$$\lim_{k \rightarrow \infty} x_{n_k} = \lim_{k \rightarrow \infty} x'_{n_k} = p,$$

together with the continuity of f , will force

$$\lim_{k \rightarrow \infty} d(f(p), f(x_{n_k})) = \lim_{k \rightarrow \infty} d(f(p), f(x'_{n_k})) = 0.$$

$\square$

REMARK 6.5. Let X be a metric space. Then any compact subset $K \subset X$ is *closed* (this is a consequence of the fact that X is Hausdorff) and *bounded*, in the sense that for every $p \in X$ we have

$$\sup_{x \in K} d(x, p) < \infty.$$

This is a consequence of the continuity (see ??) of the map

$$K \ni x \mapsto d(x, p) \in [0, \infty).$$

In general however the converse is not true, i.e. there are metric spaces in which closed bounded sets may fail to be compact.

Exercise 1. Equip $\mathbb{R}$ with the metric

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}, \quad \forall x, y \in \mathbb{R}.$$

Prove that d is indeed a metric on $\mathbb{R}$, and the metric topology on $\mathbb{R}$ defined by d is the usual topology. Prove that $\mathbb{R}$ is bounded with respect to this metric.

Exercise 2. Start with a metric space X , and let $(x_n)_{n \geq 1} \subset X$ be a sequence which is convergent to some point x . Prove that the set

$$K = \{x\} \cup \{x_n : n \geq 1\}$$

is compact in X .

DEFINITION. Let (X, d) be a metric space. For a point $x \in X$ and a non-empty subset $A \subset X$, one defines the *distance from x to A* as the number

$$d(x, A) = \inf \{d(x, a) : a \in A\}.$$

Exercise 3. Let (X, d) be a metric space, and let A be a non-empty subset of X .

- (i) For a point $x \in X$, prove that the equality $d(x, A) = 0$ is equivalent to the fact that $x \in \overline{A}$.
- (ii) Prove the inequality

$$|d(x, A) - d(y, A)| \leq d(x, y), \quad \forall x, y \in X.$$

Using (ii) conclude that the map

$$X \ni x \longmapsto d(x, A) \in [0, \infty)$$

is continuous.

PROPOSITION 6.3. *Let (X, d) be a metric space. When equipped with the metric topology, X is normal.*

PROOF. Let A and B be closed subsets of X with $A \cap B = \emptyset$. We need to find open sets $U, V \subset X$, with $U \supset A$, $V \supset B$, and $U \cap V = \emptyset$. We are going to use a converse of Urysohn Lemma. More explicitly, let us define the function $f : X \rightarrow [0, 1]$ by

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}, \quad x \in X.$$

Notice that by Exercise 3, both the numerator and denominator are continuous, and the denominator never vanishes. So f is indeed continuous. It is obvious that $f|_A = 0$ and $f|_B = 1$, so if we take the open sets $U = f^{-1}((-\infty, \frac{1}{2}))$ and $V = f^{-1}((\frac{1}{2}, \infty))$, we clearly get the desired result. $\square$

We continue now with a discussion on completeness.

DEFINITIONS. Let (X, d) be a metric space. A sequence $(x_n)_{n \geq 1} \subset X$ is said to be a *Cauchy sequence*, if it has the following property.

- (C) *For every $\varepsilon > 0$, there exists some integer $N_\varepsilon \geq 1$ such that*

$$d(x_m, x_n) < \varepsilon, \quad \forall m, n \geq N_\varepsilon.$$

The metric space (X, d) is said to be *complete*, if every Cauchy sequence is convergent.

The following result summarizes some equivalent characterizations of completeness.

PROPOSITION 6.4. *Let (X, d) be a metric space. The following are equivalent.*

- (i) *(X, d) is complete.*
- (ii) *Every sequence $(x_n)_{n \geq 1} \subset X$, with*

$$(4) \quad \sum_{n=1}^{\infty} d(x_{n+1}, x_n) < \infty,$$

is convergent.

- (iii) *Every Cauchy sequence has a convergent subsequence.*

PROOF. (i) $\Rightarrow$ (ii). Assume X is complete. Let $(x_n)_{n \geq 1} \subset X$ be a sequence with property (4). To prove (ii) it suffices to show that $(x_n)_{n \geq 1}$ is Cauchy. For every $N \geq 1$ we define

$$R_N = \sum_{n=N}^{\infty} d(x_{n+1}, x_n).$$

Using (4) we get $\lim_{N \rightarrow \infty} R_N = 0$, so for every $\varepsilon > 0$ there exists some $N(\varepsilon)$ with $R_{N(\varepsilon)} < \varepsilon$. Notice also that the sequence $(R_N)_{N \geq 1}$ is decreasing. If $m > n \geq N(\varepsilon)$, then

$$d(x_m, x_n) \leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \leq \sum_{k=n}^{\infty} d(x_{k+1}, x_k) = R_n \leq R_{N(\varepsilon)} < \varepsilon,$$

so $(x_n)_{n \geq 1}$ is indeed Cauchy.

(ii) $\Rightarrow$ (iii). Start with some Cauchy sequence $(y_k)_{k \geq 1}$. For every $n \geq 1$ choose an integer $N(n) \geq 1$ such that

$$(5) \quad d(x_k, x_\ell) < \frac{1}{2^n}, \quad \forall k, \ell \geq N(n).$$

Start with some arbitrary $k_1 \geq N(1)$ and define recursively an entire sequence $(k_n)_{n \geq 1}$ of integers, by

$$k_{n+1} = \max\{k_n + 1, N(n+1)\}, \quad n \geq 1.$$

Clearly we have $k_1 < k_2 < \dots$, and since we have

$$k_{n+1} > k_n \geq N(n), \quad \forall n \geq 1,$$

using (5), we get

$$d(y_{k_{n+1}}, y_{k_n}) < \frac{1}{2^n}, \quad \forall n \geq 1.$$

So if we define the subsequence $x_n = y_{k_n}$, $n \geq 1$, we will have

$$\sum_{n=1}^{\infty} d(x_{n+1}, x_n) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1,$$

so the subsequence $(x_n)_{n \geq 1}$ satisfies condition (4). By (ii) the subsequence $(x_n)_{n \geq 1}$ is convergent.

(iii) $\Rightarrow$ (i). Assume condition (iii) holds. Start with some Cauchy sequence $(x_n)_{n \geq 1}$. For every integer $n \geq 1$ we put

$$S_n = \sup_{\ell, m \geq n} d(x_\ell, x_m).$$

Since $(x_n)_{n \geq 1}$ is Cauchy, we have

$$(6) \quad \lim_{n \rightarrow \infty} S_n = 0.$$

Using the assumption, we can find a subsequence $(x_{k_n})_{n \geq 1}$ (defined by an increasing sequence of integers $1 \leq k_1 < k_2 < \dots$) which is convergent to some point x . We are going to prove that the entire sequence $(x_n)_{n \geq 1}$ is convergent to x . Fix for the moment $n \geq 1$. For every $m \geq n$, we have $k_m \geq m \geq n$, so we have

$$(7) \quad S_n \geq d(x_n, x_{k_m}), \quad \forall m \geq n.$$

By Remark 3.4, we also know that

$$\lim_{m \rightarrow \infty} d(x_n, x_{k_m}) = d(x_n, x),$$

so if we take $\lim_{m \rightarrow \infty}$ in (7) we will get

$$d(x_n, x) \leq S_n.$$

Since this estimate holds for arbitrary $n \geq 1$, using (6) we immediately get the fact that $(x_n)_{n \geq 1}$ is indeed convergent to x . $\square$

PROPOSITION 6.5. *Suppose (X, d) is a complete metric space, and Y is a subset of X . The following are equivalent:*

- (i) Y is complete, when equipped with the metric from X ;
- (ii) Y is closed in X , in the metric topology.

PROOF. (i) $\Rightarrow$ (ii). Assume Y is complete, and let us prove that Y is closed. Start with a point $x \in \overline{Y}$. Then there exists a sequence $(y_n)_{n \geq 1} \subset Y$ with $\lim_{n \rightarrow \infty} y_n = x$. Notice that $(y_n)_{n \geq 1}$ is Cauchy in Y , so by assumption, $(y_n)_{n \geq 1}$ is convergent to some point in Y . This will then clearly force $x \in Y$.

(ii) $\Rightarrow$ (i). Assume Y is closed, and let us prove that Y is complete. Start with a Cauchy sequence $(y_n)_{n \geq 1} \subset Y$. Since X is complete, the sequence $(y_n)_{n \geq 1}$ is convergent to some point $x \in X$. Since Y is closed, this forces $x \in Y$. $\square$

REMARK 6.6. Using Theorem 6.1, we immediately see that a metric space, which is *compact* in the metric topology, is automatically complete.

The next result identifies those complete metric spaces that are compact. In order to formulate it, we need the following:

DEFINITION. Let (X, d) be a metric space, and let $\varepsilon > 0$. A subset $A \subset X$ is said to be ε -rare, if

$$d(a, b) \geq \varepsilon, \text{ for all } a, b \in A \text{ with } a \neq b.$$

PROPOSITION 6.6. *Let (X, d) be a complete metric space. The following are equivalent:*

- (i) X is compact in the metric topology;
- (ii) for each $\varepsilon > 0$, all ε -rare subsets of X are finite;
- (iii) for any $\varepsilon > 0$, there exist finitely many points $p_1, p_2, \dots, p_n \in X$, such that

$$X = \mathcal{B}_\varepsilon(p_1) \cup \mathcal{B}_\varepsilon(p_2) \cup \dots \cup \mathcal{B}_\varepsilon(p_n).$$

PROOF. (i) $\Rightarrow$ (ii). Assume X is compact. We prove (ii) by contradiction. Assume there exists some $\varepsilon > 0$ and an infinite ε -rare set $A \subset X$. It then follows that there exists a sequence $(a_n)_{n \geq 1} \subset A$, such that

$$d(a_m, a_n) \geq \varepsilon, \quad \forall m > n \geq 1.$$

It is clear that *no subsequence of $(a_n)_{n \geq 1}$ is Cauchy*, which means that $(a_n)_{n \geq 1}$ *does not have any convergent subsequence*, thus contradicting the fact that X is compact.

(ii) $\Rightarrow$ (iii). Assume property (ii) and let us prove (iii) by contradiction. Assume there exists some $\varepsilon > 0$, such that, for every finite set $F \subset X$, one has a strict inclusion

$$\bigcup_{x \in F} \mathcal{B}_\varepsilon(x) \subsetneq X.$$

Start with some arbitrary point $a_1 \in X$, and construct recursively a sequence $(a_n)_{n \geq 1} \subset X$, by choosing

$$a_{n+1} \in X \setminus [\mathcal{B}_\varepsilon(a_1) \cup \dots \cup \mathcal{B}_\varepsilon(a_n)], \quad \forall n \geq 1.$$

This will then force

$$d(a_m, a_n) \geq \varepsilon, \quad \forall m > n \geq 1,$$

so $A = \{a_n : n \in \mathbb{N}\}$ will be an infinite ε -rare set, thus contradicting (ii).

(iii) $\Rightarrow$ (i). Assume property (iii), and let us prove that X is compact. We are going to use Theorem 6.1. Start with an arbitrary sequence $(x_n)_{n \geq 1} \subset X$, and let us construct a convergent subsequence.

Claim: There exists a sequence $(p_n)_{n \geq 1} \subset X$, such that for every integer $k \geq 1$, the set

$$M_k = \left\{ n \in \mathbb{N} : x_n \in \bigcap_{\ell=1}^k \mathcal{B}_{\frac{1}{\ell}}(p_\ell) \right\}$$

is infinite.

The sequence $(p_n)_{n \geq 1}$ is constructed recursively. To start, we use (ii) to find a finite set $F_1 \subset X$, such that

$$X = \bigcup_{p \in F_1} \mathcal{B}_1(p).$$

If we define, for each $p \in F_1$, the set

$$S_1(p) = \{n \in \mathbb{N} : x_n \in \mathcal{B}_1(p)\},$$

then we clearly have

$$\bigcup_{p \in F_1} S_1(p) = \mathbb{N},$$

so in particular one of the sets $S_1(p)$, $p \in F_1$, is infinite.

Suppose now we have constructed points $p_1, p_2, \dots, p_{m-1}$, such that, for every $k \in \{1, \dots, m-1\}$, the set

$$M_k = \left\{ n \in \mathbb{N} : x_n \in \bigcap_{\ell=1}^k \mathcal{B}_{\frac{1}{\ell}}(p_\ell) \right\}$$

is infinite, and let us indicate how the next term p_m is to be constructed. Start with a finite set $F_m \subset X$, such that

$$X = \bigcup_{p \in F_m} \mathcal{B}_{\frac{1}{m}}(p),$$

and define, for each $p \in F_m$, the set

$$S_m(p) = \{n \in M_{m-1} : x_n \in \mathcal{B}_{\frac{1}{m}}(p)\}.$$

It is clear that

$$M_{m-1} = \bigcup_{p \in F_m} S_m(p),$$

and since M_{m-1} is infinite, it follows that one of the sets $S_m(p)$, $p \in F_m$ is infinite. We then choose $p_m \in F_m$ to be one point for which $S_m(p_m)$ is infinite.

Having proven the Claim, let us construct a sequence of integers $1 \leq n_1 < n_2 < \dots$ as follows. Start with some arbitrary $n_1 \in M_1$. Once $n_1 < n_2 < \dots < n_k$ have been constructed, we choose the integer $n_{k+1} \in M_{k+1}$, such that $n_{k+1} > n_k$. (It is here that we use the fact that M_{k+1} is *infinite*.) By construction, we have $n_k \in M_k$, $\forall k \geq 1$.

Suppose $k \geq \ell \geq 1$. Then by construction we have $n_k \in M_k \subset M_\ell$ and $n_\ell \in M_\ell$. In particular we get

$$d(x_{n_k}, x_{n_\ell}) \leq d(x_{n_k}, p_\ell) + d(x_{n_\ell}, p_\ell) < \frac{2}{\ell}.$$

The above estimate clearly proves that the subsequence $(x_{n_k})_{k \geq 1}$ is Cauchy. Since X is complete, it follows that $(x_{n_k})_{k \geq 1}$ is convergent. $\square$

COROLLARY 6.4. *Let (X, d) be a complete metric space, and let A be a subset of X . The following are equivalent:*

- (i) *the closure $\overline{A}$ is compact in X ;*
- (ii) *for each $\varepsilon > 0$, all ε -rare subsets of A are finite.*

PROOF. (i) $\Rightarrow$ (ii). This is trivial from the above result.

(ii) $\Rightarrow$ (i). Assume (ii), and let us prove that $\overline{A}$ is compact. Since $\overline{A}$ is complete, it suffices to prove that, for each $\varepsilon > 0$, all ε -rare subsets of $\overline{A}$ are finite. Fix $\varepsilon > 0$, and let B be an ε -rare subset of $\overline{A}$. For each $x \in B$, let us choose a point $a_x \in A$, such that $x \in \mathcal{B}_{\varepsilon/3}(a_x)$. Suppose $x, y \in B$ are such that $x \neq y$. Then

$$d(a_x, a_y) \geq d(x, y) - d(a_x, x) - d(a_y, y) > \varepsilon - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} = \frac{\varepsilon}{3}.$$

In particular, this shows that the map

$$f : B \ni x \mapsto a_x \in A$$

is injective, and the set $f(B)$ is an $(\varepsilon/3)$ -rare subset of A . By condition (ii) this forces B to be finite. $\square$

We continue with an important construction.

DEFINITIONS. Let (X, d) be a metric space. We define

$$\text{cs}(X, d) = \{\mathbf{x} = (x_n)_{n \geq 1} : \mathbf{x} \text{ Cauchy sequence in } X\}.$$

We say that two Cauchy sequences $\mathbf{x} = (x_n)_{n \geq 1}$ and $\mathbf{y} = (y_n)_{n \geq 1}$ in X are *equivalent*, if

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0.$$

In this case we write $\mathbf{x} \sim \mathbf{y}$. (It is fairly obvious that $\sim$ is indeed an equivalence relation.) We define the quotient space

$$\tilde{X} = \text{cs}(X, d) / \sim.$$

For an element $\mathbf{x} \in \text{cs}(X, d)$, we denote its equivalence class by $\tilde{\mathbf{x}}$.

Finally, for a point $x \in X$, we define $\langle x \rangle \in \tilde{X}$, to be the equivalence class of the constant sequence x (which is obviously Cauchy).

REMARK 6.7. Let (X, d) be a metric space. If $\mathbf{x} = (x_n)_{n \geq 1}$ and $\mathbf{y} = (y_n)_{n \geq 1}$ are Cauchy sequences in X , then the sequence of real numbers $(d(x_n, y_n))_{n \geq 1}$ is convergent. Indeed, for any m, n we have

$$\begin{aligned} |d(x_m, y_m) - d(x_n, y_n)| &\leq |d(x_m, y_m) - d(x_n, y_m)| + |d(x_n, y_m) - d(x_n, y_n)| \leq \\ &\leq d(x_m, x_n) + d(y_m, y_n). \end{aligned}$$

We can then define

$$\delta(\mathbf{x}, \mathbf{y}) = \lim_{n \rightarrow \infty} d(x_n, y_n).$$

PROPOSITION 6.7. *Let (X, d) be a metric space.*

A. *The map $\delta : \text{CS}(X, d) \times \text{CS}(X, d) \rightarrow [0, \infty)$ has the following properties:*

- (i) $\delta(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{y}, \mathbf{x}), \forall \mathbf{x}, \mathbf{y} \in \text{CS}(X, d);$
- (ii) $\delta(\mathbf{x}, \mathbf{y}) \leq \delta(\mathbf{x}, \mathbf{z}) = \delta(\mathbf{z}, \mathbf{y}), \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \text{CS}(X, d);$
- (iii) $\delta(\mathbf{x}, \mathbf{y}) = 0 \Rightarrow \mathbf{x} \sim \mathbf{y};$
- (iv) *If $\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}' \in \text{CS}(X, d)$ are such that $\mathbf{x} \sim \mathbf{x}'$ and $\mathbf{y} \sim \mathbf{y}'$, then $\delta(\mathbf{y}, \mathbf{x}) = \delta(\mathbf{y}', \mathbf{x}')$.*

B. *The map $\tilde{d} : \tilde{X} \times \tilde{X} \rightarrow [0, \infty)$, correctly defined by*

$$\tilde{d}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \delta(\mathbf{x}, \mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in \text{CS}(X, d),$$

is a metric on $\tilde{X}$.

C. *The map $X \ni x \mapsto \langle x \rangle \in \tilde{X}$ is isometric, in the sense that*

$$\tilde{d}(\langle x \rangle, \langle y \rangle) = d(x, y), \quad \forall x, y \in X.$$

PROOF. A. Properties (i), (ii) and (iii) are obvious. To prove property (iv) let $\mathbf{x} = (x_n)_{n \geq 1}$, $\mathbf{x}' = (x'_n)_{n \geq 1}$, $\mathbf{y} = (y_n)_{n \geq 1}$, and $\mathbf{y}' = (y'_n)_{n \geq 1}$. The inequality

$$d(x'_n, y'_n) \leq d(x'_n, x_n) + d(x_n, y_n) + d(y_n, y'_n),$$

combined with $\lim_{n \rightarrow \infty} d(x'_n, x_n) = \lim_{n \rightarrow \infty} d(y_n, y'_n) = 0$ immediately gives

$$\delta(\mathbf{x}', \mathbf{y}') = \lim_{n \rightarrow \infty} d(x'_n, y'_n) \leq \lim_{n \rightarrow \infty} d(x_n, y_n) = \delta(\mathbf{x}, \mathbf{y}).$$

By symmetry we also have $\delta(\mathbf{x}, \mathbf{y}) \leq \delta(\mathbf{x}', \mathbf{y}')$, and we are done.

B. This is immediate from A.

C. Obvious, from the definition. □

PROPOSITION 6.8. *Let (X, d) be a metric space.*

(i) *For any Cauchy sequence $\mathbf{x} = (x_n)_{n \geq 1}$ in X , one has*

$$\lim_{n \rightarrow \infty} \langle x_n \rangle = \tilde{\mathbf{x}}, \text{ in } \tilde{X}.$$

(ii) *The metric space $(\tilde{X}, \tilde{d})$ is complete.*

PROOF. (i). For every $n \geq 1$, we have

$$(8) \quad \tilde{d}(\langle x_n \rangle, \tilde{\mathbf{x}}) = \lim_{m \rightarrow \infty} d(x_n, x_m).$$

Now if we start with some $\varepsilon > 0$, and we choose N_ε such that

$$d(x_n, x_m) < \varepsilon, \quad \forall m, n \geq N_\varepsilon,$$

then (8) shows that

$$\tilde{d}(\langle x_n \rangle, \tilde{\mathbf{x}}) \leq \varepsilon, \quad \forall n \geq N_\varepsilon,$$

so we indeed have

$$\lim_{n \rightarrow \infty} \tilde{d}(\langle x_n \rangle, \tilde{x}) = 0.$$

(ii). Let $(p_k)_{k \geq 1}$ be a Cauchy sequence in $\tilde{X}$. Using (i), we can choose, for each $k \geq 1$, an element $x_k \in X$, such that

$$\tilde{d}(\langle x_k \rangle, p_k) \leq \frac{1}{2^k}.$$

Claim 1: The sequence $x = (x_k)_{k \geq 1}$ is Cauchy in X .

Indeed, for $k \geq \ell \geq 1$ we have

$$d(x_k, x_\ell) = \tilde{d}(\langle x_k \rangle, \langle x_\ell \rangle) \leq \tilde{d}(\langle x_k \rangle, p_k) + \tilde{d}(p_k, p_\ell) + \tilde{d}(p_\ell, \langle x_\ell \rangle) \leq \tilde{d}(p_k, p_\ell) + \frac{1}{2^\ell}.$$

This clearly gives

$$\lim_{n \rightarrow \infty} \left[\sup_{k, \ell \geq N} d(x_k, x_\ell) \right] \leq \lim_{n \rightarrow \infty} \left[\sup_{k, \ell \geq N} \tilde{d}(p_k, p_\ell) \right] = 0,$$

so $x = (x_k)_{k \geq 1}$ is indeed Cauchy.

The proof of (ii) will be finished, once we prove:

Claim 2: We have $\lim_{n \rightarrow \infty} p_k = \tilde{x}$ in $\tilde{X}$.

To see this, we observe that, for $\ell \geq k \geq 1$ we have the inequality

$$(9) \quad \tilde{d}(p_k, \langle x_\ell \rangle) \leq \tilde{d}(p_k, \langle x_k \rangle) + \tilde{d}(\langle x_k \rangle, \langle x_\ell \rangle) \leq \frac{1}{2^k} + d(x_k, x_\ell).$$

If we now start with some $\varepsilon > 0$, and we choose N_ε such that

$$d(x_k, x_\ell) < \varepsilon, \quad \forall k, \ell \geq N_\varepsilon,$$

then (9) gives

$$\tilde{d}(p_k, \langle x_\ell \rangle) \leq \frac{1}{2^k} + \varepsilon, \quad \forall \ell \geq k \geq N_\varepsilon.$$

If we keep $k \geq N_\varepsilon$ fixed and take $\lim_{\ell \rightarrow \infty}$, using (i) we get

$$\tilde{d}(p_k, \tilde{x}) = \lim_{\ell \rightarrow \infty} \tilde{d}(p_k, \langle x_\ell \rangle) \leq \frac{1}{2^k} + \varepsilon, \quad \forall k \geq N_\varepsilon.$$

The above estimate clearly proves that

$$\lim_{k \rightarrow \infty} \tilde{d}(p_k, \tilde{x}) = 0,$$

so the sequence $(p_k)_{k \geq 1}$ is convergent (to $\tilde{x}$). □

DEFINITION. The metric space $(\tilde{X}, \tilde{d})$ is called the *completion* of (X, d) .

The completion has a certain universality property. In order to formulate this property we need the following

DEFINITION. Let (X, d) and (Y, ρ) be metric spaces. A map $f : X \rightarrow Y$ is said to be a *Lipschitz function*, if there exists some constant $C \geq 0$, such that

$$\rho(f(x), f(x')) \leq C \cdot d(x, x'), \quad \forall x, x' \in X.$$

Such a constant C is then called a *Lipschitz constant* for f .

PROPOSITION 6.9. *Let (X, d) be a metric space, and let $(\tilde{X}, \tilde{d})$ be its completion. If (Y, ρ) is a complete metric space, and $f : X \rightarrow Y$ is a Lipschitz function with Lipschitz constant $C \geq 0$, then there exists a unique continuous function $\tilde{f} : \tilde{X} \rightarrow Y$, such that*

$$\tilde{f}(\langle x \rangle) = f(x), \quad \forall x \in X.$$

Moreover, $\tilde{f}$ is Lipschitz, with Lipschitz constant C .

PROOF. Start with some Cauchy sequence $\mathbf{x} = (x_n)_{n \geq 1}$ in X . Using the inequality

$$\rho(f(x_m), f(x_n)) \leq C \cdot d(x_m, x_n), \quad \forall m, n \geq 1,$$

it is obvious that $(f(x_n))_{n \geq 1}$ is a Cauchy sequence in Y . Since Y is complete, this sequence is convergent. Define,

$$\phi(\mathbf{x}) = \lim_{n \rightarrow \infty} f(x_n).$$

This way we have constructed a map $\phi : \text{CS}(X, d) \rightarrow Y$.

Claim: If $\mathbf{x} \sim \mathbf{x}'$, then $\phi(\mathbf{x}) = \phi(\mathbf{x}')$.

Indeed, if $\mathbf{x} = (x_n)_{n \geq 1}$ and $\mathbf{x}' = (x'_n)_{n \geq 1}$, then the Lipschitz property will give

$$\rho(f(x_n), f(x'_n)) \leq C \cdot d(x_n, x'_n), \quad \forall n \geq 1,$$

and using the fact that $\lim_{n \rightarrow \infty} d(x_n, x'_n) = 0$, we get $\lim_{n \rightarrow \infty} \rho(f(x_n), f(x'_n)) = 0$. This clearly forces

$$\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} f(x'_n).$$

Having proven the claim, we now see that we have a correctly define map $\tilde{f} : \tilde{X} \rightarrow Y$, with the property that

$$\tilde{f}(\tilde{\mathbf{x}}) = \phi(\mathbf{x}), \quad \forall \mathbf{x} \in \text{CS}(X, d).$$

The equality

$$\tilde{f}(\langle x \rangle) = f(x), \quad \forall x \in X$$

is trivially satisfied.

Let us check now that $\tilde{f}$ is Lipschitz, with Lipschitz constant C . Start with two points $p, p' \in \tilde{X}$, represented as $p = \tilde{\mathbf{x}}$ and $p' = \tilde{\mathbf{x}'}$, for two Cauchy sequences $\mathbf{x} = (x_n)_{n \geq 1}$ and $\mathbf{x}' = (x'_n)_{n \geq 1}$ in X . Using the definition, we have

$$\tilde{f}(p) = \lim_{n \rightarrow \infty} f(x_n) \text{ and } \tilde{f}(p') = \lim_{n \rightarrow \infty} f(x'_n).$$

This will give

$$\rho(\tilde{f}(p), \tilde{f}(p')) = \lim_{n \rightarrow \infty} \rho(f(x_n), f(x'_n)).$$

Notice however that

$$\rho(f(x_n), f(x'_n)) \leq C \cdot d(x_n, x'_n), \quad \forall n \geq 1,$$

so taking the limit yields

$$\rho(\tilde{f}(p), \tilde{f}(p')) = \lim_{n \rightarrow \infty} \rho(f(x_n), f(x'_n)) \leq C \cdot \lim_{n \rightarrow \infty} d(x_n, x'_n) = C \cdot \tilde{d}(p, p').$$

Finally, let us show that $\tilde{f}$ is unique. Let $F : \tilde{X} \rightarrow Y$ be another continuous function with $F(\langle x \rangle) = f(x)$, for all $x \in X$. Start with an arbitrary point $p \in$

$\tilde{X}$, represented as $p = \mathbf{x}$, for some Cauchy sequence $\mathbf{x} = (x_n)_{n \geq 1}$ in X . Since $\lim_{n \rightarrow \infty} \langle x_n \rangle = p$ in $\tilde{X}$, by continuity we have

$$F(p) = \lim_{n \rightarrow \infty} F(\langle x_n \rangle) = \lim_{n \rightarrow \infty} f(x_n) = \phi(\mathbf{x}) = \tilde{f}(p).$$

□

COROLLARY 6.5. *Let (X, d) be a metric space, let (Y, ρ) be a complete metric space, and let $f : X \rightarrow Y$ be an isometric map, that is*

$$\rho(f(x), f(x')) = d(x, x'), \quad \forall x, x' \in X.$$

Then the map $\tilde{f} : \tilde{X} \rightarrow Y$, given by the above result, is isometric and $\tilde{f}(\tilde{X}) = \overline{f(X)}$ - the closure of $f(X)$ in Y .

PROOF. To show that $\tilde{f}(\tilde{X}) = \overline{f(X)}$, start with some arbitrary point $y \in \overline{f(X)}$. Then there exists a sequence $(x_n)_{n \geq 1} \subset X$, with $\lim_{n \rightarrow \infty} f(x_n) = y$. Since $(f(x_n))_{n \geq 1}$ is Cauchy in Y , and

$$d(x_m, x_n) = \rho(f(x_m), f(x_n)), \quad \forall m, n \geq 1,$$

it follows that the sequence $\mathbf{x} = (x_n)_{n \geq 1}$ is Cauchy in X . We then have

$$y = \lim_{n \rightarrow \infty} f(x_n) = \tilde{f}(\tilde{\mathbf{x}}).$$

Finally, we show that $\tilde{f}$ is isometric. Start with two points $p, q \in \tilde{X}$, represented as $p = \tilde{\mathbf{x}}$ and $q = \tilde{\mathbf{z}}$, for some Cauchy sequences $\mathbf{x} = (x_n)_{n \geq 1}$ and $\mathbf{z} = (z_n)_{n \geq 1}$ in X . Then by construction we have

$$\begin{aligned} \rho(\tilde{f}(p), \tilde{f}(q)) &= \lim_{n \rightarrow \infty} \rho(\tilde{f}(\langle x_n \rangle), \tilde{f}(\langle z_n \rangle)) = \lim_{n \rightarrow \infty} \rho(f(x_n), f(z_n)) = \\ &= \lim_{n \rightarrow \infty} d(x_n, z_n) = \tilde{d}(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) = \tilde{d}(p, q). \end{aligned}$$

□

COROLLARY 6.6. *If (X, d) is a complete metric space, and $\tilde{X}$ is its completion, then the map $\iota : X \ni x \mapsto \langle x \rangle \in \tilde{X}$ is bijective.*

PROOF. Apply the previous result to the map $\text{Id} : X \rightarrow X$, to get a bijective (isometric) map $\tilde{\text{Id}} : \tilde{X} \rightarrow X$. Since the map $\tilde{\text{Id}}$ is obviously a left inverse for ι , it follows that ι itself is bijective. □

In the remainder of this section we will address the following question: *Given a topological Hausdorff space X , when does there exist a metric d on X , such that the given topology coincides with the metric topology defined by d ?* A topological Hausdorff space with the above property is said to be *metrizable*. It is difficult to give non-trivial necessary and sufficient conditions for metrizability. One instance in which this is possible is the compact case (see the *Urysohn Metrization Theorem* later in these notes). Here is a useful result, which is an example of a sufficient condition for metrizability.

PROPOSITION 6.10 (Metrizability of Countable Products). *Let $(X_i, d_i)_{i \in I}$ be a countable family of metric spaces. Then the product space $X = \prod_{i \in I} X_i$, equipped with the product topology, is metrizable.*

PROOF. Denote by $\mathcal{T}$ the product topology on X . What we need is a metric d on X , such that the maps

$$\text{Id} : (X, d) \rightarrow (X, \mathcal{T}) \text{ and } \text{Id} : (X, \mathcal{T}) \rightarrow (X, d)$$

are continuous. (Here the notation (X, d) signifies that X is equipped with the metric topology defined by d .) For each $i \in I$, let $\pi_i : X \rightarrow X_i$ denote the projection onto the i^{th} coordinate.

CASE I: *Assume I is finite.* In this case we define the metric d on X as follows. If $\mathbf{x} = (x_i)_{i \in I}$ and $\mathbf{y} = (y_i)_{i \in I}$ are elements in X , we put

$$d(\mathbf{x}, \mathbf{y}) = \max_{i \in I} d_i(x_i, y_i).$$

The continuity of the map $\text{Id} : (X, d) \rightarrow (X, \mathcal{T})$ is equivalent to the fact that all maps

$$\pi_i : (X, d) \rightarrow (X_i, d_i), \quad i \in I$$

are continuous. This is obvious, because by construction we have

$$d_i(\pi_i(\mathbf{x}), \pi_i(\mathbf{y})) \leq d(\mathbf{x}, \mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in X.$$

Conversely, to prove the continuity of $\text{Id} : (X, \mathcal{T}) \rightarrow (X, d)$, we are going to prove that *every d -open set is open in the product topology*. It suffices to prove this only for open balls. Fix then $\mathbf{x} = (x_i)_{i \in I} \in \prod_{i \in I} X_i$ and $r > 0$, and consider the open ball $\mathcal{B}_r(\mathbf{x})$. If we define, for each $i \in I$, the open ball $\mathcal{B}_r^{X_i}(x_i)$, then it is obvious that

$$\mathcal{B}_r(\mathbf{x}) = \bigcap_{i \in I} \pi_i^{-1}(\mathcal{B}_r^{X_i}(x_i)),$$

and since π_i are all continuous, this proves that $\mathcal{B}_r(\mathbf{x})$ is indeed open in the product topology.

CASE II: *Assume I is infinite.* In this case we identify $I = \mathbb{N}$. For every $n \in \mathbb{N}$ we define a new metric δ_n on X_n , as follows. If

$$\sup_{p, q \in X_n} d_n(p, q) \leq 1,$$

we put $\delta_n = d_n$. Otherwise, we define

$$\delta_n(p, q) = \frac{d_n(p, q)}{1 + d_n(p, q)}, \quad \forall p, q \in X_n.$$

It is not hard to see that the metric topology defined by δ_n coincides with the one defined by d_n . The advantage is that δ_n takes values in $[0, 1]$. We define the metric $d : X \times X \rightarrow [0, \infty)$, as follows. If $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ and $\mathbf{y} = (y_n)_{n \in \mathbb{N}}$ are elements in $\prod_{n \in \mathbb{N}} X_n$, we define

$$d(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} = \sum_{n=1}^{\infty} \frac{\delta_n(x_n, y_n)}{2^n}.$$

Due to the fact that δ_n takes values in $[0, 1]$, the above series is convergent, and it obviously defines a metric on X .

As above, the continuity of the map $\text{Id} : (X, d) \rightarrow (X, \mathcal{T})$ is equivalent to the continuity of all the maps $\pi_n : (X, d) \rightarrow (X_n, d_n)$, or equivalently for $\pi_n : (X, d) \rightarrow (X_n, \delta_n)$, $n \in \mathbb{N}$. But this is an immediate consequence of the (obvious) inequalities

$$\delta_n(\pi_n(\mathbf{x}), \pi_n(\mathbf{y})) \leq 2^n \cdot d(\mathbf{x}, \mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in X.$$

As before, in order to prove the continuity of the other map $\text{Id} : (X, \mathcal{T}) \rightarrow (X, d)$, we start with some d -open set D , and we show that D is open in the product topology. Since D is a union of open balls, we need to prove that for any $\mathbf{x} \in X$ and any $r > 0$, the open ball $\mathcal{B}_r(\mathbf{x})$, in (X, d) , is a *neighborhood of $\mathbf{x}$ in the product topology*. Fix $\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} X_n$, as well as $r > 0$. Choose some integer $N \geq 1$, such that

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{r}{2},$$

and define, for each $k \in \{1, 2, \dots, N\}$ the set

$$D_k = \{\mathbf{y} = (y_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} X_n : \delta_n(x_k, y_k) < \frac{r}{2}\}.$$

It is clear that D_k is *open in the product topology*, for each $k = 1, 2, \dots, N$. (This is a consequence of the fact that $D_k = \pi_k^{-1}(\mathcal{B}_{r/2}(x_k))$, where $\mathcal{B}_{r/2}(x_k)$ is the δ_k -open ball in X_k of radius $r/2$, centered at x_k .) Then the set $D = D_1 \cap D_2 \cap \dots \cap D_N$ is also open in the product topology. Obviously we have $\mathbf{x} \in D$. We now prove that $D \subset \mathcal{B}_r(\mathbf{x})$. Start with some arbitrary $\mathbf{y} \in D$, say $\mathbf{y} = (y_n)_{n \in \mathbb{N}}$. On the one hand, we have

$$\delta_k(x_k, y_k) < \frac{r}{2}, \quad \forall k \in \{1, 2, \dots, N\},$$

so we get

$$\sum_{n=1}^N \frac{1}{2^n} \delta_n(x_n, y_n) < \frac{r}{2} \sum_{n=1}^N \frac{1}{2^n} < \frac{r}{2}.$$

On the other hand, since δ_n takes values in $[0, 1]$, we also have

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} \delta_n(x_n, y_n) < \sum_{n=1}^{\infty} \frac{1}{2^n} < \frac{r}{2},$$

so we get

$$d(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_n(x_n, y_n) < r,$$

thus proving that $\mathbf{y}$ indeed belongs to $\mathcal{B}_r(\mathbf{x})$. □

LECTURE 7

7. Baire theorem(s)

In this section we discuss some topological phenomenon that occurs in certain topological spaces. This deals with interiors of closed sets.

Exercise 1. Let X be a topological space, and let A and B be *closed* sets with the property that $\text{int}(A \cup B) \neq \emptyset$. Prove that either $\text{Int}(A) \neq \emptyset$, or $\text{Int}(B) \neq \emptyset$.

Exercise 2. Give an example of a topological space X and of two (non-closed) sets A and B such that $\text{Int}(A \cup B) \neq \emptyset$, but $\text{Int}(A) = \text{Int}(B) = \emptyset$.

THEOREM 7.1 (Baire's Theorem). *Let (X, T) be a topological Hausdorff space, which satisfies one (or both) of the following properties:*

- (A) *There exists a metric d on X , which makes (X, d) a complete metric space, and T is the metric topology.*
- (B) *X is locally compact.*

Suppose one has a sequence $(F_n)_{n \geq 1}$ of closed subsets of X , such that $X = \bigcup_{n=1}^{\infty} F_n$. Then there exists some integer $n \geq 1$, such that $\text{Int}(F_n) \neq \emptyset$.

PROOF. For every $n \geq 1$ we define the closed set $G_n = \bigcup_{k=1}^n F_k$, so that we still have $X = \bigcup_{n=1}^{\infty} G_n$, but we also have $G_1 \subset G_2 \subset \dots$. According to Exercise 1 (use an inductive argument) it suffices to show that there exists some $n \geq 1$, with $\text{Int}(G_n) \neq \emptyset$. We are going to prove this property by contradiction.

(*) *Assume $\text{Int}(G_n) = \emptyset$, for all $n \geq 1$.*

Claim: Under the assumption () there exists a sequence $(D_n)_{n \geq 1}$ of non-empty open sets, such that for all $n \geq 1$ we have:*

- (i) $D_n \cap G_n = \emptyset$;
- (ii) $\overline{D_{n+1}} \subset D_n$;
- (iii) *In case (A) we have $\text{diam}(D_n) \leq 2^{-n}$; in case (B) $\overline{D_n}$ is compact.*

The sequence is constructed recursively. To construct D_1 we use the fact that $\text{Int}(G_1) = \emptyset$ forces $X \setminus G_1 \neq \emptyset$. We then choose a point $x \in X \setminus G_1$. In case (A) we know that there exists $r > 0$ such that $\mathcal{B}_r(x) \subset X \setminus G_1$. We put $\rho = \min\{r, \frac{1}{4}\}$ and we set $D_1 = \mathcal{B}_\rho(x)$. In the case (B) we apply Lemma 5.1 to find D_1 open with $\overline{D_1}$ compact, such that $x \in D_1 \subset \overline{D_1} \subset X \setminus G_1$.

Let us assume now that we have constructed $D_1, D_2, \dots, D_k$, such that (i) and (iii) hold for all $n \in \{1, \dots, k\}$, and such that (ii) hold for all $n \in \{1, \dots, k-1\}$, and let us indicate how the next set D_{k+1} is constructed. Using the assumption that $\text{Int}(G_{k+1}) = \emptyset$, it follows that the open set $D_k \setminus G_{k+1}$ is non-empty. Choose then a point $x \in D_k \setminus G_{k+1}$. In case (A) there exists some $r > 0$ such that $\mathcal{B}_r(x) \subset D_k \setminus G_{k+1}$. We then put $\rho = \min\{\frac{r}{2}, \frac{1}{2^{k+2}}\}$, and we define $D_{k+1} = \mathcal{B}_\rho(x)$.

In case (B) we apply Lemma 5.1 and find an open set D_{k+1} with $\overline{D_{k+1}}$ compact, and $x \in D_{k+1} \subset \overline{D_{k+1}} \subset D_k \setminus G_{k+1}$. All properties (i)-(iii) are easily verified.

Having proven the Claim, let us see now that the assumption (*) produces a contradiction.

CASE (A): In this case we choose, for each $n \geq 1$ a point $x_n \in D_n$. Notice that, for every $m \geq n \geq 1$ we have

$$x_m, x_n \in D_n \text{ and } d(x_m, x_n) \leq \text{diam}(D_n) \leq \frac{1}{2^n}.$$

In particular, this proves that the sequence $(x_n)_{n \geq 1}$ is *Cauchy*, hence convergent to some point x . Since $x_m \in D_n$, $\forall m \geq n \geq 1$, we see that $x \in \overline{D_n}$, for all $n \geq 1$. In other words we get

$$(1) \quad \bigcap_{n=1}^{\infty} \overline{D_n} \neq \emptyset.$$

CASE (B): In this case we also get (1), this time as a consequence of the compactness of the sets $\overline{D_n}$ (and the finite intersection property).

Let us notice now that (1) combined with (ii) will also give $\bigcap_{n=1}^{\infty} D_n \neq \emptyset$. But this is impossible, since by (i) we have

$$\bigcap_{n=1}^{\infty} D_n \subset \bigcap_{n=1}^{\infty} (X \setminus G_n) = X \setminus \left(\bigcup_{n=1}^{\infty} G_n \right) = \emptyset.$$

□

Chapter II

Elements of Functional Analysis

LECTURE 8

1. Hahn-Banach Theorems

The result we are going to discuss is one of the most fundamental theorems in the whole field of Functional Analysis. Its statement is simple but quite technical.

DEFINITIONS. Let $\mathbb{K}$ be either of the fields $\mathbb{R}$ or $\mathbb{C}$. Suppose $\mathcal{X}$ is a $\mathbb{K}$ -vector space.

- A. A map $q : \mathcal{X} \rightarrow \mathbb{R}$ is said to be a *quasi-seminorm*, if
 - (i) $q(x + y) \leq q(x) + q(y)$, for all $x, y \in \mathcal{X}$;
 - (ii) $q(tx) = tq(x)$, for all $x \in \mathcal{X}$ and all $t \in \mathbb{R}$ with $t \geq 0$.
- B. A map $q : \mathcal{X} \rightarrow \mathbb{R}$ is said to be a *seminorm* if, in addition to the above two properties, it satisfies:
 - (ii') $q(\lambda x) = |\lambda|q(x)$, for all $x \in \mathcal{X}$ and all $\lambda \in \mathbb{K}$.

Remark that if $q : \mathcal{X} \rightarrow \mathbb{R}$ is a seminorm, then $q(x) \geq 0$, for all $x \in \mathcal{X}$. (Use $2q(x) = q(x) + q(-x) \geq q(0) = 0$.)

There are several versions of the Hahn-Banach Theorem.

THEOREM 1.1 (Hahn-Banach, $\mathbb{R}$ -version). *Let $\mathcal{X}$ be an $\mathbb{R}$ -vector space. Suppose $q : \mathcal{X} \rightarrow \mathbb{R}$ is a quasi-seminorm. Suppose also we are given a linear subspace $\mathcal{Y} \subset \mathcal{X}$ and a linear map $\phi : \mathcal{Y} \rightarrow \mathbb{R}$, such that*

$$\phi(y) \leq q(y), \text{ for all } y \in \mathcal{Y}.$$

Then there exists a linear map $\psi : \mathcal{X} \rightarrow \mathbb{R}$ such that

- (i) $\psi|_{\mathcal{Y}} = \phi$;
- (ii) $\psi(x) \leq q(x)$ for all $x \in \mathcal{X}$.

PROOF. We first prove the Theorem in the following:

Particular Case: Assume $\dim \mathcal{X}/\mathcal{Y} = 1$.

This means there exists some vector $x_0 \in \mathcal{X}$ such that

$$\mathcal{X} = \{y + sx_0 : y \in \mathcal{Y}, s \in \mathbb{R}\}.$$

What we need is to prescribe the value $\psi(x_0)$. In other words, we need a number $\alpha \in \mathbb{R}$ such that, if we define $\psi : \mathcal{X} \rightarrow \mathbb{R}$ by $\psi(y + sx_0) = \phi(y) + s\alpha$, $\forall y \in \mathcal{Y}, s \in \mathbb{R}$, then this map satisfies condition (ii). For $s > 0$, condition (ii) reads:

$$\phi(y) + s\alpha \leq q(y + sx_0), \forall y \in \mathcal{Y}, s > 0,$$

and, upon dividing by s (set $z = s^{-1}y$), is equivalent to:

$$(1) \quad \alpha \leq q(z + x_0) - \phi(z), \forall z \in \mathcal{Y}.$$

For $s < 0$, condition (ii) reads (use $t = -s$):

$$\phi(y) - t\alpha \leq q(y - tx_0), \forall y \in \mathcal{Y}, t > 0,$$

and, upon dividing by t (set $w = t^{-1}y$), is equivalent to:

$$(2) \quad \alpha \geq \phi(w) - q(w - x_0), \quad \forall w \in \mathcal{Y}.$$

Consider the sets

$$\begin{aligned} Z &= \{q(z + x_0) - \phi(z) : z \in \mathcal{Y}\} \subset \mathbb{R} \\ W &= \{\phi(w) - q(w - x_0) : w \in \mathcal{Y}\} \subset \mathbb{R}. \end{aligned}$$

The conditions (1) and (2) are equivalent to the inequalities

$$(3) \quad \sup W \leq \alpha \leq \inf Z.$$

This means that, in order to find a real number α with the desired property, it suffices to prove that $\sup W \leq \inf Z$, which in turn is equivalent to

$$(4) \quad \phi(w) - q(w - x_0) \leq q(z + x_0) - \phi(z), \quad \forall z, w \in \mathcal{Y}.$$

But the condition (4) is equivalent to

$$\phi(z + w) \leq q(z + x_0) + q(w - x_0),$$

which is obviously satisfied because

$$\phi(z + w) \leq q(z + w) = q((z + x_0) + (w - x_0)) \leq q(z + x_0) + q(w - x_0).$$

Having proved the Theorem in this particular case, let us proceed now with the general case. Let us consider the set Ξ of all pairs $(\mathcal{Z}, \nu)$ with

- $\mathcal{Z}$ is a subspace of $\mathcal{X}$ such that $\mathcal{Z} \supset \mathcal{Y}$;
- $\nu : \mathcal{Z} \rightarrow \mathbb{R}$ is a linear functional such that
 - (i) $\nu|_{\mathcal{Y}} = \phi$;
 - (ii) $\nu(z) \leq q(z)$, for all $z \in \mathcal{Z}$.

Put an order relation $\succ$ on Ξ as follows:

$$(\mathcal{Z}_1, \nu_1) \succ (\mathcal{Z}_2, \nu_2) \Leftrightarrow \begin{cases} \mathcal{Z}_1 \supset \mathcal{Z}_2 \\ \nu_1|_{\mathcal{Z}_2} = \nu_2 \end{cases}$$

Using Zorn's Lemma, Ξ possesses a maximal element $(\mathcal{Z}, \psi)$. The proof of the Theorem is finished once we prove that $\mathcal{Z} = \mathcal{X}$. Assume $\mathcal{Z} \subsetneq \mathcal{X}$ and choose a vector $x_0 \in \mathcal{X} \setminus \mathcal{Z}$. Form the subspace $\mathcal{V} = \{z + tx_0 : z \in \mathcal{Z}, t \in \mathbb{R}\}$ and apply the particular case of the Theorem for the inclusion $\mathcal{Z} \subset \mathcal{V}$, for $\psi : \mathcal{Z} \rightarrow \mathbb{R}$ and for the quasi-seminorm $q|_{\mathcal{V}} : \mathcal{V} \rightarrow \mathbb{R}$. It follows that there exists some linear functional $\eta : \mathcal{V} \rightarrow \mathbb{R}$ such that

- (i) $\eta|_{\mathcal{Z}} = \psi$ (in particular we will also have $\eta|_{\mathcal{Y}} = \phi$);
- (ii) $\eta(v) \leq q(v)$, for all $v \in \mathcal{V}$.

But then the element $(\mathcal{V}, \eta) \in \Xi$ will contradict the maximality of $(\mathcal{Z}, \psi)$. $\square$

THEOREM 1.2 (Hahn-Banach, $\mathbb{C}$ -version). *Let $\mathcal{X}$ be an $\mathbb{C}$ -vector space. Suppose $q : \mathcal{X} \rightarrow \mathbb{R}$ is a quasi-seminorm. Suppose also we are given a linear subspace $\mathcal{Y} \subset \mathcal{X}$ and a linear map $\phi : \mathcal{Y} \rightarrow \mathbb{C}$, such that*

$$\operatorname{Re} \phi(y) \leq q(y), \quad \text{for all } y \in \mathcal{Y}.$$

Then there exists a linear map $\psi : \mathcal{X} \rightarrow \mathbb{C}$ such that

- (i) $\psi|_{\mathcal{Y}} = \phi$;
- (ii) $\operatorname{Re} \psi(x) \leq q(x)$ for all $x \in \mathcal{X}$.

PROOF. Regard for the moment both $\mathcal{X}$ and $\mathcal{Y}$ as $\mathbb{R}$ -vector spaces. Define the $\mathbb{R}$ -linear map $\phi_1 : \mathcal{Y} \rightarrow \mathbb{R}$ by $\phi_1(y) = \operatorname{Re} \phi(y)$, for all $y \in \mathcal{Y}$, so that we have

$$\phi_1(y) \leq q(y), \quad \forall y \in \mathcal{Y}.$$

Use Theorem 1 to find an $\mathbb{R}$ -linear map $\psi_1 : \mathcal{X} \rightarrow \mathbb{R}$ such that

- (i) $\psi_1|_{\mathcal{Y}} = \phi_1$;
- (ii) $\psi_1(x) \leq q(x)$, for all $x \in \mathcal{X}$.

Define the map $\psi : \mathcal{X} \rightarrow \mathbb{C}$ by

$$\psi(x) = \psi_1(x) - i\psi_1(ix), \quad \text{for all } x \in \mathcal{X}.$$

Claim 1: ψ is $\mathbb{C}$ -linear.

It is obvious that ψ is $\mathbb{R}$ -linear, so the only thing to prove is that $\psi(ix) = i\psi(x)$, for all $x \in \mathcal{X}$. But this is quite obvious:

$$\begin{aligned} \psi(ix) &= \psi_1(ix) - i\psi_1(i^2x) = \psi_1(ix) - i\psi_1(-x) = \\ &= -i^2\psi_1(ix) + i\psi_1(x) = i(\psi_1(x) - i\psi_1(ix)) = i\psi(x), \quad \forall x \in \mathcal{X}. \end{aligned}$$

Because of the way ψ is defined, and because ψ_1 is real-valued, condition (ii) in the Theorem follows immediately

$$\operatorname{Re} \psi(x) = \psi_1(x) \leq q(x), \quad \forall x \in \mathcal{X},$$

so in order to finish the proof, we need to prove condition (i) in the Theorem, (i.e. $\psi|_{\mathcal{Y}} = \phi$). This follows from the fact that $\phi_1 = \psi_1|_{\mathcal{Y}}$, and from:

Claim 2: For every $y \in \mathcal{Y}$, we have $\phi(y) = \phi_1(y) - i\phi_1(iy)$.

But this is quite obvious, because

$$\operatorname{Im} \phi(y) = -\operatorname{Re} (i\phi(y)) = -\operatorname{Re} \phi(iy) = -\phi_1(iy), \quad \forall y \in \mathcal{Y}.$$

□

THEOREM 1.3 (Hahn-Banach, for seminorms). *Let $\mathcal{X}$ be a $\mathbb{K}$ -vector space ($\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$). Suppose q is a seminorm on $\mathcal{X}$. Suppose also we are given a linear subspace $\mathcal{Y} \subset \mathcal{X}$ and a linear map $\phi : \mathcal{Y} \rightarrow \mathbb{K}$, such that*

$$|\phi(y)| \leq q(y), \quad \text{for all } y \in \mathcal{Y}.$$

Then there exists a linear map $\psi : \mathcal{X} \rightarrow \mathbb{K}$ such that

- (i) $\psi|_{\mathcal{Y}} = \phi$;
- (ii) $|\psi(x)| \leq q(x)$ for all $x \in \mathcal{X}$.

PROOF. We are going to apply Theorems 1 and 2, using the fact that q is also a quasi-seminorm.

THE CASE $\mathbb{K} = \mathbb{R}$. Remark that

$$\phi(y) \leq |\phi(y)| \leq q(y), \quad \forall y \in \mathcal{Y}.$$

So we can apply Theorem 1 and find $\psi : \mathcal{X} \rightarrow \mathbb{R}$ with

- (i) $\psi|_{\mathcal{Y}} = \phi$;
- (ii) $\psi(x) \leq q(x)$, for all $x \in \mathcal{X}$.

Using condition (ii) we also get

$$-\psi(x) = \psi(-x) \leq q(-x) = q(x), \text{ for all } x \in \mathcal{X}.$$

In other words we get

$$\pm\psi(x) \leq q(x), \text{ for all } x \in \mathcal{X},$$

which of course gives the desired property (ii) in the Theorem.

THE CASE $\mathbb{K} = \mathbb{C}$. Remark that

$$\operatorname{Re} \phi(y) \leq |\phi(y)| \leq q(y), \quad \forall y \in \mathcal{Y}.$$

So we can apply Theorem 2 and find $\psi : \mathcal{X} \rightarrow \mathbb{R}$ with

- (i) $\psi|_{\mathcal{Y}} = \phi$;
- (ii) $\operatorname{Re} \psi(x) \leq q(x)$, for all $x \in \mathcal{X}$.

Using condition (ii) we also get

$$(5) \quad \operatorname{Re} (\lambda\psi(x)) = \operatorname{Re} \psi(\lambda x) \leq q(\lambda x) = q(x), \text{ for all } x \in \mathcal{X} \text{ and all } \lambda \in \mathbb{T}.$$

(Here $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.) Fix for the moment $x \in \mathcal{X}$. There exists some $\lambda \in \mathbb{T}$ such that $|\psi(x)| = \lambda\psi(x)$. For this particular λ we will have $\operatorname{Re} (\lambda\psi(x)) = |\psi(x)|$, so the inequality (5) will give

$$|\psi(x)| \leq q(x).$$

□

In the remainder of this section we will discuss the geometric form of the Hahn-Banach theorems. We begin by describing a method of constructing quasi-seminorms.

PROPOSITION 1.1. *Let $\mathcal{X}$ be a real vector space. Suppose $\mathcal{C} \subset \mathcal{X}$ is a convex subset, which contains 0, and has the property*

$$(6) \quad \bigcup_{t>0} t\mathcal{C} = \mathcal{X}.$$

For every $x \in \mathcal{X}$ we define

$$Q_{\mathcal{C}}(x) = \inf\{t > 0 : x \in t\mathcal{C}\}.$$

(By (6) the set in the right hand side is non-empty.) Then the map $Q_{\mathcal{C}} : \mathcal{X} \rightarrow \mathbb{R}$ is a quasi-seminorm.

PROOF. For every $x \in \mathcal{X}$, let us define the set

$$T_{\mathcal{C}}(x) = \{t > 0 : x \in t\mathcal{C}\}.$$

It is pretty clear that, since $0 \in \mathcal{C}$, we have

$$T_{\mathcal{C}}(0) = (0, \infty),$$

so we get

$$Q_{\mathcal{C}}(0) = \inf T_{\mathcal{C}}(0) = 0.$$

Claim 1: For every $x \in \mathcal{X}$ and every $\lambda > 0$, one has the equality

$$T_{\mathcal{C}}(\lambda x) = \lambda T_{\mathcal{C}}(x).$$

Indeed, if $t \in T_{\mathcal{C}}(\lambda x)$, we have $\lambda x \in t\mathcal{C}$, which means that $\lambda^{-1}tx \in \mathcal{C}$, i.e. $\lambda^{-1}t \in T_{\mathcal{C}}(x)$. Consequently we have

$$t = \lambda(\lambda^{-1}t) \in \lambda T_{\mathcal{C}}(x),$$

which proves the inclusion

$$T_{\mathcal{C}}(\lambda x) \subset \lambda T_{\mathcal{C}}(x).$$

To prove the other inclusion, we start with some $s \in \lambda T_{\mathcal{C}}(x)$, which means that there exists some $t \in T_{\mathcal{C}}(x)$ with $\lambda t = s$. The fact that $t = \lambda^{-1}s$ belongs to $T_{\mathcal{C}}(x)$ means that $x \in \lambda^{-1}s\mathcal{C}$, so get $\lambda x \in s\mathcal{C}$, so s indeed belongs to $T_{\mathcal{C}}(\lambda x)$.

Claim 2:: For every $x, y \in \mathcal{X}$, one has the inclusion⁴

$$T_{\mathcal{C}}(x + y) \supset T_{\mathcal{C}}(x) + T_{\mathcal{C}}(y).$$

Start with some $t \in T_{\mathcal{C}}(x)$ and some $s \in T_{\mathcal{C}}(y)$. Define the elements $u = t^{-1}x$ and $v = s^{-1}y$. Since $u, v \in \mathcal{C}$, and $\mathcal{C}$ is convex, it follows that $\mathcal{C}$ contains the element

$$\frac{t}{t+s}u + \frac{s}{t+s}v = \frac{1}{t+s}(x+y),$$

which means that $x+y \in (t+s)\mathcal{C}$, so $t+s$ indeed belongs to $T_{\mathcal{C}}(x+y)$.

We can now conclude the proof. If $x \in \mathcal{X}$ and $\lambda > 0$, then the equality

$$Q_{\mathcal{C}}(\lambda x) = \lambda Q_{\mathcal{C}}(x)$$

is an immediate consequence of Claim 1. If $x, y \in \mathcal{X}$, then the inequality

$$Q_{\mathcal{C}}(x+y) \leq \lambda Q_{\mathcal{C}}(x) + Q_{\mathcal{C}}(y)$$

is an immediate consequence of Claim 2. □

DEFINITION. Under the hypothesis of the above proposition, the quasi-seminorm $Q_{\mathcal{C}}$ is called the *Minkowski functional* associated with the set $\mathcal{C}$.

REMARK 1.1. Let $\mathcal{X}$ be a real vector space. Suppose $\mathcal{C} \subset \mathcal{X}$ is a convex subset, which contains 0, and has the property (6). Then one has the inclusions

$$\{x \in \mathcal{X} : Q_{\mathcal{C}}(x) < 1\} \subset \mathcal{C} \subset \{x \in \mathcal{X} : Q_{\mathcal{C}}(x) \leq 1\}.$$

The second inclusion is pretty obvious, since if we start with some $x \in \mathcal{C}$, using the notations from the proof of Proposition 2.1, we have $1 \in T_{\mathcal{C}}(x)$, so

$$Q_{\mathcal{C}}(x) = \inf T_{\mathcal{C}}(x) \leq 1.$$

To prove the first inclusion, start with some $x \in \mathcal{X}$ with $Q_{\mathcal{C}}(x) < 1$. In particular this means that there exists some $t \in (0, 1)$ such that $x \in t\mathcal{C}$. Define the vector $y = t^{-1}x \in \mathcal{C}$ and notice now that, since $\mathcal{C}$ is convex, it will contain the convex combination $ty + (1-t)0 = x$.

Exercise 1. Let $\mathcal{X}$ be a real vector space, and let $q : \mathcal{X} \rightarrow \mathbb{R}$ be a quasi-seminorm. Define the sets

$$\begin{aligned} \mathcal{C}_0 &= \{x \in \mathcal{X} : q(x) < 1\}, \\ \mathcal{C}_1 &= \{x \in \mathcal{X} : q(x) \leq 1\}. \end{aligned}$$

- (i) Prove that $\mathcal{C}_0$ and $\mathcal{C}_1$ are both convex, they contain 0, and they both have property (6).

⁴For subsets $T, S \subset \mathbb{R}$ we define $T + S = \{t + s : t \in T, s \in S\}$.

(ii) Let $\mathcal{C}$ is any convex set with

$$\mathcal{C}_0 \subset \mathcal{C} \subset \mathcal{C}_1.$$

Analyze the relationship between $Q_{\mathcal{C}}$ and q .

DEFINITION. A *topological vector space* is a vector space $\mathcal{X}$ over $\mathbb{K}$ (which is either $\mathbb{R}$ or $\mathbb{C}$), which is also a topological space, such that the maps

$$\mathcal{X} \times \mathcal{X} \ni (x, y) \longmapsto x + y \in \mathcal{X}$$

$$\mathbb{K} \times \mathcal{X} \ni (\lambda, x) \longmapsto \lambda x \in \mathcal{X}$$

are continuous.

REMARK 1.2. Let $\mathcal{X}$ be a real topological vector space. Suppose $\mathcal{C} \subset \mathcal{X}$ is a convex *open* subset, which contains 0. Then $\mathcal{C}$ has the property (6). Moreover (compare with Remark 2.1), one has the equality

$$(7) \quad \{x \in \mathcal{X} : Q_{\mathcal{C}}(x) < 1\} = \mathcal{C}.$$

To prove this remark, we define for each $x \in \mathcal{X}$, the function

$$F_x : \mathbb{R} \ni t \longmapsto tx \in \mathcal{X}.$$

Since $\mathcal{X}$ is a topological vector space, the map F_x , $x \in \mathcal{X}$ are continuous. To prove the property (6) we start with an arbitrary $x \in \mathcal{X}$, and we use the continuity of the map F_x at 0. Since $\mathcal{C}$ is a neighborhood of 0, there exists some $\rho > 0$ such that

$$F_x(t) \in \mathcal{C}, \quad \forall t \in [-\rho, \rho].$$

In particular we get $\rho x \in \mathcal{C}$, which means that $x \in \rho^{-1}\mathcal{C}$.

To prove the equality (7) we only need to prove the inclusion “ $\supset$ ” (since the inclusion “ $\subset$ ” holds in general, by Remark 2.1). Start with some element $x \in \mathcal{C}$. Using the continuity of the map F_x at 1, plus the fact that $F_x(1) = x \in \mathcal{C}$, there exists some $\varepsilon > 0$, such that

$$F_x(t) \in \mathcal{C}, \quad \forall t \in [1 - \varepsilon, 1 + \varepsilon].$$

In particular, we have $F(1 + \varepsilon) \in \mathcal{C}$, which means precisely that

$$x \in (1 + \varepsilon)^{-1}\mathcal{C}.$$

This gives the inequality

$$Q_{\mathcal{C}}(x) \leq (1 + \varepsilon)^{-1},$$

so we indeed get $Q_{\mathcal{C}}(x) < 1$.

The first geometric version of the Hahn-Banach Theorem is:

LEMMA 1.1. *Let $\mathcal{X}$ be a real topological vector space, and let $\mathcal{C} \subset \mathcal{X}$ be a convex open set which contains 0. If $x_0 \in \mathcal{X}$ is some point which does not belong to $\mathcal{C}$, then there exists a linear continuous map $\phi : \mathcal{X} \rightarrow \mathbb{R}$, such that*

- $\phi(x_0) = 1$;
- $\phi(v) < 1$, $\forall v \in \mathcal{C}$.

PROOF. Consider the linear subspace

$$\mathcal{Y} = \mathbb{R}x_0 = \{tx_0 : t \in \mathbb{R}\},$$

and define $\psi : \mathcal{Y} \rightarrow \mathbb{R}$ by

$$\psi(tx_0) = t, \quad \forall t \in \mathbb{R}.$$

It is obvious that ψ is linear, and $\psi(x_0) = 1$.

Claim: One has the inequality

$$\psi(y) \leq Q_{\mathcal{C}}(y), \quad \forall y \in \mathcal{Y}.$$

Let y be represented as $y = tx_0$ for some $t \in \mathbb{R}$. If $t \leq 0$, the inequality is clear, because $\psi(y) = t \leq 0$ and the right hand side $Q_{\mathcal{C}}(y)$ is always non-negative. Assume $t > 0$. Since $Q_{\mathcal{C}}$ is a quasi-seminorm, we have

$$(8) \quad Q_{\mathcal{C}}(y) = Q_{\mathcal{C}}(tx_0) = tQ_{\mathcal{C}}(x_0),$$

and the fact that $x_0 \notin \mathcal{C}$ will give (by Remark 2.2) the inequality $Q_{\mathcal{C}}(x_0) \geq 1$. Since $t > 0$, the computation (8) can be continued with

$$Q_{\mathcal{C}}(y) = tQ_{\mathcal{C}}(x_0) \geq t = \psi(y),$$

so the Claim follows also in this case.

Use now the Hahn-Banach Theorem, to find a linear map $\phi : \mathcal{X} \rightarrow \mathbb{R}$ such that

- (i) $\phi|_{\mathcal{Y}} = \psi$;
- (ii) $\phi(x) \leq Q_{\mathcal{C}}(x), \quad \forall x \in \mathcal{X}$.

It is obvious that (i) gives $\phi(x_0) = \psi(x_0) = 1$. If $v \in \mathcal{C}$, then by Remark 2.2 we have $Q_{\mathcal{C}}(v) < 1$, so by (ii) we also get $\phi(v) < 1$. This means that the only thing that remains to be proven is the continuity of ϕ . Since ϕ is linear, we only need to prove that ϕ is continuous at 0. Start with some $\varepsilon > 0$. We must find some open set $\mathcal{U}_{\varepsilon} \subset \mathcal{X}$, with $\mathcal{U}_{\varepsilon} \ni 0$, such that

$$|\phi(u)| < \varepsilon, \quad \forall u \in \mathcal{U}_{\varepsilon}.$$

We take $\mathcal{U}_{\varepsilon} = (\varepsilon\mathcal{C}) \cap (-\varepsilon\mathcal{C})$. Notice that, for every $u \in \mathcal{U}_{\varepsilon}$, we have $\pm u \in \varepsilon\mathcal{C}$, which gives $\varepsilon^{-1}(\pm u) \in \mathcal{C}$. By Remark 2.2 this gives $Q_{\mathcal{C}}(\varepsilon^{-1}(\pm u)) < 1$, which gives

$$Q_{\mathcal{C}}(\pm u) < \varepsilon.$$

Then using property (ii) we immediately get

$$\phi(\pm u) < \varepsilon,$$

and we are done. □

It turns out that the above result is a particular case of a more general result:

THEOREM 1.4 (Hahn-Banach Separation Theorem - real case). *Let $\mathcal{X}$ be a real topological vector space, let $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$ be non-empty convex sets with $\mathcal{A}$ open, and $\mathcal{A} \cap \mathcal{B} = \emptyset$. Then there exists a linear continuous map $\phi : \mathcal{X} \rightarrow \mathbb{R}$, and a real number α , such that*

$$\phi(a) < \alpha \leq \phi(b), \quad \forall a \in \mathcal{A}, b \in \mathcal{B}.$$

PROOF. Fix some points $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$, and define the set

$$\mathcal{C} = \mathcal{A} - \mathcal{B} + b_0 - a_0 = \{a - b + b_0 - a_0 : a \in \mathcal{A}, b \in \mathcal{B}\}.$$

It is straightforward that $\mathcal{C}$ is convex and contains 0. The equality

$$\mathcal{C} = \bigcup_{b \in \mathcal{B}} (\mathcal{A} + b_0 - a_0 - b)$$

shows that $\mathcal{C}$ is also open. Define the vector $x_0 = b_0 - a_0$. Since $\mathcal{A} \cap \mathcal{B} = \emptyset$, it is clear that $x_0 \notin \mathcal{C}$.

Use Lemma 2.1 to produce a linear continuous map $\phi : \mathcal{X} \rightarrow \mathbb{R}$ such that

- (i) $\phi(x_0) = 1$;

(ii) $\phi(v) < 1, \forall v \in \mathcal{C}$.

By the definition of x_0 and $\mathcal{C}$, we have $\phi(b_0) = \phi(a_0) + 1$, and

$$\phi(a) < \phi(b) + \phi(a_0) - \phi(b_0) + 1, \quad \forall a \in \mathcal{A}, b \in \mathcal{B},$$

which gives

$$(9) \quad \phi(a) < \phi(b), \quad \forall a \in \mathcal{A}, b \in \mathcal{B}.$$

Put

$$\alpha = \inf_{b \in \mathcal{B}} \phi(b).$$

The inequalities (9) give

$$(10) \quad \phi(a) \leq \alpha \leq \phi(b), \quad \forall a \in \mathcal{A}, b \in \mathcal{B}.$$

The proof will be complete once we prove the following

Claim: One has the inequality

$$\phi(a) < \alpha, \quad \forall a \in \mathcal{A}.$$

Suppose the contrary, i.e. there exists some $a_1 \in \mathcal{A}$ with $\phi(a_1) = \alpha$. Using the continuity of the map

$$\mathbb{R} \ni t \longmapsto a_1 + tx_0 \in \mathcal{X}$$

there exists some $\varepsilon > 0$ such that

$$a_1 + tx_0 \in \mathcal{A}, \quad \forall t \in [-\varepsilon, \varepsilon].$$

In particular, by (10) one has

$$\phi(a_1 + \varepsilon x_0) \leq \alpha,$$

which means that

$$\alpha + \varepsilon \leq \alpha,$$

which is clearly impossible. $\square$

THEOREM 1.5 (Hahn-Banach Separation Theorem - complex case). *Let $\mathcal{X}$ be a complex topological vector space, let $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$ be non-empty convex sets with $\mathcal{A}$ open, and $\mathcal{A} \cap \mathcal{B} = \emptyset$. Then there exists a linear continuous map $\phi : \mathcal{X} \rightarrow \mathbb{C}$, and a real number α , such that*

$$\operatorname{Re} \phi(a) < \alpha \leq \operatorname{Im} \phi(b), \quad \forall a \in \mathcal{A}, b \in \mathcal{B}.$$

PROOF. Regard $\mathcal{X}$ as a real topological vector space, and apply the real version to produce an $\mathbb{R}$ -linear continuous map $\phi_1 : \mathcal{X} \rightarrow \mathbb{R}$, and a real number α , such that

$$\phi_1(a) < \alpha \leq \phi_1(b), \quad \forall a \in \mathcal{A}, b \in \mathcal{B}.$$

Then the function $\phi : \mathcal{X} \rightarrow \mathbb{C}$ defined by

$$\phi(x) = \phi_1(x) - i\phi_1(ix), \quad x \in \mathcal{X}$$

will clearly satisfy the desired properties. $\square$

There is another version of the Hahn-Banach Separation Theorem, which holds for a special type of topological vector spaces. Before we discuss these, we shall need a technical result.

LEMMA 1.2. *Let $\mathcal{X}$ be a topological vector space, let $\mathcal{C} \subset \mathcal{X}$ be a compact set, and let $\mathcal{D} \subset \mathcal{X}$ be a closed set. Then the set*

$$\mathcal{C} + \mathcal{D} = \{x + y : x \in \mathcal{C}, y \in \mathcal{D}\}$$

is closed.

PROOF. Start with some point $p \in \overline{\mathcal{C} + \mathcal{D}}$, and let us prove that $p \in \mathcal{C} + \mathcal{D}$. For every neighborhood $\mathcal{U}$ of 0, the set $p + \mathcal{U}$ is a neighborhood of p , so by assumption, we have

$$(11) \quad (p + \mathcal{U}) \cap (\mathcal{C} + \mathcal{D}) \neq \emptyset.$$

Define, for each neighborhood $\mathcal{U}$ of 0, the set

$$\mathcal{A}_{\mathcal{U}} = (p + \mathcal{U} - \mathcal{D}) \cap \mathcal{C}.$$

Using (11), it is clear that $\mathcal{A}_{\mathcal{U}}$ is non-empty. It is also clear that, if $\mathcal{U}_1 \subset \mathcal{U}_2$, then $\mathcal{A}_{\mathcal{U}_1} \subset \mathcal{A}_{\mathcal{U}_2}$. Using the compactness of $\mathcal{C}$, it follows that

$$\bigcap_{\substack{\mathcal{U} \text{ neighborhood} \\ \text{of } 0}} \overline{\mathcal{A}_{\mathcal{U}}} \neq \emptyset.$$

Choose then a point q in the above intersection. It follows that

$$(q + \mathcal{V}) \cap \mathcal{A}_{\mathcal{U}} \neq \emptyset,$$

for any two neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of 0. In other words, for any two such neighborhoods of 0, we have

$$(12) \quad (q + \mathcal{V} - \mathcal{U}) \cap (p - \mathcal{D}) \neq \emptyset.$$

Fix now an arbitrary neighborhood $\mathcal{W}$ of 0. Using the continuity of the map

$$\mathcal{X} \times \mathcal{X} \ni (x_1, x_2) \longmapsto x_1 - x_2 \in \mathcal{X},$$

there exist neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of 0, such that $\mathcal{U} - \mathcal{V} \subset \mathcal{W}$. Then $q + \mathcal{V} - \mathcal{U} \subset q - \mathcal{W}$, so (12) gives

$$(q - \mathcal{W}) \cap (p - \mathcal{D}) \neq \emptyset,$$

which yields

$$(p - q + \mathcal{W}) \cap \mathcal{D} \neq \emptyset.$$

Since this is true for all neighborhoods $\mathcal{W}$ of 0, we get $p - q \in \overline{\mathcal{D}}$, and since $\mathcal{D}$ is closed, we finally get $p - q \in \mathcal{D}$. Since, by construction we have $q \in \mathcal{C}$, it follows that the point $p = q + (p - q)$ indeed belongs to $\mathcal{C} + \mathcal{D}$. $\square$

DEFINITION. A topological vector space $\mathcal{X}$ is said to be *locally convex*, if every point has a fundamental system of convex open neighborhoods. This means that for every $x \in \mathcal{X}$ and every neighborhood N of x , there exists a convex open set D , with $x \in D \subset N$.

THEOREM 1.6 (Hahn-Banach Separation Theorem for Locally Convex Spaces). *Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $\mathcal{X}$ be a locally convex $\mathbb{K}$ -vector space. Suppose $\mathcal{C}, \mathcal{D} \subset \mathcal{X}$ are convex sets, with $\mathcal{C}$ compact, $\mathcal{D}$ closed, and $\mathcal{C} \cap \mathcal{D} = \emptyset$. Then there exists a linear continuous map $\phi : \mathcal{X} \rightarrow \mathbb{K}$, and two numbers $\alpha, \beta \in \mathbb{R}$, such that*

$$\operatorname{Re} \phi(x) \leq \alpha < \beta \leq \operatorname{Re} \phi(y), \quad \forall x \in \mathcal{C}, y \in \mathcal{D}.$$

PROOF. Consider the convex set $\mathcal{B} = \mathcal{D} - \mathcal{C}$. By Lemma ??, $\mathcal{B}$ is closed. Since $\mathcal{C} \cap \mathcal{D} = \emptyset$, we have $0 \notin \mathcal{B}$. Since $\mathcal{B}$ is closed, its complement $\mathcal{X} \setminus \mathcal{B}$ will then be a neighborhood of 0. Since $\mathcal{X}$ is locally convex, there exists a convex open set $\mathcal{A}$, with $0 \in \mathcal{A} \subset \mathcal{X} \setminus \mathcal{B}$. In particular we have $\mathcal{A} \cap \mathcal{B} = \emptyset$. Applying the suitable version of the Hahn-Banach Theorem (real or complex case), we find a linear continuous map $\phi : \mathcal{X} \rightarrow \mathbb{K}$, and a real number ρ , such that

$$\operatorname{Re} \phi(a) < \rho \leq \operatorname{Re} \phi(b), \quad \forall a \in \mathcal{A}, \quad b \in \mathcal{B}.$$

Notice that, since $\mathcal{A} \ni 0$, we get $\rho > 0$. Then the inequality

$$\rho \leq \operatorname{Re} \phi(b), \quad b \in \mathcal{B}$$

gives

$$\operatorname{Re} \phi(y) - \operatorname{Re} \phi(x) \geq \rho > 0, \quad \forall x \in \mathcal{C}, \quad y \in \mathcal{D}.$$

Then if we define

$$\beta = \inf_{y \in \mathcal{D}} \operatorname{Re} \phi(y) \quad \text{and} \quad \alpha = \sup_{x \in \mathcal{C}} \operatorname{Re} \phi(x),$$

we get $\beta \geq \alpha + \rho$, and we are done. □

LECTURES 9-11

2. Normed vector spaces

DEFINITION. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $\mathcal{X}$ be a $\mathbb{K}$ -vector space. A *norm* on $\mathcal{X}$ is a map

$$\mathcal{X} \ni x \longmapsto \|x\| \in [0, \infty)$$

with the following properties

- (i) $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in \mathcal{X}$;
- (ii) $\|\lambda x\| = |\lambda| \cdot \|x\|$, $\forall x \in \mathcal{X}$, $\lambda \in \mathbb{K}$;
- (iii) $\|x\| = 0 \implies x = 0$.

(Note that conditions (i) and (ii) state that $\|\cdot\|$ is a seminorm.)

EXAMPLE 2.1. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Fix some non-empty set I , and define

$$c_0^{\mathbb{K}}(I) = \left\{ \alpha : I \rightarrow \mathbb{K} : \inf_{\substack{F \subset I \\ \text{finite}}} \left[\sup_{i \in I \setminus F} |\alpha(i)| \right] = 0 \right\}.$$

Remark that for a function $\alpha : I \rightarrow \mathbb{K}$, the fact that α belongs to $c_0^{\mathbb{K}}(I)$ is equivalent to the following condition:

- For every $\varepsilon > 0$, there exists some finite set $F \subset I$, such that

$$|\alpha(i)| < \varepsilon, \quad \forall i \in I \setminus F.$$

We equip the space $c_0^{\mathbb{K}}(I)$ with the $\mathbb{K}$ -vector space structure defined by point-wise addition and point-wise scalar multiplication. We also define the norm $\|\cdot\|_{\infty}$ by

$$\|\alpha\| = \sup_{i \in I} |\alpha(i)|, \quad \alpha \in c_0^{\mathbb{K}}(I).$$

When $\mathbb{K} = \mathbb{C}$, the space $c_0^{\mathbb{C}}(I)$ is simply denoted by $c_0(I)$. When $I = \mathbb{N}$ - the set of natural numbers - the space $c_0^{\mathbb{K}}(\mathbb{N})$ can be equivalently described as

$$c_0^{\mathbb{K}}(\mathbb{N}) = \{ \alpha = (\alpha_n)_{n \geq 1} \subset \mathbb{K} : \lim_{n \rightarrow \infty} \alpha_n = 0 \}.$$

In this case instead of $c_0^{\mathbb{R}}(\mathbb{N})$ we simply write $c_0^{\mathbb{R}}$, and instead of $c_0(\mathbb{N})$ we simply write c_0 .

Exercise 1. Prove that $\|\cdot\|_{\infty}$ is indeed a norm on $c_0^{\mathbb{K}}(I)$.

EXAMPLE 2.2. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, and let I be a non-empty set. We define the space

$$fin_{\mathbb{K}}(I) = \{ \alpha : I \rightarrow \mathbb{K} : \text{the set } \{i \in I : \alpha(i) \neq 0\} \text{ is finite} \}.$$

Then $fin_{\mathbb{K}}(I)$ is a linear subspace in $c_0^{\mathbb{K}}(I)$.

DEFINITION. Suppose $\mathcal{X}$ is a normed vector space, with norm $\|\cdot\|$. Then there is a natural metric d on $\mathcal{X}$, defined by

$$d(x, y) = \|x - y\|, \quad x, y \in \mathcal{X}.$$

The topology on $\mathcal{X}$, defined by this metric, is called the *norm topology*.

Exercise 2. Let $\mathcal{X}$ be a normed vector space, over $\mathbb{K}(= \mathbb{R}, \mathbb{C})$. Prove that, when equipped with the norm topology, $\mathcal{X}$ becomes a topological vector space. That is, the maps

$$\mathcal{X} \times \mathcal{X} \ni (x, y) \longmapsto x + y \in \mathcal{X}$$

$$\mathbb{K} \times \mathcal{X} \ni (\lambda, x) \longmapsto \lambda x \in \mathcal{X}$$

are continuous.

Exercise 3. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let I be a non-empty set. Prove that $\text{fin}_{\mathbb{K}}(I)$ is dense in $c_0^{\mathbb{K}}(I)$ in the norm topology.

EXAMPLE 2.3. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let I be a non-empty set. Define

$$\ell_{\mathbb{K}}^{\infty}(I) = \left\{ \alpha : I \rightarrow \mathbb{K} : \sup_{i \in I} |\alpha(i)| < \infty \right\}.$$

We equip the space $\ell_{\mathbb{K}}^{\infty}(I)$ with the $\mathbb{K}$ -vector space structure defined by point-wise addition and point-wise scalar multiplication. We also define the norm $\|\cdot\|_{\infty}$ by

$$\|\alpha\|_{\infty} = \sup_{i \in I} |\alpha(i)|, \quad \alpha \in \ell_{\mathbb{K}}^{\infty}(I).$$

When $\mathbb{K} = \mathbb{C}$, the space $\ell_{\mathbb{C}}^{\infty}(I)$ is simply denoted by $\ell^{\infty}(I)$. When $I = \mathbb{N}$ - the set of natural numbers - instead of $\ell_{\mathbb{R}}^{\infty}(\mathbb{N})$ we simply write $\ell_{\mathbb{R}}^{\infty}$, and instead of $\ell^{\infty}(\mathbb{N})$ we simply write ℓ^{∞} .

Exercise 4. Prove that $\|\cdot\|_{\infty}$ is indeed a norm on $\ell_{\mathbb{K}}^{\infty}(I)$.

Exercise 5. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let I be a non-empty set. Prove that $c_0^{\mathbb{K}}(I)$ is a linear subspace in $\ell_{\mathbb{K}}^{\infty}(I)$, which is *closed* in the norm topology.

In preparation for the next class of examples, we introduce the following:

DEFINITION. A map $\alpha : I \rightarrow \mathbb{K}$ is said to be *summable*, if there exists some number $s \in \mathbb{K}$ such that

(s) *for every $\varepsilon > 0$ there exists some finite set $F_{\varepsilon} \subset I$ such that*

$$\left| s - \sum_{i \in F} \alpha(i) \right| < \varepsilon, \text{ for all finite sets } F \text{ with } F_{\varepsilon} \subset F \subset I.$$

If such an s exists, then it is unique, and it is denoted by $\sum_{i \in I} \alpha(i)$. In the case when I is *finite*, every map $\alpha : I \rightarrow \mathbb{K}$ is summable, and the above notation agrees with the usual notation for the sum.

Exercise 6. Assume $\alpha : I \rightarrow \mathbb{K}$ is summable. Prove that, for every $\lambda \in \mathbb{K}$, the map $\lambda\alpha : I \rightarrow \mathbb{K}$ is summable, and

$$\sum_{i \in I} \lambda\alpha(i) = \lambda \sum_{i \in I} \alpha(i).$$

If $\beta : I \rightarrow \mathbb{K}$ is another summable map, prove that $\alpha + \beta : I \rightarrow \mathbb{K}$ is summable, and

$$\sum_{i \in I} [\alpha(i) + \beta(i)] = \left[\sum_{i \in I} \alpha(i) \right] + \left[\sum_{i \in I} \beta(i) \right].$$

The following result characterizes summability for non-negative terms

LEMMA 2.1. *Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, let I be a non-empty set, and let $\alpha : I \rightarrow [0, \infty)$. The following are equivalent:*

- (i) α is summable;
- (ii) $\sup \left\{ \sum_{i \in F} \alpha(i) : F \subset I, \text{ finite} \right\} < \infty$.

Moreover, in this case we have

$$\sup \left\{ \sum_{i \in F} \alpha(i) : F \subset I, \text{ finite} \right\} = \sum_{i \in I} \alpha(i).$$

PROOF. We denote the quantity $\sup \left\{ \sum_{i \in F} \alpha(i) : F \subset I, \text{ finite} \right\}$ simply by t .

(i) $\Rightarrow$ (ii). Assume α is summable, and denote $\sum_{i \in I} \alpha(i)$ simply by s . Choose, for each $\varepsilon > 0$ a finite set $F_\varepsilon \subset I$ such that

$$\left| s - \sum_{i \in F} \alpha(i) \right| < \varepsilon, \text{ for all finite subsets } F \subset I \text{ with } F \supset F_\varepsilon.$$

Claim: For any finite set $G \subset I$, and any $\varepsilon > 0$, one has the inequality

$$\sum_{i \in G} \alpha(i) < s + \varepsilon.$$

Indeed, if we take the finite set $G \cup F_\varepsilon$, then using the fact that all α 's are non-negative, we get

$$\sum_{i \in G} \alpha(i) \leq \sum_{i \in G \cup F_\varepsilon} \alpha(i) < s + \varepsilon.$$

Using the Claim, which holds for *any* $\varepsilon > 0$, we immediately get

$$\sum_{i \in G} \alpha(i) \leq s, \text{ for all finite subsets } G \subset I,$$

so taking supremum yields $t \leq s$, in particular $t < \infty$.

(ii) $\Rightarrow$ (i). Assume condition (ii) is true. We are going to show that α is summable, by proving that the number t satisfies the definition of summability. Consider the set

$$S = \left\{ \sum_{i \in F} \alpha(i) : F \text{ finite subset of } I \right\},$$

so that $\sup S = t < \infty$. Start with some $\varepsilon > 0$. Since $t - \varepsilon$ is no longer an upper bound for S , there exists some finite set $F_\varepsilon \subset I$, such that $\sum_{i \in F_\varepsilon} \alpha(i) > t - \varepsilon$. Notice that, for any finite set $F \subset I$ with $F \supset F_\varepsilon$, we have

$$t - \varepsilon < \sum_{i \in F_\varepsilon} \alpha(i) \leq \sum_{i \in F} \alpha(i) \leq t,$$

so we immediately get

$$\left| t - \sum_{i \in F} \alpha(i) \right| < \varepsilon.$$

□

Exercise 7. Let $\alpha : I \rightarrow [0, \infty)$ be summable. Prove that every map $\beta : I \rightarrow [0, \infty)$ with

$$\beta(j) \leq \alpha(j), \quad \forall j \in I,$$

is summable, and $\sum_{j \in I} \beta(j) \leq \sum_{j \in I} \alpha(j)$.

REMARK 2.1. It is obvious that the above result has a version for non-positive maps as well. More explicitly, for a map $\alpha : I \rightarrow (-\infty, 0]$ the following are equivalent:

- (i) α is summable;
- (ii) $\inf \left\{ \sum_{i \in F} \alpha(i) : F \subset I, \text{ finite} \right\} > -\infty$.

Moreover, in this case we have

$$\inf \left\{ \sum_{i \in F} \alpha(i) : F \subset I, \text{ finite} \right\} = \sum_{i \in I} \alpha(i).$$

LEMMA 2.2. Let I be a non-empty set. For a function $\alpha : I \rightarrow \mathbb{C}$, the following are equivalent:

- (i) α is summable;
- (ii) both functions $\operatorname{Re} \alpha, \operatorname{Im} \alpha : I \rightarrow \mathbb{R}$ are summable.

Moreover, in this case we have the equality

$$\sum_{j \in I} \alpha(j) = \sum_{j \in I} \operatorname{Re} \alpha(j) + i \sum_{j \in I} \operatorname{Im} \alpha(j).$$

PROOF. $(i) \Rightarrow (ii)$. Assume α is summable. Denote the sum $\sum_{j \in I} \alpha(j)$ simply by s . For every $\varepsilon > 0$ choose a finite set $F_\varepsilon \subset I$ such that

$$\left| s - \sum_{j \in F} \alpha(j) \right| < \varepsilon, \text{ for all finite sets } F \subset I \text{ with } F \supset F_\varepsilon.$$

Using the inequality

$$\max \{ |\operatorname{Re} z|, |\operatorname{Im} z| \} \leq |z|, \quad \forall z \in \mathbb{C},$$

we immediately get the inequalities

$$\begin{aligned} \left| \operatorname{Re} s - \sum_{j \in F} \operatorname{Re} \alpha(j) \right| &= \left| \operatorname{Re} \left[s - \sum_{j \in F} \alpha(j) \right] \right| \leq \left| s - \sum_{j \in F} \alpha(j) \right| < \varepsilon, \\ \left| \operatorname{Im} s - \sum_{j \in F} \operatorname{Im} \alpha(j) \right| &= \left| \operatorname{Im} \left[s - \sum_{j \in F} \alpha(j) \right] \right| \leq \left| s - \sum_{j \in F} \alpha(j) \right| < \varepsilon, \end{aligned}$$

for all finite sets $F \subset I$ with $F \supset F_\varepsilon$,

so $\operatorname{Re} \alpha$ and $\operatorname{Im} \alpha$ are indeed summable and moreover, we have

$$\sum_{j \in I} \operatorname{Re} \alpha(j) = \operatorname{Re} s \text{ and } \sum_{j \in I} \operatorname{Im} \alpha(j) = \operatorname{Im} s.$$

$(ii) \Rightarrow (i)$. Assume $\operatorname{Re} \alpha$ and $\operatorname{Im} \alpha$ are both summable. Denote $\sum_{j \in I} \operatorname{Re} \alpha(j)$ by u and denote $\sum_{j \in I} \operatorname{Im} \alpha(j)$ by v . Fix some $\varepsilon > 0$. Choose finite sets $E_\varepsilon, G_\varepsilon \subset I$

such that

$$\left| u - \sum_{j \in E} \operatorname{Re} \alpha(j) \right| < \frac{\varepsilon}{2}, \text{ for all finite sets } E \subset I \text{ with } E \supset E_\varepsilon,$$

$$\left| v - \sum_{j \in G} \operatorname{Im} \alpha(j) \right| < \frac{\varepsilon}{2}, \text{ for all finite sets } G \subset I \text{ with } G \supset G_\varepsilon.$$

Put $F_\varepsilon = E_\varepsilon \cup G_\varepsilon$. Suppose $F \subset I$ is a finite set with $F \supset F_\varepsilon$. Using the inclusions $F \supset E_\varepsilon$ and $F \supset G_\varepsilon$, we then get

$$\left| u - \sum_{j \in F} \operatorname{Re} \alpha(j) \right| < \frac{\varepsilon}{2} \text{ and } \left| v - \sum_{j \in F} \operatorname{Im} \alpha(j) \right| < \frac{\varepsilon}{2},$$

so we get

$$\begin{aligned} \left| [u + iv] - \sum_{j \in F} \alpha(j) \right| &= \left| \left[u - \sum_{j \in F} \operatorname{Re} \alpha(j) \right] + i \left[v - \sum_{j \in F} \operatorname{Im} \alpha(j) \right] \right| \leq \\ &\leq \left| u - \sum_{j \in F} \operatorname{Re} \alpha(j) \right| + \left| v - \sum_{j \in F} \operatorname{Im} \alpha(j) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This proves that α is indeed summable, and $\sum_{j \in I} \alpha(j) = u + iv$. $\square$

Exercise 8. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let I be a non-empty set. Suppose one has two non-empty sets I_1, I_2 with $I = I_1 \cup I_2$ and $I_1 \cap I_2 = \emptyset$. Suppose $\alpha : I \rightarrow \mathbb{K}$ has the property that both $\alpha|_{I_1} : I_1 \rightarrow \mathbb{K}$ and $\alpha|_{I_2} : I_2 \rightarrow \mathbb{K}$ are summable. Prove that α is summable, and

$$\sum_{j \in I} \alpha(j) = \sum_{j \in I_1} \alpha(j) + \sum_{j \in I_2} \alpha(j).$$

PROPOSITION 2.1. *Let I be a non-empty set, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. For a map $\alpha : I \rightarrow \mathbb{K}$, the following are equivalent:*

- (i) α is summable;
- (ii) $|\alpha|$ is summable.

Moreover, in this case one has the inequality

$$(1) \quad \left| \sum_{j \in I} \alpha(j) \right| \leq \sum_{j \in I} |\alpha(j)|.$$

PROOF. (i) $\Rightarrow$ (ii). Assume α is summable. We divide the proof in two cases:
Case $\mathbb{K} = \mathbb{R}$. Define the sets

$$\begin{aligned} I^+ &= \{j \in I : \alpha(j) > 0\}, \\ I^- &= \{j \in I : \alpha(j) < 0\}, \\ I^0 &= \{j \in I : \alpha(j) = 0\}. \end{aligned}$$

More generally, for any subset $F \subset I$ we define $F^\pm = F \cap I^\pm$ and $F^0 = F \cap I^0$.

Claim: Both maps $\alpha|_{I^+} : I^+ \rightarrow \mathbb{R}$ and $\alpha|_{I^-} : I^- \rightarrow \mathbb{R}$ are summable.
Moreover, one has the equality

$$(2) \quad \sum_{j \in I} \alpha(j) = \sum_{j \in I^+} \alpha(j) + \sum_{j \in I^-} \alpha(j).$$

Denote the sum $\sum_{j \in I} \alpha(j)$ simply by s . Start by choosing some finite set $F \subset I$ such that

$$\left| s - \sum_{j \in G} \alpha(j) \right| < 1, \text{ for all finite sets } G \subset I \text{ with } G \supset F.$$

Let $E \subset I^+$ be a finite subset. Then the set $\tilde{E} = E \cup F$ will be a finite subset of I with $\tilde{E} \supset F$, so we will have

$$\left| s - \sum_{j \in \tilde{E}} \alpha(j) \right| < 1,$$

so we get

$$\begin{aligned} \sum_{j \in E} \alpha(j) &\leq \sum_{j \in E \cup F^+} \alpha(j) = \left[\sum_{j \in E \cup F^+} \alpha(j) + \sum_{j \in F^0 \cup F^-} \alpha(j) \right] - \left[\sum_{j \in F^0 \cup F^-} \alpha(j) \right] = \\ &= \left[\sum_{j \in \tilde{E}} \alpha(j) \right] - \left[\sum_{j \in F^-} \alpha(j) \right] < s + 1 - \left[\sum_{j \in F^-} \alpha(j) \right]. \end{aligned}$$

In particular this gives

$$\sup \left\{ \sum_{j \in E} \alpha(j) : E \subset I^+, \text{ finite} \right\} \leq s + 1 - \left[\sum_{j \in F^-} \alpha(j) \right],$$

so by Lemma ??, the map $\alpha|_{I^+} : I^+ \rightarrow [0, \infty)$ is indeed summable. The fact that the map $\alpha|_{I^-} : I^- \rightarrow (-\infty, 0]$ is summable is proven the exact same way. The equality (2) follows from Exercise ??

Having proven the Claim, we notice now that the map $-\alpha|_{I^-} : I^- \rightarrow [0, \infty)$ is also summable. Using Exercise ??, it is clear then that the map $|\alpha| : I \rightarrow [0, \infty)$ is summable, simply because all the three maps $|\alpha|_{I^+} = \alpha|_{I^+}$, $|\alpha|_{I^-} = -\alpha|_{I^-}$, and $|\alpha|_{I^0} = 0$ are all summable.

Case $\mathbb{K} = \mathbb{C}$. By Lemma ?? we know that the maps $\operatorname{Re} \alpha, \operatorname{Im} \alpha : I \rightarrow \mathbb{R}$ are summable. In particular, using the real case, we get the fact that the maps $|\operatorname{Re} \alpha|, |\operatorname{Im} \alpha| : I \rightarrow [0, \infty)$ are summable. Using the obvious inequality

$$|z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|, \quad \forall z \in \mathbb{C},$$

we get

$$\sum_{j \in F} |\alpha(j)| \leq \sum_{j \in F} |\operatorname{Re} \alpha(j)| + \sum_{j \in F} |\operatorname{Im} \alpha(j)| \leq \sum_{j \in I} |\operatorname{Re} \alpha(j)| + \sum_{j \in I} |\operatorname{Im} \alpha(j)|,$$

for every finite subset $F \subset I$. Then we get

$$\sup \left\{ \sum_{j \in F} |\alpha(j)| : F \subset I, \text{ finite} \right\} \leq \sum_{j \in I} |\operatorname{Re} \alpha(j)| + \sum_{j \in I} |\operatorname{Im} \alpha(j)| < \infty,$$

so $|\alpha| : I \rightarrow [0, \infty)$ is indeed summable.

Having proven the implication (i) $\Rightarrow$ (ii), let us prove the inequality (1). If s denotes the sum $\sum_{j \in I} \alpha(j)$, then for every $\varepsilon > 0$ there exists $F_\varepsilon \subset I$ finite such that

$$\left| s - \sum_{j \in F} \alpha(j) \right| < \varepsilon, \text{ for all finite sets } F \subset I \text{ with } F \supset F_\varepsilon.$$

In particular, we get

$$|s| \leq \varepsilon + \left| \sum_{j \in F_\varepsilon} \alpha(j) \right| \leq \varepsilon + \sum_{j \in F_\varepsilon} |\alpha(j)| \leq \varepsilon + \sum_{j \in I} |\alpha(j)|.$$

Since this inequality holds for all $\varepsilon > 0$, we then get

$$|s| \leq \sum_{j \in F_\varepsilon} |\alpha(j)|.$$

(ii) $\Rightarrow$ (i). Assume now $|\alpha| : I \rightarrow [0, \infty)$ is summable.

Case $\mathbb{K} = \mathbb{R}$. It is obvious that $|\alpha|_J : J \rightarrow [0, \infty)$ is summable, for any subset $J \subset I$. In particular, using the notations from the proof of (i) $\Rightarrow$ (ii), it follows that $\alpha|_{I^+} = |\alpha|_{I^+}$, $\alpha|_{I^-} = -|\alpha|_{I^-}$, and $\alpha|_{I^0} = 0$ are all summable. Then the summability of α follows from Exercise ??.

Case $\mathbb{K} = \mathbb{C}$. Using the inequality

$$\max \{ |\operatorname{Re} z|, |\operatorname{Im} z| \} \leq |z|, \quad \forall z \in \mathbb{C},$$

combined with Exercise ??, it follows that both maps $|\operatorname{Re} z|, |\operatorname{Im} z| : I \rightarrow [0, \infty)$ are summable. Using the real case it then follows that both maps $\operatorname{Re} \alpha, \operatorname{Im} \alpha : I \rightarrow \mathbb{R}$ are summable. Then the summability of α follows from Lemma ??. $\square$

The following result shows that summability is essentially the same as the summability of series.

PROPOSITION 2.2. *Suppose $\alpha : I \rightarrow \mathbb{K}$ is summable. Then the support set*

$$[[\alpha]] = \{j \in I : \alpha(j) \neq 0\}$$

is at most countable.

PROOF. For every integer $n \geq 1$, we define the set $J_n = \{j \in I : |\alpha(j)| \geq \frac{1}{n}\}$. Since $|\alpha|$ is summable, the sets $J_n, n \geq 1$ are all finite. The desired result then follows from the obvious equality $[[\alpha]] = \bigcup_{n=1}^{\infty} J_n$. $\square$

We are now ready to discuss our next class of examples.

EXAMPLE 2.4. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, let I be a non-empty set, and let $p \in [1, \infty)$ be a real number. We define

$$\ell_{\mathbb{K}}^p(I) = \{\alpha : I \rightarrow \mathbb{K} : |\alpha|^p : I \rightarrow [0, \infty) \text{ summable}\}.$$

For $\alpha \in \ell_{\mathbb{K}}^p(I)$ we define

$$\|\alpha\|_p = \left[\sum_{j \in I} |\alpha(j)|^p \right]^{\frac{1}{p}}.$$

When $\mathbb{K} = \mathbb{C}$, the space $\ell_{\mathbb{C}}^{\infty}(I)$ is simply denoted by $\ell^{\infty}(I)$. When $I = \mathbb{N}$ - the set of natural numbers - instead of $\ell_{\mathbb{R}}^{\infty}(\mathbb{N})$ we simply write $\ell_{\mathbb{R}}^{\infty}$, and instead of $\ell^{\infty}(\mathbb{N})$ we simply write ℓ^{∞} .

In order to show that the ℓ^p spaces ($1 \leq p < \infty$) are normed vector spaces, we will need several preliminary results. The first result we are going to need is the (classical) Hölder inequality.

Exercise 9. Let $q > 1$ and let $u, v \geq 0$. Define the function $f : [0, 1] \rightarrow \mathbb{R}$ by

$$f(t) = ut + v(1 - t^q)^{\frac{1}{q}}, \quad t \in [0, 1].$$

Prove that

$$\max_{t \in [0, 1]} f(t) = (u^p + v^p)^{\frac{1}{p}},$$

where $p = \frac{q}{q-1}$. Prove that, unless $u = v = 0$, there exists a unique $s \in [0, 1]$ such that

$$f(s) = \max_{t \in [0, 1]} f(t).$$

HINT: Analyze the derivative: $f'(t) = u - v \left(\frac{t^q}{1 - t^q} \right)^{\frac{1}{p}}, \quad t \in (0, 1)$.

LEMMA 2.3 (Hölder's inequality). Let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be non-negative numbers. Let $p, q > 1$ be real number with the property $\frac{1}{p} + \frac{1}{q} = 1$. Then:

$$(3) \quad \sum_{j=1}^n a_j b_j \leq \left(\sum_{j=1}^n a_j^p \right)^{\frac{1}{p}} \cdot \left(\sum_{j=1}^n b_j^q \right)^{\frac{1}{q}}.$$

Moreover, one has equality only when the sequences $(a_1^p, \dots, a_n^p)$ and $(b_1^q, \dots, b_n^q)$ are proportional.

PROOF. The proof will be carried on by induction on n . The case $n = 1$ is trivial.

Case $n = 2$.

Assume $(b_1, b_2) \neq (0, 0)$. (Otherwise everything is trivial). Define the number

$$r = \frac{b_1}{(b_1^q + b_2^q)^{1/q}}.$$

Notice that $r \in [0, 1]$, and we have

$$\frac{b_2}{(b_1^q + b_2^q)^{1/q}} = (1 - r^q)^{1/q}.$$

Notice also that, upon dividing by $(b_1^q + b_2^q)^{1/q}$, the desired inequality

$$(4) \quad a_1 b_1 + a_2 b_2 \leq (a_1^p + a_2^p)^{\frac{1}{p}} (b_1^q + b_2^q)^{\frac{1}{q}}$$

reads

$$a_1 r + a_2 (1 - r^q)^{1/q} \leq (a_1^p + a_2^p)^{1/p},$$

and it follows immediately from the exercise, applied to the function

$$f(t) = a_1 t + a_2 (1 - t^q)^{1/q}, \quad t \in [0, 1].$$

Let us examine when equality holds. If $a_1 = a_2 = 0$, the equality obviously holds, and in this case (a_1, a_2) is clearly proportional to (b_1, b_2) . Assume $(a_1, a_2) \neq (0, 0)$. Put

$$s = \frac{a_1^{p/q}}{(a_1^p + a_2^p)^{1/q}},$$

and notice that

$$(1 - s^q)^{1/q} = \left(1 - \frac{a_1^p}{a_1^p + a_2^p} \right)^{1/q} = \frac{a_2^{p/q}}{(a_1^p + a_2^p)^{1/q}},$$

so we have

$$f(s) = \frac{a_1^{1+\frac{p}{q}} + a_2^{1+\frac{p}{q}}}{(a_1^p + a_2^p)^{\frac{1}{q}}} = \frac{a_1^p + a_2^p}{(a_1^p + a_2^p)^{\frac{1}{q}}} = (a_1^p + a_2^p)^{1-\frac{1}{q}} = (a_1^p + a_2^p)^{\frac{1}{p}} = \max_{t \in [0,1]} f(t).$$

By the exercise, it follows that we have equality in (4) precisely when $r = s$, i.e.

$$\frac{b_1}{(b_1^q + b_2^q)^{\frac{1}{q}}} = \frac{a_1^{\frac{p}{q}}}{(a_1^p + a_2^p)^{\frac{1}{q}}},$$

or equivalently

$$\frac{b_1^q}{b_1^q + b_2^q} = \frac{a_1^p}{a_1^p + a_2^p}.$$

Obviously this forces

$$\frac{b_2^q}{b_1^q + b_2^q} = \frac{a_2^p}{a_1^p + a_2^p},$$

so indeed (a_1^p, a_2^p) and (b_1^q, b_2^q) are proportional.

Having proven the case $n = 2$, we now proceed with the proof of:

The implication: Case $n = k \Rightarrow$ Case $n = k + 1$.

Start with two sequences $(a_1, a_2, \dots, a_k, a_{k+1})$ and $(b_1, b_2, \dots, b_k, b_{k+1})$. Define the numbers

$$a = \left(\sum_{j=1}^k a_j^p \right)^{\frac{1}{p}} \text{ and } b = \left(\sum_{j=1}^k b_j^q \right)^{\frac{1}{q}}.$$

Using the assumption that the case $n = k$ holds, we have

$$(5) \quad \sum_{j=1}^{k+1} a_j b_j \leq \left(\sum_{j=1}^k a_j^p \right)^{\frac{1}{p}} \cdot \left(\sum_{j=1}^k b_j^q \right)^{\frac{1}{q}} + a_{k+1} b_{k+1} = ab + a_{k+1} b_{k+1}.$$

Using the case $n = 2$ we also have

$$(6) \quad ab + a_{k+1} b_{k+1} \leq (a^p + a_{k+1}^p)^{\frac{1}{p}} \cdot (b^q + b_{k+1}^q)^{\frac{1}{q}} = \left(\sum_{j=1}^{k+1} a_j^p \right)^{\frac{1}{p}} \cdot \left(\sum_{j=1}^{k+1} b_j^q \right)^{\frac{1}{q}},$$

so combining with (5) we see that the desired inequality (3) holds for $n = k + 1$.

Assume now we have equality. Then we must have equality in both (5) and in (6). On the one hand, the equality in (5) forces $(a_1^p, a_2^p, \dots, a_k^p)$ and $(b_1^q, b_2^q, \dots, b_k^q)$ to be proportional (since we assume the case $n = k$). On the other hand, the equality in (6) forces (a^p, a_{k+1}^p) and (b^q, b_{k+1}^q) to be proportional (by the case $n = 2$). Since

$$a^p = \sum_{j=1}^k a_j^p \text{ and } b^q = \sum_{j=1}^k b_j^q,$$

it is clear that $(a_1^p, a_2^p, \dots, a_k^p, a_{k+1}^p)$ and $(b_1^q, b_2^q, \dots, b_k^q, b_{k+1}^q)$ are proportional. $\square$

DEFINITION. Two numbers $p, q \in [1, \infty)$ are said to be *Hölder conjugate*, if $\frac{1}{p} + \frac{1}{q} = 1$. Here we use the convention $\frac{1}{\infty} = 0$.

PROPOSITION 2.3. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, let I be a non-empty set, and let $p, q \in [1, \infty]$ be two Hölder conjugate numbers. If $\alpha \in \ell_{\mathbb{K}}^p(I)$ and $\beta \in \ell_{\mathbb{K}}^q(I)$, then $\alpha\beta \in \ell_{\mathbb{K}}^1(I)$, and

$$\|\alpha\beta\|_1 \leq \|\alpha\|_p \cdot \|\beta\|_q.$$

PROOF. Using Lemma ??, it suffices to prove the inequality

$$(7) \quad \sum_{j \in F} |\alpha(j)\beta(j)| \leq \|\alpha\|_p \cdot \|\beta\|_q,$$

for every finite set $F \subset I$.

Fix for the moment a finite subset $F \subset I$. Assume $p, q \in (1, \infty)$, using Hölder's inequality we have

$$(8) \quad \sum_{j \in F} |\alpha(j)\beta(j)| = \sum_{j \in F} |\alpha(j)| \cdot |\beta(j)| \leq \left[\sum_{j \in F} |\alpha(j)|^p \right]^{\frac{1}{p}} \cdot \left[\sum_{j \in F} |\beta(j)|^q \right]^{\frac{1}{q}}.$$

Notice however that

$$\begin{aligned} \sum_{j \in F} |\alpha(j)|^p &\leq \sum_{j \in I} |\alpha(j)|^p = (\|\alpha\|_p)^p, \\ \sum_{j \in F} |\beta(j)|^q &\leq \sum_{j \in I} |\beta(j)|^q = (\|\beta\|_q)^q, \end{aligned}$$

so we get

$$\left[\sum_{j \in F} |\alpha(j)|^p \right]^{\frac{1}{p}} \leq \|\alpha\|_p \text{ and } \left[\sum_{j \in F} |\beta(j)|^q \right]^{\frac{1}{q}} \leq \|\beta\|_q,$$

so when we go back to (8) we immediately get the desired inequality (7)

In the case when $p = 1$, we immediately have

$$\begin{aligned} \sum_{j \in F} |\alpha(j)\beta(j)| &\leq \left[\sum_{j \in F} |\alpha(j)| \right] \cdot \left[\max_{j \in F} |\beta(j)| \right] \leq \\ &\leq \left[\sum_{j \in I} |\alpha(j)| \right] \cdot \left[\sup_{j \in I} |\beta(j)| \right] = \|\alpha\|_1 \cdot \|\beta\|_\infty. \end{aligned}$$

The case $p = \infty$ is proven in the exact same way. $\square$

REMARK 2.2. Suppose $p, q \in [1, \infty]$ are Hölder conjugate numbers. For any $\alpha \in \ell_{\mathbb{K}}^p(I)$ and $\beta \in \ell_{\mathbb{K}}^q(I)$, the map $\alpha\beta$ is summable (by Proposition ??). In particular, one can define the number

$$\langle \alpha, \beta \rangle = \sum_{j \in I} \alpha(j)\beta(j) \in \mathbb{K}.$$

As a consequence we get the inequality

$$|\langle \alpha, \beta \rangle| \leq \|\alpha\|_p \cdot \|\beta\|_q, \quad \forall \alpha \in \ell_{\mathbb{K}}^p(I), \beta \in \ell_{\mathbb{K}}^q(I).$$

NOTATIONS. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, let I be a non-empty set, and let $q \in [1, \infty]$ be a real number. We define

$$\mathcal{B}_{\mathbb{K}}^q(I) = \{ \alpha \in \text{fin}\mathbb{K}(I) : \|\alpha\|_q \leq 1 \}.$$

(remark that $\text{fin}_{\mathbb{K}}(I) \subset \ell_{\mathbb{K}}^q(I)$, for all $q \in [1, \infty]$.)

THEOREM 2.1 (Dual definition of ℓ^p spaces). *Let $p, q \in (1, \infty)$ be Hölder conjugate numbers, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let I be a non-empty set. For a function $\alpha : I \rightarrow \mathbb{K}$, the following are equivalent:*

- (i) $\alpha \in \ell_{\mathbb{K}}^p(I)$;
- (ii) $\sup_{\beta \in \mathcal{B}_{\mathbb{K}}^q(I)} |\langle \alpha, \beta \rangle| < \infty$.

Moreover, one has the equality

$$(9) \quad \sup_{\beta \in \mathcal{B}_{\mathbb{K}}^q(I)} |\langle \alpha, \beta \rangle| = \|\alpha\|_p, \quad \forall \alpha \in \ell_{\mathbb{K}}^p(I).$$

PROOF. It will be convenient to introduce several notations. Given a function $\alpha : I \rightarrow \mathbb{K}$, and a finite set $F \subset I$, we define the function $\beta_{\alpha}^F : I \rightarrow \mathbb{K}$, as follows:

$$\beta_{\alpha}^F(i) = \begin{cases} \frac{|\alpha(i)|^{1+\frac{p}{q}}}{\alpha(i) \cdot (\sum_{j \in F} |\alpha(j)|^p)^{1/q}} & \text{if } i \in F \text{ and } \alpha(i) \neq 0 \\ 0 & \text{if } i \notin F \text{ or } \alpha(i) = 0 \end{cases}$$

Notice that $[[\beta_{\alpha}^F]] \subset F$, and unless β_{α}^F is identically zero, we have

$$\sum_{i \in [[\beta_{\alpha}^F]]} |\beta_{\alpha}^F(i)|^q = 1.$$

So in any case we have $\beta_{\alpha}^F \in \mathcal{B}_{\mathbb{K}}^q(I)$. Notice also that, unless β_{α}^F is identically zero, we have

$$(10) \quad \begin{aligned} \langle \alpha, \beta_{\alpha}^F \rangle &= \sum_{i \in F} \alpha(i) \beta_{\alpha}^F(i) = \frac{\sum_{i \in F} |\alpha(i)|^{1+\frac{p}{q}}}{(\sum_{j \in F} |\alpha(j)|^p)^{1/q}} = \frac{\sum_{i \in F} |\alpha(i)|^p}{(\sum_{j \in F} |\alpha(j)|^p)^{1/q}} = \\ &= (\sum_{i \in F} |\alpha(i)|^p)^{1-\frac{1}{q}} = (\sum_{i \in F} |\alpha(i)|^p)^{1/p}. \end{aligned}$$

It is clear that the equality (10) actually holds even when β_{α}^F is identically zero.

To make the exposition a bit clearer, we denote the quantity $\sup_{\beta \in \mathcal{B}_{\mathbb{K}}^q(I)} |\langle \alpha, \beta \rangle|$

simply by $|||\alpha|||$.

We now proceed with the proof of the Theorem.

(i) $\Rightarrow$ (ii). Assume $\alpha \in \ell_{\mathbb{K}}^p(I)$. In order to prove (ii) it suffices to prove the inequality

$$(11) \quad |||\alpha||| \leq \|\alpha\|_p.$$

Start with some arbitrary $\beta \in \mathcal{B}_{\mathbb{K}}^q(I)$. Using Hölder inequality we have

$$\begin{aligned} |\langle \alpha, \beta \rangle| &= \left| \sum_{j \in [[\beta]]} \alpha(j) \beta(j) \right| \leq \sum_{j \in [[\beta]]} |\alpha(j)| \cdot |\beta(j)| \leq \\ &\leq \left(\sum_{j \in [[\beta]]} |\alpha(j)|^p \right)^{1/p} \cdot \left(\sum_{j \in [[\beta]]} |\beta(j)|^q \right)^{1/q} \leq \left(\sup_{\substack{F \subset I \\ \text{finite}}} \left[\sum_{i \in F} |\alpha(i)|^p \right] \right)^{1/p} = \|\alpha\|_p. \end{aligned}$$

Since this inequality holds for all $\beta \in \mathcal{B}_{\mathbb{K}}^q(I)$, the inequality (11) follows.

(ii) $\Rightarrow$ (i). Assume now $|||\alpha||| < \infty$. In order to prove condition (i) it suffices to prove that

$$(12) \quad \sum_{i \in F} |\alpha(i)|^p \leq |||\alpha|||^p, \text{ for every finite subset } F \subset I.$$

By (10) we know that for every finite subset $F \subset I$ we have

$$(13) \quad \sum_{i \in F} |\alpha(i)|^p \leq |||\alpha|||^p = \langle \alpha, \beta_{\alpha}^F \rangle^p.$$

In particular we get the fact that $\langle \alpha, \beta_\alpha^F \rangle = |\langle \alpha, \beta_\alpha^F \rangle|$, and the fact that β_α^F belongs to $\mathcal{B}_\mathbb{K}^q(I)$, combined with (13) will give

$$\sum_{i \in F} |\alpha(i)|^p = |\langle \alpha, \beta_\alpha^F \rangle|^p \leq \left(\sup_{\beta \in \mathcal{B}_\mathbb{K}^q(I)} |\langle \alpha, \beta \rangle| \right)^p = \|\alpha\|^p.$$

Having proven the equivalence (i) $\Leftrightarrow$ (ii), let us now observe that (9) is an immediate consequence of (11) and (12). $\square$

Exercise 10. Prove that Theorem 9.1 holds also in the cases $(p, q) = (1, \infty)$ and $(p, q) = (\infty, 1)$.

COROLLARY 2.1. *Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, let I be a non-empty set, and let $p \geq 1$.*

- (i) *When equipped with point-wise addition and scalar multiplication, the set $\ell_\mathbb{K}^p(I)$ is a $\mathbb{K}$ -vector space.*
- (ii) *The map*

$$\ell_\mathbb{K}^p(I) \ni \alpha \longmapsto \|\alpha\|_p \in [0, \infty)$$

is a norm.

PROOF. Let q be the Hölder conjugate of p . If $\alpha \in \ell_\mathbb{K}^p(I)$, and $\lambda \in \mathbb{K}$, then

$$\langle \lambda \alpha, \beta \rangle = \lambda \langle \alpha, \beta \rangle, \quad \forall \beta \in \text{fin}_\mathbb{K}(I),$$

so we get

$$\sup_{\beta \in \mathcal{B}_\mathbb{K}^q(I)} |\langle \lambda \alpha, \beta \rangle| = |\lambda| \cdot \sup_{\beta \in \mathcal{B}_\mathbb{K}^q(I)} |\langle \alpha, \beta \rangle|,$$

which gives the fact that $\lambda \alpha \in \ell_\mathbb{K}^p(I)$, as well as the equality $\|\lambda \alpha\|_p = |\lambda| \cdot \|\alpha\|_p$.

If $\alpha_1, \alpha_2 \in \ell_\mathbb{K}^p(I)$, then

$$\langle \alpha_1 + \alpha_2, \beta \rangle = \langle \alpha_1, \beta \rangle + \langle \alpha_2, \beta \rangle, \quad \forall \beta \in \text{fin}_\mathbb{K}(I),$$

so we get

$$\begin{aligned} \sup_{\beta \in \mathcal{B}_\mathbb{K}^q(I)} |\langle \alpha_1 + \alpha_2, \beta \rangle| &= \sup_{\beta \in \mathcal{B}_\mathbb{K}^q(I)} |\langle \alpha_1, \beta \rangle + \langle \alpha_2, \beta \rangle| \leq \\ &\leq \sup_{\beta \in \mathcal{B}_\mathbb{K}^q(I)} (|\langle \alpha_1, \beta \rangle| + |\langle \alpha_2, \beta \rangle|) \leq \sup_{\beta \in \mathcal{B}_\mathbb{K}^q(I)} |\langle \alpha_1, \beta \rangle| + \sup_{\beta \in \mathcal{B}_\mathbb{K}^q(I)} |\langle \alpha_2, \beta \rangle|, \end{aligned}$$

which gives the fact that $\alpha_1 + \alpha_2 \in \ell_\mathbb{K}^p(I)$, as well as the inequality

$$\|\alpha_1 + \alpha_2\|_p \leq \|\alpha_1\|_p + \|\alpha_2\|_p.$$

The implication $\|\alpha\|_p = 0 \Rightarrow \alpha = 0$ is obvious. $\square$

Exercise 11. Let $p \geq 1$ be a real number, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let I be a non-empty set. Prove that $\text{fin}_\mathbb{K}(I)$ is a *dense* linear subspace in $\ell_\mathbb{K}^p(I)$.

REMARK 2.3. Let $p, q \in [1, \infty]$ be Hölder conjugate. Then the map

$$\ell_\mathbb{K}^p(I) \times \ell_\mathbb{K}^q(I) \ni (\alpha, \beta) \longmapsto \langle \alpha, \beta \rangle \in \mathbb{K}$$

is *bilinear*, in the sense that for any $\gamma \in \ell_\mathbb{K}^p(I)$ and any $\eta \in \ell_\mathbb{K}^q(I)$, the maps

$$\ell_\mathbb{K}^p(I) \ni \alpha \longmapsto \langle \alpha, \eta \rangle \in \mathbb{K},$$

$$\ell_\mathbb{K}^q(I) \ni \beta \longmapsto \langle \gamma, \beta \rangle \in \mathbb{K}$$

are linear. These facts follow immediately from Exercise ??

We now examine linear continuous maps between normed spaces.

PROPOSITION 2.4. *Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, let $\mathcal{X}$ and $\mathcal{Y}$ be normed $\mathbb{K}$ -vector spaces, and let $T : \mathcal{X} \rightarrow \mathcal{Y}$ be a $\mathbb{K}$ -linear map. The following are equivalent:*

- (i) *T is continuous;*
- (ii) $\sup \{ \|Tx\| : x \in \mathcal{X}, \|x\| \leq 1 \} < \infty$;
- (iii) $\sup \{ \|Tx\| : x \in \mathcal{X}, \|x\| = 1 \} < \infty$;
- (iv) *T is continuous at 0.*

PROOF. (i) $\Rightarrow$ (ii). Assume T is continuous, but

$$\sup \{ \|Tx\| : x \in \mathcal{X}, \|x\| \leq 1 \} < \infty,$$

which means there exists some sequence $(x_n)_{n \geq 1} \subset \mathcal{X}$ such that

- (a) $\|x_n\| \leq 1, \forall n \geq 1$;
- (b) $\lim_{n \rightarrow \infty} \|Tx_n\| = \infty$.

Put

$$z_n = \|Tx_n\|^{-1} x_n, \quad \forall n \geq 1.$$

On the one hand, we have

$$\|z_n\| = \frac{\|x_n\|}{\|Tx_n\|} \leq \frac{1}{\|Tx_n\|}, \quad \forall n \geq 1,$$

which gives $\lim_{n \rightarrow \infty} \|z_n\| = 0$, i.e. $\lim_{n \rightarrow \infty} z_n = 0$. Since T is assumed to be continuous, we will get

$$(14) \quad \lim_{n \rightarrow \infty} Tz_n = T0 = 0.$$

On the other hand, since T is linear, we have $Tz_n = \|Tx_n\|^{-1} Tx_n$, so in particular we get

$$\|Tz_n\| = 1, \quad \forall n \geq 1,$$

which clearly contradicts (14).

(ii) $\Rightarrow$ (iii). This is obvious, since the supremum in (iii) is taken over a subset of the set used in (ii).

(iii) $\Rightarrow$ (iv). Let $(x_n)_{n \geq 1} \subset \mathcal{X}$ be a sequence with $\lim_{n \rightarrow \infty} x_n = 0$. For each $n \geq 1$, define

$$u_n = \begin{cases} \|x_n\|^{-1} x_n, & \text{if } x_n \neq 0 \\ \text{any vector of norm 1,} & \text{if } x_n = 0 \end{cases}$$

so that we have

$$\|u_n\| = 1 \text{ and } x_n = \|x_n\| u_n, \quad \forall n \geq 1.$$

Since T is linear, we have

$$(15) \quad Tx_n = \|x_n\| Tu_n, \quad \forall n \geq 1.$$

If we define $M = \sup \{ \|Tx\| : x \in \mathcal{X}, \|x\| = 1 \}$, then $\|Tu_n\| \leq M, \forall n \geq 1$, so (15) will give

$$\|Tx_n\| \leq M \cdot \|x_n\|, \quad \forall n \geq 1,$$

and the condition $\lim_{n \rightarrow \infty} x_n = 0$ will force $\lim_{n \rightarrow \infty} Tx_n = 0$.

(iv) $\Rightarrow$ (i). Assume T is continuous at 0, and let us prove that T is continuous at any point. Start with some arbitrary $x \in \mathcal{X}$ and an arbitrary sequence $(x_n)_{n \geq 1} \subset \mathcal{X}$ with $\lim_{n \rightarrow \infty} x_n = x$. Put $z_n = x_n - x$, so that $\lim_{n \rightarrow \infty} z_n = 0$. Then we will have $\lim_{n \rightarrow \infty} Tz_n = 0$, which (use the linearity of T) means that

$$0 = \lim_{n \rightarrow \infty} \|Tz_n\| = \lim_{n \rightarrow \infty} \|Tx_n - Tx\|,$$

thus proving that $\lim_{n \rightarrow \infty} Tx_n = Tx$. $\square$

REMARK 2.4. Using the notations above, the quantities in (ii) and (iii) are in fact equal. Indeed, if we define

$$M_1 = \sup \{ \|Tx\| : x \in \mathcal{X}, \|x\| \leq 1 \},$$

$$M_2 = \sup \{ \|Tx\| : x \in \mathcal{X}, \|x\| = 1 \},$$

then as observed during the proof, we have $M_2 \leq M_1$. Conversely, if we start with some arbitrary $x \in \mathcal{X}$ with $\|x\| \leq 1$, then we can always write $x = \|x\|u$, for some $u \in \mathcal{X}$ with $\|u\| = 1$. In particular we will get

$$\|Tx\| = \|x\| \cdot \|Tu\| \leq \|x\| \cdot M_2 \leq M_1.$$

Taking supremum in the above inequality, over all $x \in \mathcal{X}$ with $\|x\| \leq 1$, will then give the inequality $M_1 \leq M_2$.

NOTATIONS. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, and let $\mathcal{X}$ and $\mathcal{Y}$ be normed $\mathbb{K}$ -vector spaces. We define

$$\mathcal{L}(\mathcal{X}, \mathcal{Y}) = \{ T : \mathcal{X} \rightarrow \mathcal{Y} : T \text{ } \mathbb{K}\text{-linear and continuous} \}.$$

For $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ we define (see the above remark)

$$\|T\| = \sup \{ \|Tx\| : x \in \mathcal{X}, \|x\| \leq 1 \} = \sup \{ \|Tx\| : x \in \mathcal{X}, \|x\| = 1 \}$$

When $\mathcal{Y} = \mathbb{K}$ (equipped with the absolute value as the norm), the space $\mathcal{L}(\mathcal{X}, \mathbb{K})$ will be denoted simply by $\mathcal{X}^*$, and will be called the *topological dual* of $\mathcal{X}$.

PROPOSITION 2.5. *Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, and let $\mathcal{X}$ and $\mathcal{Y}$ be normed $\mathbb{K}$ -vector spaces.*

- (i) *The space $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a $\mathbb{K}$ -vector space.*
- (ii) *For $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ we have*

$$(16) \quad \|T\| = \min \{ C \geq 0 : \|Tx\| \leq C\|x\|, \forall x \in \mathcal{X} \}.$$

In particular one has

$$(17) \quad \|Tx\| \leq \|T\| \cdot \|x\|, \forall x \in \mathcal{X}.$$

- (iii) *The map $\mathcal{L}(\mathcal{X}, \mathcal{Y}) \ni T \mapsto \|T\| \in [0, \infty)$ is a norm.*

PROOF. The fact that $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a vector space is clear.

(ii). Assume $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. We begin by proving (17). Start with some arbitrary $x \in \mathcal{X}$, and write it as $x = \|x\|u$, for some $u \in \mathcal{X}$ with $\|u\| = 1$. Then by definition we have $\|Tu\| \leq \|T\|$, and by linearity we have

$$\|Tx\| = \|x\| \cdot \|Tu\| \leq \|x\| \cdot \|T\|.$$

To prove the equality (16) let us define the set

$$\mathcal{C}_T = \{ C \geq 0 : \|Tx\| \leq C\|x\|, \forall x \in \mathcal{X} \}.$$

On the one hand, by (17) we know that $\|T\| \in \mathcal{C}_T$. On the other hand, if we take an arbitrary $C \in \mathcal{C}_T$, then for every $u \in \mathcal{X}$ with $\|u\| = 1$, we will have

$$\|Tu\| \leq C\|u\| = C,$$

so taking supremum, over all u with $\|u\| = 1$, will immediately give $\|T\| \leq C$. Since we now have

$$\|T\| \leq C, \forall C \in \mathcal{C}_T,$$

we clearly get $\|T\| = \min \mathcal{C}_T$.

(iii). Let $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Using (17), we have

$$\|(T + S)x\| = \|Tx + Sx\| \leq \|Tx\| + \|Sx\| \leq (\|T\| + \|S\|) \cdot \|x\|, \quad \forall x \in \mathcal{X}.$$

Then using (16) we get

$$\|T + S\| \leq \|T\| + \|S\|.$$

If $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $\lambda \in \mathbb{K}$, then the equality

$$\|(\lambda T)x\| = |\lambda| \cdot \|Tx\|, \quad x \in \mathcal{X}$$

will immediately give $\|\lambda T\| = |\lambda| \cdot \|T\|$.

Finally if $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has $\|T\| = 0$, then using (17) one immediately gets $T = 0$. $\square$

NOTATION. Let I be a non-empty set, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $p \in [1, \infty]$. Let q be the Hölder conjugate of p . For every element $\alpha \in \ell_{\mathbb{K}}^p(I)$ we define the map $\theta_{\alpha} : \ell_{\mathbb{K}}^q(I) \rightarrow \mathbb{K}$ by

$$\theta_{\alpha}(\beta) = \langle \alpha, \beta \rangle = \sum_{i \in I} \alpha(i)\beta(i), \quad \beta \in \ell_{\mathbb{K}}^q(I).$$

We know that θ_{α} is linear, and by Remark 9.2, we have

$$|\theta_{\alpha}(\beta)| \leq \|\alpha\|_p \cdot \|\beta\|_q, \quad \forall \beta \in \ell_{\mathbb{K}}^q(I),$$

so θ_{α} is continuous, and we have the inequality

$$(18) \quad \|\theta_{\alpha}\| \leq \|\alpha\|_p.$$

PROPOSITION 2.6. *Using the above notations, but assuming $p \in (1, \infty]$, the map*

$$\Theta : \ell_{\mathbb{K}}^p(I) \ni \alpha \longmapsto \theta_{\alpha} \in (\ell_{\mathbb{K}}^q(I))^*$$

is a linear isomorphism of $\mathbb{K}$ -vector spaces. Moreover, Θ is isometric, in the sense that

$$(19) \quad \|\Theta\alpha\| = \|\alpha\|_p, \quad \forall \alpha \in \ell_{\mathbb{K}}^p(I).$$

PROOF. We begin by proving (19). Since we have the inclusion

$$\{\beta \in \ell_{\mathbb{K}}^q(I) : \|\beta\|_q \leq 1\} \supset \mathcal{B}_{\mathbb{K}}^q(I),$$

it follows that

$$(20) \quad \|\theta_{\alpha}\| = \sup \{|\theta_{\alpha}(\beta)| : \beta \in \ell_{\mathbb{K}}^q(I), \|\beta\|_q \leq 1\} \geq \sup \{|\theta_{\alpha}(\beta)| : \beta \in \mathcal{B}_{\mathbb{K}}^q(I)\}.$$

We know however (see Theorem 9 and Exercise 7) that

$$\sup \{|\theta_{\alpha}(\beta)| : \beta \in \mathcal{B}_{\mathbb{K}}^q(I)\} = \|\alpha\|_p,$$

so using (20) we get

$$\|\theta_{\alpha}\| \geq \|\alpha\|_p.$$

Combining this with (18) yields the desired equality.

The fact that Θ is linear is pretty obvious. Notice now that since Θ is isometric, it is clear that Θ is injective, so the only thing we need to prove is the fact that Θ is surjective. Start with an arbitrary linear continuous map $\phi : \ell_{\mathbb{K}}^q(I) \rightarrow \mathbb{K}$. For every $i \in I$ we define the function $\delta^i : I \rightarrow \mathbb{K}$ by

$$\delta^i(j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

It is clear that $\delta^i \in \ell_{\mathbb{K}}^q(I)$, for all $i \in I$. (In fact $\delta^i \in \text{fin}_{\mathbb{K}}(I)$.) We define $\alpha : I \rightarrow \mathbb{K}$ by

$$\alpha(i) = \phi(\delta^i), \quad \forall i \in I.$$

Notice that, for every $\beta \in \text{fin}_{\mathbb{K}}$, we have

$$(21) \quad \sum_{i \in I} \alpha(i)\beta(i) = \sum_{i \in I} \beta(i)\phi(\delta^i) = \phi\left(\sum_{i \in [\beta]} \beta(i)\delta^i\right) = \phi(\beta),$$

where $[\beta] = \{i \in I : \beta(i) \neq 0\}$. (Since $\beta \in \text{fin}_{\mathbb{K}}(I)$, the set $[\beta]$ is finite.) Using Hölder's inequality, the above computation shows that

$$|\langle \alpha, \beta \rangle| \leq \|\phi\| \cdot \|\beta\|_q, \quad \forall \beta \in \text{fin}_{\mathbb{K}}(I).$$

By Theorem 9.1 and Exercise 7, this proves that $\alpha \in \ell_{\mathbb{K}}^p(I)$. Going back to (21) we now have

$$\theta_{\alpha}(\beta) = \phi(\beta), \quad \forall \beta \in \text{fin}_{\mathbb{K}}(I).$$

Since both θ_{α} and ϕ are continuous, and $\text{fin}_{\mathbb{K}}(I)$ is dense in $\ell_{\mathbb{K}}^q(I)$ (by Exercise 10), it follows that $\phi = \theta_{\alpha}$. $\square$

REMARK 2.5. In the case $p = 1$, the map

$$\Theta : \ell_{\mathbb{K}}^1(I) \ni \alpha \longmapsto \theta_{\alpha} \in (\ell_{\mathbb{K}}^{\infty}(I))^*$$

is still isometric, but it is no longer surjective, unless I is finite. The explanation is the fact that when I is infinite, the subspace $\text{fin}_{\mathbb{K}}(I)$ is *not* dense in $\ell_{\mathbb{K}}^{\infty}(I)$. For example, if we take $1 \in \ell_{\mathbb{K}}^{\infty}(I)$ to be the constant function 1, then it is pretty obvious that

$$\|1 - \beta\| \geq 1, \quad \forall \beta \in \text{fin}_{\mathbb{K}}(I).$$

The above equality can be immediately extended to

$$(22) \quad \|\lambda 1 + \beta\| \geq |\lambda|, \quad \forall \lambda \in \mathbb{K}, \beta \in \text{fin}_{\mathbb{K}}(I).$$

If we then consider the subspace

$$\widetilde{\text{fin}}_{\mathbb{K}}(I) = \{\lambda 1 + \beta : \beta \in \text{fin}_{\mathbb{K}}(I), \lambda \in \mathbb{K}\},$$

we see that the map

$$\phi_0 : \widetilde{\text{fin}}_{\mathbb{K}}(I) \ni \lambda 1 + \beta \longmapsto \lambda \in \mathbb{K}$$

is linear, continuous, and has the property that

$$(23) \quad \phi_0|_{\text{fin}_{\mathbb{K}}(I)} = 0, \quad \phi_0(1) = 1,$$

$$(24) \quad |\phi_0(\gamma)| \leq \|\gamma\|, \quad \forall \gamma \in \widetilde{\text{fin}}_{\mathbb{K}}(I).$$

Using the Hahn-Banach Theorem, we can then extend ϕ_0 to a linear map $\phi : \ell_{\mathbb{K}}^{\infty}(I) \rightarrow \mathbb{K}$ which will still satisfy (23) and (24), in particular we have $\phi \in (\ell_{\mathbb{K}}^{\infty}(I))^*$. Notice however that if we had $\phi = \theta_{\alpha}$, for some $\alpha \in \ell_{\mathbb{K}}^1(I)$, then we must have $\alpha(i) = \phi(\delta^i) = 0$, for all $i \in I$, so this would force $\phi = 0$, which is impossible, since $\phi(1) = 1$.

Exercise 12. Use the notations above. For every $\alpha \in \ell_{\mathbb{K}}^1(I)$, define

$$\sigma_{\alpha} = \theta_{\alpha}|_{c_0^{\mathbb{K}}(I)} : c_0^{\mathbb{K}}(I) \rightarrow \mathbb{K}.$$

Prove that σ_{α} is linear and continuous. Prove that the map

$$\Sigma : \ell_{\mathbb{K}}^1(I) \ni \alpha \longmapsto \sigma_{\alpha} \in (c_0^{\mathbb{K}}(I))^*$$

is an isometric linear isomorphism of $\mathbb{K}$ -vector spaces.

LECTURE 12

3. Banach spaces

DEFINITION. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. A *Banach space over $\mathbb{K}$* is a normed $\mathbb{K}$ -vector space $(\mathcal{X}, \|\cdot\|)$, which is complete with respect to the metric

$$d(x, y) = \|x - y\|, \quad x, y \in \mathcal{X}.$$

EXAMPLE 3.1. The field $\mathbb{K}$, equipped with the absolute value norm, is a Banach space. More generally, the vector space $\mathbb{K}^n$, equipped with any of the norms

$$\begin{aligned} \|(\lambda_1, \dots, \lambda_n)\|_\infty &= \max\{|\lambda_1|, \dots, |\lambda_n|\}, \\ \|(\lambda_1, \dots, \lambda_n)\|_p &= [|\lambda_1|^p + \dots + |\lambda_n|^p]^{1/p}, \quad p \geq 1, \end{aligned}$$

is a Banach space.

REMARK 3.1. Using the facts from the general theory of metric spaces, we know that for a normed vector space $(\mathcal{X}, \|\cdot\|)$, the following are equivalent:

- (i) $\mathcal{X}$ is a Banach space;
- (ii) given any sequence $(x_n)_{n \geq 1} \subset \mathcal{X}$ with $\sum_{n=1}^{\infty} \|x_n\| < \infty$, the sequence $(y_n)_{n \geq 1}$ of partial sums, defined by $y_n = \sum_{k=1}^n x_k$, is convergent;
- (iii) every Cauchy sequence in $\mathcal{X}$ has a convergent subsequence.

This is pretty obvious, since the sequence of partial sums has the property that

$$d(y_{n+1}, y_n) = \|y_{n+1} - y_n\| = \|x_{n+1}\|, \quad \forall n \geq 1.$$

*Exercise 1**. Let $\mathcal{X}$ be a *finite dimensional* normed vector space. Prove that $\mathcal{X}$ is a Banach space.

HINTS: Use induction on $\dim \mathcal{X}$. The case $\dim \mathcal{X} = 1$ is trivial. Assume the statement is true for all normed vector spaces of dimension d , and let us prove it for a normed vector space of dimension $d+1$. Fix such an $\mathcal{X}$, and a linear basis $\{e_1, e_2, \dots, e_n, e_{d+1}\}$ for $\mathcal{X}$. Start with a Cauchy sequence $(x_n)_{n \geq 1} \subset \mathcal{X}$. Write each term as

$$x_n = \sum_{k=1}^{d+1} \alpha_n(k) e_k.$$

Prove first that $(\alpha_n(d+1))_{n \geq 1} \subset \mathbb{K}$ is *bounded*. Then extract a subsequence $(x_{n_p})_{p \geq 1}$ such that $(\alpha_{n_p}(d+1))_{p \geq 1}$ is convergent. If we take $\alpha(d+1) = \lim_{p \rightarrow \infty} \alpha_{n_p}(d+1)$, then prove that the sequence $(x_{n_p} - \alpha_{n_p}(d+1)e_{d+1})_{p \geq 1}$ is Cauchy in the space $\text{Span}\{e_1, \dots, e_d\}$. Using the inductive hypothesis, conclude that $(x_{n_p})_{p \geq 1}$ is convergent in $\mathcal{X}$. Thus, *every Cauchy sequence in $\mathcal{X}$ has a convergent subsequence*, hence $\mathcal{X}$ is Banach.

*Exercise 2**. Let $n \geq 1$ be an integer, and let $\|\cdot\|$ be a norm on $\mathbb{K}^n$. Prove that there exist constants $C, D > 0$, such that

$$C\|x\|_\infty \leq \|x\| \leq D\|x\|_\infty, \quad \forall x \in \mathbb{K}^n.$$

HINT: Let $e_1, \dots, e_n$ be the standard basis vectors for $\mathbb{K}^n$, so that

$$\alpha_1 e_1 + \dots + \alpha_n e_n = (\alpha_1, \dots, \alpha_n), \quad \forall (\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n.$$

Define $D = \|e_1\| + \dots + \|e_n\|$. The existence of C is equivalent to the existence of some $C' > 0$ such that

$$\|x\|_\infty \leq C' \|x\|, \quad \forall x \in \mathbb{K}^n.$$

(If such a C' exists, then we take $C = 1/C'$.) To prove the existence of C' as above, we consider the set $T = \{x \in \mathbb{K}^n : \|x\| \leq 1\}$, and we need to prove that

$$\sup_{x \in T} \|x\|_\infty < \infty.$$

Argue by contradiction (see also the hint from the preceding exercise).

Exercise 3. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed vector spaces. Consider the product $\mathcal{X} \times \mathcal{Y}$, equipped with the natural vector space structure.

- (i) Prove that $\|(x, y)\| = \|x\| + \|y\|$, $(x, y) \in \mathcal{X} \times \mathcal{Y}$ defines a norm on $\mathcal{X} \times \mathcal{Y}$.
- (ii) Prove that, when equipped with the above norm, $\mathcal{X} \times \mathcal{Y}$ is a Banach space, if and only if both $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces.

There are two key constructions which enable one to construct new Banach space out of old ones.

PROPOSITION 3.1. *Let $\mathcal{X}$ be a normed vector space, and let $\mathcal{Y}$ be a Banach space. Then $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a Banach space, when equipped with the operator norm.*

PROOF. Start with a Cauchy sequence $(T_n)_{n \geq 1} \subset \mathcal{L}(\mathcal{X}, \mathcal{Y})$. This means that for every $\varepsilon > 0$, there exists some N_ε such that

$$(1) \quad \|T_m - T_n\| < \varepsilon, \quad \forall m, n \geq N_\varepsilon.$$

Notice that, if one takes for example $\varepsilon = 1$, and we define

$$C = 1 + \max\{\|T_1\|, \|T_2\|, \dots, \|T_{N_1}\|\},$$

then we clearly have

$$(2) \quad \|T_n\| \leq C, \quad \forall n \geq 1.$$

Notice that, using (1), we have

$$(3) \quad \|T_m x - T_n x\| \leq \varepsilon \|x\|, \quad \forall m, n \geq N_\varepsilon, \quad x \in \mathcal{X},$$

which proves that

- for every $x \in \mathcal{X}$, the sequence $(T_n x)_{n \geq 1} \subset \mathcal{Y}$ is Cauchy.

Since $\mathcal{Y}$ is a Banach space, for each $x \in \mathcal{X}$, the sequence $(T_n)_{n \geq 1}$ will be convergent. We define the map $T : \mathcal{X} \rightarrow \mathcal{Y}$ by

$$Tx = \lim_{n \rightarrow \infty} T_n x, \quad x \in \mathcal{X}.$$

Using (2) we immediately get

$$\|Tx\| \leq C\|x\|, \quad \forall x \in \mathcal{X}.$$

Since T is obviously linear, this prove that T is continuous. Finally, if we fix $n \geq N_\varepsilon$ and we take $\lim_{m \rightarrow \infty}$ in (3), we get

$$\|T_n x - Tx\| \leq \varepsilon \|x\|, \quad \forall n \geq N_\varepsilon, \quad x \in \mathcal{X},$$

which proves precisely that we have the inequality

$$\|T_n - T\| \leq \varepsilon, \quad \forall n \geq N_\varepsilon,$$

hence $(T_n)_{n \geq 1}$ is convergent to T in the norm topology. $\square$

COROLLARY 3.1. *If $\mathcal{X}$ is a normed vector space, then its topological dual $\mathcal{X}^* = \mathcal{L}(\mathcal{X}, \mathbb{K})$ is a Banach space.*

PROOF. Immediate from the fact that $\mathbb{K}$ is a Banach space. $\square$

As a direct application of the above result we get

COROLLARY 3.2. *If I is a non-empty set, if $p \in [1, \infty]$, then $\ell_{\mathbb{K}}^p(I)$ is a Banach space.*

PROOF. For $p = 1$ we know that $\ell^1 \simeq (c_0)^*$. For $p \in (1, \infty]$, we know that $\ell^p \simeq (\ell^q)^*$, where q is Hölder conjugate to p . $\square$

PROPOSITION 3.2. *Let $\mathcal{X}$ be a Banach space, and let $\mathcal{Z} \subset \mathcal{X}$ be a linear subspace. The following are equivalent:*

- (i) $\mathcal{Z}$ is a Banach space, when equipped with the norm from $\mathcal{X}$;
- (ii) $\mathcal{Z}$ is closed in $\mathcal{X}$, in the norm topology.

PROOF. This is a particular case of a general result from the theory of complete metric spaces. $\square$

COROLLARY 3.3. *Let I be a non-empty set, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. Then $c_0^{\mathbb{K}}(I)$ is a Banach space.*

PROOF. Use the fact that $c_0^{\mathbb{K}}(I)$ is closed in $\ell_{\mathbb{K}}^{\infty}(I)$. $\square$

Exercise 4.* Let $\mathcal{X}$ be an infinite dimensional Banach space, and let B be a linear basis for $\mathcal{X}$. Prove that B is uncountable.

HINT: If B is countable, say $B = \{b_n : n \in \mathbb{N}\}$, then

$$\mathcal{X} = \bigcup_{n=1}^{\infty} F_n,$$

where $F_n = \text{Span}(b_1, b_2, \dots, b_n)$. Since the F_n 's are finite dimensional linear subspaces, they will be closed. Use Baire's Theorem to get a contradiction.

COMMENTS. A third method of constructing Banach spaces is the completion. If we start with a normed $\mathbb{K}$ -vector space $\mathcal{X}$, when we regard $\mathcal{X}$ as a metric space, its completion $\tilde{\mathcal{X}}$ is constructed as follows. One defines

$$\text{cs}(\mathcal{X}) = \{\mathbf{x} = (x_n)_{n \geq 1} : (x_n)_{n \geq 1} \text{ Cauchy sequence in } \mathcal{X}\}.$$

Two Cauchy sequences $\mathbf{x} = (x_n)_{n \geq 1}$ and $\mathbf{x}' = (x'_n)_{n \geq 1}$ are said to be equivalent, if $\lim_{n \rightarrow \infty} \|x_n - x'_n\| = 0$. In this case one writes $\mathbf{x} \sim \mathbf{x}'$. The completion $\tilde{\mathcal{X}}$ is then defined as the space

$$\tilde{\mathcal{X}} = \text{cs}(\mathcal{X}) / \sim$$

of equivalence classes. For $\mathbf{x} \in \text{cs}(\mathcal{X})$, one denotes by $\tilde{\mathbf{x}}$ its equivalence class in $\tilde{\mathcal{X}}$. Finally for an element $x \in \mathcal{X}$ one denotes by $\langle x \rangle \in \tilde{\mathcal{X}}$ the equivalence class of the constant sequence x .

We know from general theory that $\tilde{\mathcal{X}}$ is a complete metric space, with the distance $\tilde{d}$ (correctly) defined by

$$\tilde{d}(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}') = \lim_{n \rightarrow \infty} \|x_n - x'_n\|,$$

for any two Cauchy sequences $\mathbf{x} = (x_n)_{n \geq 1}$ and $\mathbf{x}' = (x'_n)_{n \geq 1}$.

It turns out that, in our situation, the space $\text{CS}(\mathcal{X})$ carries a natural vector space structure, defined by pointwise addition and scalar multiplication. Moreover, the space $\tilde{\mathcal{X}}$ is identified as a quotient vector space

$$\tilde{\mathcal{X}} = \text{CS}(\mathcal{X})/\text{NS}(\mathcal{X}),$$

where

$$\text{NS}(\mathcal{X}) = \{ \mathbf{x} = (x_n)_{n \geq 1} : (x_n)_{n \geq 1} \text{ sequence in } \mathcal{X} \text{ with } \lim_{n \rightarrow \infty} x_n = 0 \}$$

is the linear subspace of null sequences. It then follows that $\tilde{\mathcal{X}}$ carries a natural vector space structure. More explicitly, if we start with a scalar $\lambda \in \mathbb{K}$, and with two elements $p, q \in \tilde{\mathcal{X}}$, which are represented as $p = \tilde{\mathbf{x}}$ and $q = \tilde{\mathbf{y}}$, for two Cauchy sequences $\mathbf{x} = (x_n)_{n \geq 1}$ and $\mathbf{y} = (y_n)_{n \geq 1}$ in $\mathcal{X}$, then the sequence

$$\mathbf{w} = (\lambda x_n + y_n)_{n \geq 1}$$

is Cauchy in $\mathcal{X}$, and the element $\lambda p + q \in \tilde{\mathcal{X}}$ is then defined as $\lambda p + q = \tilde{\mathbf{w}}$.

Finally, there is a natural norm on $\tilde{\mathcal{X}}$, (correctly) defined by

$$\|\tilde{\mathbf{x}}\| = \tilde{d}(\tilde{\mathbf{x}}, \langle 0 \rangle) = \lim_{n \rightarrow \infty} \|x_n\|,$$

for all Cauchy sequences $\mathbf{x} = (x_n)_{n \geq 1}$. These considerations then prove that $\tilde{\mathcal{X}}$ is a Banach space, and the map

$$\mathcal{X} \ni x \longmapsto \langle x \rangle \in \tilde{\mathcal{X}}$$

is linear and isometric, in the sense that

$$\|\langle x \rangle\| = \|x\|, \quad \forall x \in \mathcal{X}.$$

In the context of normed vector spaces, the universality property of the completion is stated as follows:

PROPOSITION 3.3. *Let $\mathcal{X}$ be a normed vector space, let $\tilde{\mathcal{X}}$ denote its completion, and let $\mathcal{Y}$ be a Banach space. For every linear continuous map $T : \mathcal{X} \rightarrow \mathcal{Y}$, there exists a unique linear continuous map $\tilde{T} : \tilde{\mathcal{X}} \rightarrow \mathcal{Y}$, such that*

$$\tilde{T}\langle x \rangle = Tx, \quad \forall x \in \mathcal{X}.$$

Moreover the map

$$\mathcal{L}(\mathcal{X}, \mathcal{Y}) \ni T \longmapsto \tilde{T} \in \mathcal{L}(\tilde{\mathcal{X}}, \mathcal{Y})$$

is an isometric linear isomorphism.

PROOF. If $T : \mathcal{X} \rightarrow \mathcal{Y}$ is linear and continuous, then T is a Lipschitz map with Lipschitz constant $\|T\|$, because

$$\|Tx - Tx'\| \leq \|T\| \cdot \|x - x'\|, \quad \forall x, x' \in \mathcal{X}.$$

We know, from the theory of metric spaces, that there exists a unique continuous map $\tilde{T} : \tilde{\mathcal{X}} \rightarrow \mathcal{Y}$, such that

$$\tilde{T}\langle x \rangle = Tx, \quad \forall x \in \mathcal{X}.$$

We also know that $\tilde{T}$ is Lipschitz, with Lipschitz constant $\|T\|$. The only thing we need to prove is the fact that $\tilde{T}$ is linear. Start with two points $p, q \in \tilde{\mathcal{X}}$, represented as $p = \tilde{\mathbf{x}}$ and $q = \tilde{\mathbf{z}}$, for some Cauchy sequences $\mathbf{x} = (x_n)_{n \geq 1}$ and $\mathbf{z} = (z_n)_{n \geq 1}$ in $\mathcal{X}$. If $\lambda \in \mathbb{K}$, then $\lambda p + q = \tilde{\mathbf{w}}$, where $\mathbf{w} = (\lambda x_n + z_n)_{n \geq 1}$. We then have

$$\tilde{T}(\lambda p + q) = \lim_{n \rightarrow \infty} T(\lambda x_n + z_n) = \left[\lambda \cdot \lim_{n \rightarrow \infty} Tx_n \right] + \left[\lim_{n \rightarrow \infty} Tz_n \right] = \lambda \tilde{T}p + \tilde{T}q.$$

Let us prove now that $\|\tilde{T}\| = \|T\|$. Since $\tilde{T}$ is Lipschitz, with Lipschitz constant $\|\tilde{T}\|$, we will have $\|\tilde{T}\| \leq \|T\|$. To prove the other inequality, let us consider the sets

$$\mathcal{B}_0 = \{p \in \tilde{\mathcal{X}} : \|p\| \leq 1\}, \mathcal{B}_1 = \{\langle x \rangle : x \in \mathcal{X}, \|x\| \leq 1\}.$$

By definition, we have

$$\|\tilde{T}\| = \sup_{p \in \mathcal{B}_0} \|\tilde{T}p\|.$$

Since we clearly have $\mathcal{B}_0 \supset \mathcal{B}_1$, we get

$$\begin{aligned} \|\tilde{T}\| &= \sup_{p \in \mathcal{B}_1} \|\tilde{T}p\| \geq \sup \{\|\tilde{T}\langle x \rangle\| : x \in \mathcal{X}, \|x\| \leq 1\} = \\ &= \sup \{\|Tx\| : x \in \mathcal{X}, \|x\| \leq 1\} = \|T\|. \end{aligned}$$

The fact that the map $\mathcal{L}(\mathcal{X}, \mathcal{Y}) \ni T \longmapsto \tilde{T} \in \mathcal{L}(\tilde{\mathcal{X}}, \mathcal{Y})$ is linear is obvious.

To prove the surjectivity, start with some $S \in \mathcal{L}(\tilde{\mathcal{X}}, \mathcal{Y})$. Consider the map

$$\iota : \mathcal{X} \ni x \longmapsto \langle x \rangle \in \tilde{\mathcal{X}}.$$

Since ι is linear and isometric, in particular it is continuous, so the composition $T = S \circ \iota$ is linear and continuous. Notice that

$$S\langle x \rangle = S(\iota(x)) = (S \circ \iota)x = Tx, \quad \forall x \in \mathcal{X},$$

so by uniqueness we have $S = \tilde{T}$. $\square$

COROLLARY 3.4. *Let $\mathcal{X}$ be a normed space, let $\mathcal{Y}$ be a Banach space, and let $T : \mathcal{X} \rightarrow \mathcal{Y}$ be an isometric linear map.*

- (i) *Let $\tilde{T} : \tilde{\mathcal{X}} \rightarrow \mathcal{Y}$ be the linear continuous map defined in the previous result. Then $\tilde{T}$ is linear, isometric, and $\tilde{T}(\tilde{\mathcal{X}}) = \overline{T(\mathcal{X})}$.*
- (ii) *$\mathcal{X}$ is complete, if and only if $T(\mathcal{X})$ is closed in $\mathcal{Y}$.*

PROOF. (i). The fact that $\tilde{T}$ is isometric, and has the range equal to $\overline{T(\mathcal{X})}$ is true in general (i.e. for $\mathcal{X}$ metric space, and $\mathcal{Y}$ complete metric space). The linearity follows from the previous result.

(ii). This is obvious. $\square$

EXAMPLE 3.2. Let $\mathcal{X}$ be a normed vector space. For every $x \in \mathcal{X}$ define the map $\epsilon_x : \mathcal{X}^* \rightarrow \mathbb{K}$ by

$$\epsilon_x(\phi) = \phi(x), \quad \forall \phi \in \mathcal{X}^*.$$

Then ϵ_x is a linear and continuous. This is an immediate consequence of the inequality

$$|\epsilon_x(\phi)| = |\phi(x)| \leq \|x\| \cdot \|\phi\|, \quad \forall \phi \in \mathcal{X}^*.$$

Notice that this also proves

$$\|\epsilon_x\| \leq \|x\|, \quad \forall x \in \mathcal{X}.$$

Interestingly enough, we actually have

$$(4) \quad \|\epsilon_x\| = \|x\|, \quad \forall x \in \mathcal{X}.$$

To prove this fact, we start with an arbitrary $x \in \mathcal{X}$, and we consider the linear subspace

$$\mathcal{Y} = \mathbb{K}x = \{\lambda x : \lambda \in \mathbb{K}\}.$$

If we define $\phi_0 : \mathcal{Y} \rightarrow \mathbb{K}$, by

$$\phi_0(\lambda x) = \lambda \|x\|, \quad \forall \lambda \in \mathbb{K},$$

then it is clear that $\phi_0(x) = \|x\|$, and

$$|\phi_0(y)| \leq \|y\|, \quad \forall y \in \mathcal{Y}.$$

Use then the Hahn-Banach Theorem to find $\phi : \mathcal{X} \rightarrow \mathbb{K}$ such that $\phi|_{\mathcal{Y}} = \phi_0$, and

$$|\phi(z)| \leq \|z\|, \quad \forall z \in \mathcal{X}.$$

This will clearly imply $\|\phi\| \leq 1$, while the first condition will give $\phi(x) = \phi_0(x) = \|x\|$. In particular, we will have

$$\|x\| = |\phi(x)| = |\epsilon_x(\phi)| \leq \|\epsilon_x\| \cdot \|\phi\| \leq \|\epsilon_x\|.$$

Having proven (4), we now have a linear isometric map

$$E : \mathcal{X} \ni x \mapsto \epsilon_x \in \mathcal{X}^{**}.$$

Since $\mathcal{X}^{**}$ is a Banach space, we now see that $\tilde{E} : \tilde{\mathcal{X}} \rightarrow \overline{E(\mathcal{X})}$ is an isometric linear isomorphism. In particular, $\mathcal{X}$ is Banach, if and only if $E(\mathcal{X})$ is closed in $\mathcal{X}^{**}$.

We conclude with a series of results, which are often regarded as the “principles of Banach space theory.” These results are consequences of Baire Theorem.

THEOREM 3.1 (Uniform Boundedness Principle). *Let $\mathcal{X}$ be a Banach space, let $\mathcal{Y}$ be normed vector space, and let $\mathcal{M} \subset \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The following are equivalent*

- (i) $\sup \{\|T\| : T \in \mathcal{M}\} < \infty$;
- (ii) $\sup \{\|Tx\| : T \in \mathcal{M}\} < \infty, \forall x \in \mathcal{X}$.

PROOF. The implication (i) $\Rightarrow$ (ii) is trivial, because if we define

$$M = \sup \{\|T\| : T \in \mathcal{M}\},$$

then by the definition of the norm, we clearly have

$$\sup \{\|Tx\| : T \in \mathcal{M}\} \leq M\|x\|, \quad \forall x \in \mathcal{X}.$$

(ii) $\Rightarrow$ (i). Assume $\mathcal{M}$ satisfies condition (ii). For each integer $n \geq 1$, let us define the set

$$\mathcal{F}_n = \{x \in \mathcal{X} : \|Tx\| \leq n, \forall T \in \mathcal{M}\}.$$

It is obvious that $\mathcal{F}_n$ is a closed subset of $\mathcal{X}$, for each $n \geq 1$. Moreover, by (ii) we clearly have $\bigcup_{n=1}^{\infty} \mathcal{F}_n = \mathcal{X}$. Using Baire’s Theorem, there exists some $n \geq 1$, such that $\text{Int}(\mathcal{F}_n) \neq \emptyset$. This means that there exists some $x_0 \in \mathcal{X}$ and some $r > 0$, such that

$$\mathcal{F}_n \supset \bar{\mathcal{B}}_r(x_0) = \{y \in \mathcal{X} : \|x - x_0\| \leq r\}.$$

Put $M_0 = \sup \{\|Tx_0\| : T \in \mathcal{M}\}$. Fix for the moment some arbitrary $x \in \mathcal{X}$, with $\|x\| \leq 1$, and some arbitrary element $T \in \mathcal{M}$. The vector $y = x_0 + rx$ clearly belongs to $\bar{\mathcal{B}}_r(x_0)$, so we have $\|Ty\| \leq n$. We then get

$$\|Tx\| = \left\| T\left(\frac{1}{r}(y - x_0)\right) \right\| = \frac{1}{r} \|Ty - Tx_0\| \leq \frac{1}{r} (\|Ty\| + \|Tx_0\|) \leq \frac{1}{r} (n + M_0).$$

Keep T fixed, and use the above estimate, which gives

$$\sup \{\|Tx\| : x \in \mathcal{X}, \|x\| \leq 1\} \leq \frac{n + M_0}{r},$$

to conclude that $\|T\| \leq \frac{n + M_0}{r}$. Since $T \in \mathcal{M}$ is arbitrary, we finally get

$$\sup \{\|T\| : T \in \mathcal{M}\} \leq \frac{n + M_0}{r} < \infty. \quad \square$$

THEOREM 3.2 (Inverse Mapping Theorem). *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let $T : \mathcal{X} \rightarrow \mathcal{Y}$ be a bijective linear continuous map. Then the linear map $T^{-1} : \mathcal{Y} \rightarrow \mathcal{X}$ is also continuous.*

PROOF. Let us denote by $\mathcal{A}$ the open unit ball in $\mathcal{X}$ centered at the origin, i.e.

$$\mathcal{A} = \{x \in \mathcal{X} : \|x\| < 1\}.$$

The first step in the proof is contained in the following.

Claim 1: The closure $\overline{T(\mathcal{A})}$ is a neighborhood of 0 in $\mathcal{Y}$.

Consider the sequence of closed sets $(\overline{kT(\mathcal{A})})_{k=1}^{\infty}$. (Here we use the notation $k\mathcal{M} = \{kv : v \in \mathcal{M}\}$.) Since the map $v \mapsto kv$ is a homeomorphism, one has the equalities

$$\overline{kT(\mathcal{A})} = \overline{kT(\mathcal{A})} = \overline{T(k\mathcal{A})}, \quad \forall k \geq 1.$$

In particular, we have

$$\bigcup_{k=1}^{\infty} \overline{kT(\mathcal{A})} = \bigcup_{k=1}^{\infty} \overline{T(k\mathcal{A})} \supset \bigcup_{k=1}^{\infty} T(k\mathcal{A}) = T\left(\bigcup_{k=1}^{\infty} k\mathcal{A}\right).$$

Since we obviously have $\bigcup_{k=1}^{\infty} k\mathcal{A} = \mathcal{X}$, and T is surjective, the above equality shows that $\bigcup_{k=1}^{\infty} \overline{kT(\mathcal{A})} = \mathcal{Y}$. Using Baire's Theorem, there exists some $k \geq 1$, such that $\text{Int}\left(\overline{kT(\mathcal{A})}\right) \neq \emptyset$. Again using the fact that $v \mapsto kv$ is a homeomorphism, this gives $\text{Int}\left(\overline{T(\mathcal{A})}\right) \neq \emptyset$. Fix now some point $y \in \text{Int}\left(\overline{T(\mathcal{A})}\right)$, and some $r > 0$, such that $\overline{T(\mathcal{A})}$ contains the open ball

$$(5) \quad \mathcal{B}_r(y) = \{z \in \mathcal{Y} : \|z - y\| < r\}.$$

The proof of the Claim is then finished, once we prove the inclusion

$$\overline{T(\mathcal{A})} \supset \mathcal{B}_{\frac{r}{2}}(0).$$

To prove this inclusion, start with some arbitrary $v \in \mathcal{B}_{\frac{r}{2}}(0)$, i.e. $v \in \mathcal{Y}$ and $\|v\| < \frac{r}{2}$. Since $\|(2v + y) - y\| = 2\|v\| < r$, using (5) it follows that $2v + y \in \overline{T(\mathcal{A})}$. i.e. there exists a sequence $(x_n)_{n=1}^{\infty} \subset \mathcal{X}$ with $\|x_n\| < 1$, $\forall n \geq 1$, and $2v + y = \lim_{n \rightarrow \infty} Tx_n$. Since y itself belongs to $\overline{T(\mathcal{A})}$, there also exists some sequence $(z_n)_{n=1}^{\infty} \subset \mathcal{X}$, with $\|z_n\| < 1$, $\forall n \geq 1$, and $y = \lim_{n \rightarrow \infty} Tz_n$. On the one hand, if we consider the sequence $(u_n)_{n=1}^{\infty} \subset \mathcal{X}$ given by $u_n = \frac{1}{2}(x_n - z_n)$, then it is clear that

$$\|u_n\| \leq \frac{1}{2}(\|x_n\| + \|z_n\|) < 1, \quad \forall n \geq 1,$$

i.e. $(u_n)_{n=1}^{\infty} \subset \mathcal{A}$. On the other hand, we have

$$\lim_{n \rightarrow \infty} Tu_n = \lim_{n \rightarrow \infty} \frac{1}{2}(Tx_n - Tz_n) = \frac{1}{2}(2v + y - y) = v,$$

so v indeed belongs to $\overline{T(\mathcal{A})}$.

The next step is a slight (but crucial) improvement of Claim 1.

Claim 2: $T(\mathcal{A})$ is a neighborhood of 0.

Start off by choosing $\varepsilon > 0$, such that

$$(6) \quad \overline{T(\mathcal{A})} \supset \mathcal{B}_{\varepsilon}(0).$$

The Claim will follow, once we prove the inclusion

$$(7) \quad T(\mathcal{A}) \supset \mathcal{B}_{\frac{\varepsilon}{2}}(0).$$

To prove this inclusion, we start with some arbitrary $y \in \mathcal{B}_\varepsilon(0)$. We want to construct a sequence of vectors $(x_n)_{n=1}^\infty \subset \mathcal{A}$, such that, for every $n \geq 1$, we have the inequality

$$(8) \quad \left\| y - \sum_{k=1}^n T\left(\frac{1}{2^k} x_k\right) \right\| \leq \frac{\varepsilon}{2^{n+1}}.$$

This sequence is constructed inductively as follows. We start by using (6), and we pick $x_1 \in \mathcal{A}$ such that $\|2y - Tx_1\| < \frac{\varepsilon}{2}$. Once $x_1, \dots, x_p$ are constructed, such that (8) holds with $n = p$, we consider the vector

$$z = 2^{p+1} \left[y - \sum_{k=1}^p T\left(\frac{1}{2^k} x_k\right) \right] \in \mathcal{B}_\varepsilon(0),$$

and we use again (6) to find $x_{p+1} \in \mathcal{A}$, such that $\|z - Tx_{p+1}\| \leq \frac{\varepsilon}{2}$. We then clearly have

$$\left\| y - \sum_{k=1}^{p+1} T\left(\frac{1}{2^k} x_k\right) \right\| = \frac{\|z - Tx_{p+1}\|}{2^{p+1}} \leq \frac{\varepsilon}{2^{p+2}},$$

Consider now the series $\sum_{k=1}^\infty \frac{1}{2^k} x_k$. Since $\|x_k\| < 1$, $\forall k \geq 1$, and $\mathcal{X}$ is a Banach space, by Remark 3.1, the sequence of $(w_n)_{n=1}^\infty \subset \mathcal{X}$ of partial sums

$$w_n = \sum_{k=1}^n \frac{1}{2^k} x_k, \quad n \geq 1,$$

is convergent to some point $x \in \mathcal{X}$. Moreover, since we have

$$\|w_n\| \leq \sum_{k=1}^n \frac{\|x_k\|}{2^k} \leq \sum_{k=1}^\infty \frac{\|x_k\|}{2^k}, \quad \forall n \geq 1,$$

we get the inequality

$$\|x\| \leq \sum_{k=1}^\infty \frac{\|x_k\|}{2^k} < 1,$$

which means that $x \in \mathcal{A}$. Note also that using these partial sums, the inequality (8) reads

$$\|y - Tw_n\| \leq \frac{\varepsilon}{2^{n+2}}, \quad \forall n \geq 1,$$

so by the continuity of T , we have $y = Tx \in T(\mathcal{A})$.

Let us show now that T^{-1} is continuous. Use Claim 2, to find some $r > 0$ such that

$$(9) \quad T(\mathcal{A}) \supset \mathcal{B}_r(0),$$

and let $y \in \mathcal{Y}$ be an arbitrary vector with $\|y\| \leq 1$. Consider the vector $v = \frac{r}{2}y$, which has $\|v\| \leq \frac{r}{2} < r$. By (9), there exists $x \in \mathcal{A}$, such that $Tx = v$, which means that $T^{-1}y = \frac{2}{r}x$. This forces $\|T^{-1}y\| \leq \frac{2}{r}$. This argument shows that

$$\sup \{ \|T^{-1}y\| : y \in \mathcal{Y}, \|y\| \leq 1 \} \leq \frac{2}{r} < \infty,$$

and the continuity of T^{-1} follows from Proposition 2.4. $\square$

The following two exercises deal with two more “principles of Banach space theory.”

Exercise 5 $\diamond$. (Closed Graph Theorem). Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let $T : \mathcal{X} \rightarrow \mathcal{Y}$ be a linear map. Prove that the following are equivalent:

- (i) T is continuous.
- (ii) The *graph* of T

$$\mathcal{G}_T = \{(x, Tx) : x \in \mathcal{X}\}$$

is a closed subset of $\mathcal{X} \times \mathcal{Y}$, in the product topology.

HINT: For the implication (ii) $\Rightarrow$ (i), use Exercise 3, to get the fact that $\mathcal{G}_T$ is a Banach space. Then T is exactly the inverse of $\pi_{\mathcal{X}}|_{\mathcal{G}_T}$, where $\pi_{\mathcal{X}} : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X}$ is the projection onto the first coordinate. Use Theorem 3.2.

Exercise 6 $\diamond$. (Open Mapping Theorem). Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let $T : \mathcal{X} \rightarrow \mathcal{Y}$ be a *surjective* linear continuous map. Prove that T is an *open* map, in the sense that

- whenever $\mathcal{D} \subset \mathcal{X}$ is open, it follows that $T(\mathcal{D})$ is open in $\mathcal{Y}$.

HINT: Consider the linear map

$$S : \mathcal{X} \times \mathcal{Y} \ni (x, y) \mapsto (x, Tx + y) \in \mathcal{X} \times \mathcal{Y}.$$

Prove that S is linear, continuous, bijective, hence by Theorem 3.2, it is a homeomorphism. Use this fact to prove that for every open set $\mathcal{D} \subset \mathcal{X}$, there exists some open set $\mathcal{E} \subset \mathcal{X} \times \mathcal{Y}$, such that $T(\mathcal{D}) = \pi_{\mathcal{Y}}(\mathcal{E})$, where $\pi_{\mathcal{Y}} : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Y}$ is the projection onto the second coordinate. This reduces the problem to proving the fact that $\pi_{\mathcal{Y}}$ is an open map.

LECTURE 13

4. The weak dual topology

In this section we examine the topological duals of normed vector spaces. Besides the norm topology, there is another natural topology which is constructed as follows.

DEFINITION. Let $\mathcal{X}$ be a normed vector space over $\mathbb{K}(= \mathbb{R}, \mathbb{C})$. For every $x \in \mathcal{X}$, let $\epsilon_x : \mathcal{X}^* \rightarrow \mathbb{K}$ be the linear map defined by

$$\epsilon_x(\phi) = \phi(x), \quad \forall \phi \in \mathcal{X}^*.$$

We equip the vector space $\mathcal{X}^*$ with the weak topology defined by the family $\Xi = (\epsilon_x)_{x \in \mathcal{X}}$. This topology is called the *weak dual topology*, which is denoted by w^* . Recall (see Section 3) that this topology is characterized by the following property

(w^*) *Given a topological space T , a map $f : T \rightarrow \mathcal{X}^*$ is continuous with respect to the w^* topology, if and only if $\epsilon_x \circ f : T \rightarrow \mathbb{K}$ is continuous, for each $x \in \mathcal{X}$.*

Remark that all the maps $\epsilon_x : \mathcal{X}^* \rightarrow \mathbb{K}$, $x \in \mathcal{X}$ are already continuous with respect to the *norm* topology. This gives the fact that

- *the w^* topology on $\mathcal{X}^*$ is weaker than the norm topology.*

REMARK 4.1. The w^* topology is Hausdorff. Indeed, if $\phi, \psi \in \mathcal{X}^*$ are such that $\phi \neq \psi$, then there exists some $x \in \mathcal{X}$ such that

$$\epsilon_x(\phi) = \phi(x) \neq \psi(x) = \epsilon_x(\psi).$$

PROPOSITION 4.1. *Let $\mathcal{X}$ be a normed vector space over $\mathbb{K}$. For every $\varepsilon > 0$, $\phi \in \mathcal{X}^*$, and $x \in \mathcal{X}$, define the set*

$$W(\phi; x, \varepsilon) = \{ \psi \in \mathcal{X}^* : |\psi(x) - \phi(x)| < \varepsilon \}.$$

Then the collection

$$\mathcal{W} = \{ W(\phi; x, \varepsilon) : \varepsilon > 0, \phi \in \mathcal{X}^*, x \in \mathcal{X} \}$$

is a subbase for the w^ topology. More precisely, given $\phi \in \mathcal{X}^*$, a set $N \subset \mathcal{X}^*$ is a neighborhood of ϕ with respect to the w^* topology, if and only if, there exist $\varepsilon > 0$ and $x_1, \dots, x_n \in \mathcal{X}$, such that*

$$N \supset W(\phi; \varepsilon, x_1) \cap \dots \cap W(\phi; \varepsilon, x_n).$$

PROOF. It is clearly sufficient to prove the second assertion, because it would imply the fact that any w^* open set is a union of finite intersections of sets in $\mathcal{W}$.

If we define the collection

$$\mathcal{S} = \{ \epsilon_x^{-1}(D) : x \in \mathcal{X}, D \subset \mathbb{K} \text{ open} \},$$

then we know that $\mathcal{S}$ is a subbase for the w^* topology.

Fix $\phi \in \mathcal{X}^*$. Start with some w^* neighborhood N of ϕ , so there exists some w^* open set E with $\phi \in E \subset N$. Using the fact that $\mathcal{S}$ is a subbase for the w^* topology, there exist open sets $D_1, \dots, D_n \subset \mathbb{K}$, and points $x_1, \dots, x_n$, such that

$$\phi \in \bigcap_{k=1}^n \epsilon_{x_k}^{-1}(D_k) \subset E.$$

Fix for the moment $k \in \{1, \dots, n\}$. The fact that $\phi \in \epsilon_{x_k}^{-1}(D_k)$ means that $\phi(x_k) \in D_k$. Since D_k is open in $\mathbb{K}$, there exists some $\varepsilon_k > 0$, such that

$$D_k \supset \mathcal{B}_{\varepsilon_k}(\phi(x_k)).$$

Then if we have an arbitrary $\psi \in W(\phi; \varepsilon_k, x_k)$, we will have

$$|\psi(x_k) - \phi(x_k)| < \varepsilon_k,$$

which gives $\psi \in \epsilon_{x_k}^{-1}(D_k)$. This proves that

$$W(\phi; \varepsilon_k, x_k) \subset \epsilon_{x_k}^{-1}(D_k).$$

Notice that, if one takes $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$, then we clearly have the inclusions

$$W(\phi; \varepsilon, x_k) \subset W(\phi; \varepsilon_k, x_k) \subset \epsilon_{x_k}^{-1}(D_k).$$

We then immediately get

$$W(\phi; \varepsilon, x_k) \subset \bigcap_{k=1}^n \epsilon_{x_k}^{-1}(D_k) \subset E \subset N,$$

and we are done. $\square$

COROLLARY 4.1. *Let $\mathcal{X}$ be a normed vector space. Then the w^* topology on $\mathcal{X}^*$ is locally convex, i.e.*

- for every $\phi \in \mathcal{X}^*$ and every w^* -neighborhood N of ϕ , there exists a convex w^* -open set D such that $\phi \in D \subset N$.

PROOF. Apply the second part of the proposition, together with the obvious fact that each of the sets $W(\phi; \varepsilon, x)$ is convex and w^* -open. $\square$

PROPOSITION 4.2. *Let $\mathcal{X}$ be a normed vector space. When equipped with the w^* topology, the space $\mathcal{X}^*$ is a topological vector space. This means that the maps*

$$\mathcal{X}^* \times \mathcal{X}^* \ni (\phi, \psi) \longmapsto \phi + \psi \in \mathcal{X}^*$$

$$\mathbb{K} \times \mathcal{X}^* \ni (\lambda, \phi) \longmapsto \lambda\phi \in \mathcal{X}^*$$

are continuous with respect to the w^ topology on the target space, and the w^* product topology on the domain.*

PROOF. According to the definition of the w^* topology, it suffices to prove that, for every $x \in \mathcal{X}$, the maps

$$\sigma_x : \mathcal{X}^* \times \mathcal{X}^* \ni (\phi, \psi) \longmapsto \gamma_x : \epsilon_x(\phi + \psi) \in \mathbb{K}$$

$$\mathbb{K} \times \mathcal{X}^* \ni (\lambda, \phi) \longmapsto \epsilon_x(\lambda\phi) \in \mathbb{K}$$

are continuous. But the continuity of σ_x and γ_x is obvious, since we have

$$\sigma_x(\phi, \psi) = \phi(x) + \psi(x) = \epsilon_x(\phi) + \epsilon_x(\psi), \quad \forall (\phi, \psi) \in \mathcal{X}^* \times \mathcal{X}^*;$$

$$\gamma_x(\lambda, \phi) = \lambda\phi(x) = \lambda\epsilon_x(\phi), \quad \forall (\lambda, \phi, \psi) \in \mathbb{K} \times \mathcal{X}^*.$$

$\square$

Our next goal will be to describe the linear maps $\mathcal{X}^* \rightarrow \mathbb{K}$, which are continuous in the w^* topology.

PROPOSITION 4.3. *Let $\mathcal{X}$ be a normed vector space over $\mathbb{K}$. For a linear map $\omega : \mathcal{X}^* \rightarrow \mathbb{K}$, the following are equivalent:*

- (i) ω is continuous with respect to the w^* topology;
- (ii) there exists some $x \in \mathcal{X}$, such that

$$\omega(\phi) = \phi(x), \quad \forall \phi \in \mathcal{X}^*.$$

PROOF. The implication (ii) $\Rightarrow$ (i) is trivial, since condition (ii) gives $\omega = \epsilon_x$.
(i) $\Rightarrow$ (ii). Suppose ω is continuous. In particular, ω is continuous at 0, so if we take the set

$$D = \{\lambda \in \mathbb{K} : |\lambda| < 1\},$$

the set

$$\omega^{-1}(D) = \{\phi \in \mathcal{X}^* : |\omega(\phi)| < 1\}$$

is an open neighborhood of 0 in the w^* topology. By Proposition ?? there exist $x_1, \dots, x_n \in \mathcal{X}$, and $\varepsilon > 0$, such that

$$(1) \quad W(0; \varepsilon, x_1) \cap \dots \cap W(0; \varepsilon, x_n) \subset D.$$

Claim 1: One has the inequality

$$|\omega(\phi)| \leq \varepsilon^{-1} \cdot \max \{|\phi(x_1)|, \dots, |\phi(x_n)|\}, \quad \forall \phi \in \mathcal{X}^*.$$

Fix an arbitrary $\phi \in \mathcal{X}^*$, and put $M = \max \{|\phi(x_1)|, \dots, |\phi(x_n)|\}$. For every integer $k \geq 1$, define

$$\phi_k = \varepsilon(M + \frac{1}{k})^{-1} \phi,$$

so that

$$|\phi_k(x_j)| = \varepsilon(M + \frac{1}{k})^{-1} |\phi(x_j)| \leq \varepsilon M (M + \frac{1}{k})^{-1} < \varepsilon, \quad \forall k \geq 1, j \in \{1, \dots, n\}.$$

This proves that $\phi_k \in W(0; \varepsilon, x_j)$, for all $k \geq 1$, and all $j \in \{1, \dots, n\}$. By (1) this will give

$$|\omega(\phi_k)| < 1, \quad \forall k \geq 1,$$

which reads

$$\varepsilon(M + \frac{1}{k})^{-1} |\omega(\phi)| < 1, \quad \forall k \geq 1.$$

This gives

$$|\omega(\phi)| \leq \varepsilon^{-1} (M + \frac{1}{k}), \quad \forall k \geq 1,$$

and it will obviously force

$$|\omega(\phi)| \leq \varepsilon^{-1} M.$$

Having proven the Claim, we now define the linear map $T : \mathcal{X}^* \rightarrow \mathbb{K}^n$, by

$$T\phi = (\phi(x_1), \dots, \phi(x_n)), \quad \forall \phi \in \mathcal{X}^*.$$

Claim 2: There exists a linear map $\sigma : \mathbb{K}^n \rightarrow \mathbb{K}$, such that $\omega = \sigma \circ T$.

First we show that we have the inclusion

$$\text{Ker } \omega \supset \text{Ker } T.$$

If we start with $\phi \in \text{Ker } T$, then $\phi(x_1) = \dots = \phi(x_n) = 0$, and then by Claim 1 we immediately get $\omega(\phi) = 0$, so ϕ indeed belongs to $\text{Ker } \omega$. We use now a bit of linear algebra. On the one hand, since $\omega|_{\text{Ker } T} = 0$, there exists a linear map $\hat{\omega} : \mathcal{X}/\text{Ker } T \rightarrow \mathbb{K}$, such that $\omega = \hat{\omega} \circ \pi$, where $\pi : \mathcal{X} \rightarrow \mathcal{X}/\text{Ker } T$ denotes the quotient map. On the other hand, by the Isomorphism Theorem for linear maps,

there exists a linear isomorphism $\hat{T} : \mathcal{X}/\text{Ker } T \xrightarrow{\sim} \text{Ran } T$, such that $\hat{T} \circ \pi = T$. We then define

$$\sigma_0 = \hat{\omega} \circ \hat{T}^{-1} : \text{Ran } T \rightarrow \mathbb{K},$$

and we will have

$$\sigma_0 \circ T = (\hat{\omega} \circ \hat{T}^{-1}) \circ (\hat{T} \circ \pi) = \hat{\omega} \circ \pi = \omega.$$

We finally extend⁵ $\sigma_0 : \text{Ran } T \rightarrow \mathbb{K}$ to a linear map $\sigma : \mathbb{K}^n \rightarrow \mathbb{K}$.

Having proven Claim 2, we choose scalars $\alpha_1, \dots, \alpha_n \in \mathbb{K}$, such that

$$\sigma(\lambda_1, \dots, \lambda_n) = \alpha_1 \lambda_1 + \dots + \alpha_n \lambda_n, \quad \forall (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n.$$

We now have

$$\omega(\phi) = \sigma(T\phi) = \sigma(\phi(x_1), \dots, \phi(x_n)) = \alpha_1 \phi(x_1) + \dots + \alpha_n \phi(x_n), \quad \forall \phi \in \mathcal{X}^*,$$

so if we define $x = \alpha_1 x_1 + \dots + \alpha_n x_n$, we clearly have

$$\omega(\phi) = \phi(x), \quad \forall \phi \in \mathcal{X}^*.$$

(ii) $\Rightarrow$ (i). This implication is trivial. $\square$

COROLLARY 4.2. *Let $\mathcal{X}$ be a normed vector space, let $\mathcal{C} \subset \mathcal{X}^*$ be a convex set, and let $\phi \in \mathcal{X}^* \setminus \overline{\mathcal{C}}^{w^*}$. (Here $\overline{\mathcal{C}}^{w^*}$ denotes the w^* -closure of $\mathcal{C}$.) Then there exists an element $x \in \mathcal{X}$, and a real number α , such that*

$$\text{Re } \phi(x) < \alpha \leq \text{Re } \psi(x), \quad \forall \psi \in \mathcal{C}.$$

PROOF. Since the w^* topology on $\mathcal{X}^*$ is locally convex, there exists a convex w^* -open set $\mathcal{A} \subset \mathcal{X}^*$, such that $\phi \in \mathcal{A} \subset \mathcal{X}^* \setminus \overline{\mathcal{C}}^{w^*}$. In particular, we have $\mathcal{A} \cap \mathcal{C} = \emptyset$. Apply the Hahn-Banach separation theorem to find a linear map $\omega : \mathcal{X}^* \rightarrow \mathbb{K}$, which is w^* -continuous, and a real number α , such that

$$\text{Re } \omega(\rho) < \alpha \leq \text{Re } \omega(\psi), \quad \forall \rho \in \mathcal{A}, \psi \in \mathcal{C}.$$

We then apply the above Proposition. $\square$

COMMENTS. The definition of the w^* topology can be used in a more general setting, when $\mathcal{X}$ is just a topological vector space. The above results are still valid in this general setting.

In general the unit ball

$$(\mathcal{X}^*)_1 = \{\phi \in \mathcal{X}^* : \|\phi\| \leq 1\},$$

although bounded and closed, is *not* compact in the *norm* topology. However, when the w^* topology is used, we have

THEOREM 4.1 (Alaoglu). *If $\mathcal{X}$ is a normed vector space, then the unit ball $(\mathcal{X}^*)_1$, in the topological dual space, is compact in the w^* topology.*

PROOF. Let us consider the unit ball in $\mathbb{K}$:

$$\mathbb{B} = \{\lambda \in \mathbb{K} : |\lambda| \leq 1\}.$$

Let us also consider the unit ball in $\mathcal{X}$:

$$(\mathcal{X})_1 = \{x \in \mathcal{X} : \|x\| \leq 1\}.$$

⁵ One can invoke the Hahn-Banach Theorem here. In fact this is not necessary, since $\text{Ran } T \subset \mathbb{K}^n$ are finite dimensional vector spaces.

Define the product space

$$P = \prod_{x \in (\mathcal{X})_1} \mathbb{B},$$

identified equivalently as the space of maps $(\mathcal{X})_1 \rightarrow \mathbb{B}$. By Tihonov's Theorem, when we equip P with the *product topology*, it will become a *compact* topological space. We denote by $\pi_x : P \rightarrow \mathbb{B}$, $x \in (\mathcal{X})_1$, the projection onto the factor with label x . By definition of the product topology π_x is continuous.

For any $x, y \in (\mathcal{X})_1$ define the map $\Delta_{x,y} : P \rightarrow \mathbb{K}$ by

$$\Delta_{x,y}(f) = \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right), \quad \forall f \in P.$$

Note that

$$\Delta_{x,y} = \frac{1}{2}(\pi_x + \pi_y) - \pi_{(x+y)/2},$$

so $\Delta_{x,y} : P \rightarrow \mathbb{K}$ is obviously continuous. In particular, the set

$$A_{x,y} = \Delta_{x,y}^{-1}(\{0\}) = \left\{ f \in P : \frac{f(x) + f(y)}{2} = f\left(\frac{x+y}{2}\right) \right\}$$

is closed in P , for every $x, y \in (\mathcal{X})_1$.

Similarly, for every $x \in (\mathcal{X})_1$ and every $\lambda \in \mathbb{B}$, we define the map $\Sigma_{\lambda,x} : P \rightarrow \mathbb{K}$ by

$$\Sigma_{\lambda,x}(f) = f(\lambda x) - \lambda f(x), \quad \forall f \in P,$$

then $\Sigma_{\lambda,x}$ is continuous, so the set

$$B_{x,y} = \Sigma_{x,y}^{-1}(\{0\}) = \{ f \in P : f(\lambda x) = \lambda f(x) \}$$

is closed in P , for every $\lambda \in \mathbb{B}$, $x \in (\mathcal{X})_1$.

Define the set

$$L = \left(\bigcap_{x,y \in (\mathcal{X})_1} A_{x,y} \right) \cap \left(\bigcap_{\substack{\lambda \in \mathbb{B} \\ x \in (\mathcal{X})_1}} B_{\lambda,x} \right).$$

Since L is an intersection of closed sets, it follows that L itself is closed. In particular, L is *compact*. By construction, we have

$$L = \left\{ f : (\mathcal{X})_1 \rightarrow \mathbb{B} \left| \begin{array}{l} \frac{1}{2}[f(x) + f(y)] = f\left(\frac{1}{2}[x+y]\right) \text{ and} \\ f(\lambda x) = \lambda f(x), \quad \forall x, y \in (\mathcal{X})_1, \lambda \in \mathbb{B} \end{array} \right. \right\}.$$

For any $f \in L$, we define the map $\psi_f : \mathcal{X} \rightarrow \mathbb{K}$ by

$$\psi_f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \|x\| \cdot f\left(\frac{x}{\|x\|}\right) & \text{if } x \neq 0 \end{cases}$$

Claim 1: For any $f \in L$, the map $\psi_f : \mathcal{X} \rightarrow \mathbb{K}$ is linear, and satisfies

$$\psi_f|_{(\mathcal{X})_1} = f.$$

Fix $f \in L$. Start with some $x \in \mathcal{X}$ and some $\lambda \in \mathbb{K}$. We have $\|\lambda x\| = |\lambda| \cdot \|x\|$, so we get

$$\psi_f(\lambda x) = \begin{cases} 0 & \text{if either } x = 0, \text{ or } \lambda = 0 \\ |\lambda| \cdot \|x\| \cdot f\left(\frac{\lambda}{|\lambda|} \cdot \frac{x}{\|x\|}\right) & \text{if } \lambda \neq 0 \text{ and } x \neq 0 \end{cases}$$

If $\lambda \neq 0$ and $x \neq 0$, we put

$$\mu = \frac{\lambda}{|\lambda|} \text{ and } y = \frac{x}{\|x\|},$$

and the fact that $\mu \in \mathbb{B}$, $y \in (\mathcal{X})_1$, and $f \in B_{\mu,y}$, will give

$$f\left(\frac{\lambda}{|\lambda|} \cdot \frac{x}{\|x\|}\right) = f(\mu y) = \mu f(y) = \frac{\lambda}{|\lambda|} \cdot f\left(\frac{x}{\|x\|}\right) = \frac{\lambda}{|\lambda| \cdot \|x\|} \psi_f(x),$$

so in this case we get

$$\psi_f(\lambda x) = |\lambda| \cdot \|x\| \cdot f\left(\frac{\lambda}{|\lambda|} \cdot \frac{x}{\|x\|}\right) = |\lambda| \cdot \|x\| \cdot \frac{\lambda}{|\lambda| \cdot \|x\|} \psi_f(x) = \lambda \psi_f(x).$$

In the case when either $\lambda = 0$ or $x = 0$, we also get the equality

$$\psi_f(\lambda x) = 0 = \lambda \psi_f(x).$$

This way we have proven the homogeneity of ψ_f

$$(2) \quad \psi_f(\lambda x) = \lambda \psi_f(x), \quad \forall \lambda \in \mathbb{K}, x \in \mathcal{X}.$$

Let us prove now that $\psi_f|_{(\mathcal{X})_1} = f$. If $x = 0$, then using the property

$$(3) \quad f(\mu y) = \mu f(y), \quad \forall \mu \in \mathbb{B}, y \in (\mathcal{X})_1$$

with $\mu = 0$ and $y = 0$, we immediately get $f(x) = 0 = \psi_f(x)$. If $x \neq 0$, we use (3)

with $\mu = \|x\|$ and $y = \frac{x}{\|x\|}$ and we again get

$$f(x) = f(\|x\| \cdot y) = \|x\| \cdot f(y) = \|x\| \cdot f\left(\frac{x}{\|x\|}\right) = \psi_f(x).$$

We now prove that ψ_f is additive. Start with two elements $x, y \in \mathcal{X}$. Define

$$v = \frac{x}{\|x\| + \|y\| + 1} \text{ and } w = \frac{y}{\|x\| + \|y\| + 1},$$

so that we obviously have $v, w \in (\mathcal{X})_1$ and

$$x = \{\|x\| + \|y\| + 1\} \cdot v \text{ and } y = \{\|x\| + \|y\| + 1\} \cdot w.$$

By homogeneity, we have

$$\psi_f(x + y) = \psi_f(2\{\|x\| + \|y\| + 1\} \cdot \frac{1}{2}[v + w]) = 2\{\|x\| + \|y\| + 1\} \cdot f\left(\frac{1}{2}[v + w]\right).$$

Using the fact that $f \in A_{v,w}$ the above computation can be continued to give:

$$\begin{aligned} \psi_f(x + y) &= 2\{\|x\| + \|y\| + 1\} \cdot f\left(\frac{1}{2}[v + w]\right) = \\ &= 2\{\|x\| + \|y\| + 1\} \cdot \frac{1}{2}[f(v) + f(w)] = \\ &= \{\|x\| + \|y\| + 1\} \cdot f(v) + \{\|x\| + \|y\| + 1\} \cdot f(w). \end{aligned}$$

Using the fact that $\psi_f|_{(\mathcal{X})_1} = f$, the above equality gives

$$\psi_f(x + y) = \{\|x\| + \|y\| + 1\} \cdot \psi_f(v) + \{\|x\| + \|y\| + 1\} \cdot \psi_f(w).$$

Finally, using the homogeneity property (2) we get

$$\psi_f(x + y) = \psi_f(\{\|x\| + \|y\| + 1\} \cdot v) + \psi_f(\{\|x\| + \|y\| + 1\} \cdot w) = \psi_f(x) + \psi_f(y).$$

Having proven the Claim, let us now observe that, for $f \in L$, the fact that

$$\psi_f(x) = f(x) \in \mathbb{B}, \quad \forall x \in (\mathcal{X})_1,$$

shows that ψ_f is *continuous*, and $\|\psi_f\| \leq 1$. Therefore we have a correctly defined map

$$\Psi : L \ni f \longmapsto \psi_f \in (\mathcal{X}^*)_1.$$

Claim 2: When $(\mathcal{X}^)_1$ is equipped with the w^* topology, the map Ψ is continuous.*

By the definition of the w^* topology, we need to prove that $\epsilon_x \circ \Psi : L \rightarrow \mathbb{K}$ is continuous, for every $x \in \mathcal{X}$. If $x = 0$, the composition $\epsilon_x \circ \Psi$ is the constant map 0, so there is nothing to prove. If $x \neq 0$, we define

$$y = \frac{x}{\|x\|} \in (\mathcal{S})_1,$$

and using Claim 1, we have

$$(\epsilon_x \circ \Psi)(f) = \epsilon_x(\psi_f) = \psi_f(x) = \|x\| \cdot \psi_f\left(\frac{x}{\|x\|}\right) = \|x\| \cdot \psi_f(y) = \|x\| \cdot f(y), \quad \forall f \in L.$$

This proves that

$$\epsilon_x \circ \Psi = \|x\| \cdot \pi_y,$$

and since $\pi_y : P \rightarrow \mathbb{B}$ is continuous, the continuity of $\epsilon_x \circ \Psi$ follows.

In order to finish the proof of the Theorem, it then suffices to prove

Claim 3: The map $\Psi : L \rightarrow (\mathcal{X}^)_1$ is surjective.*

Start with an arbitrary $\phi \in (\mathcal{X}^*)_1$, which means that $\phi : \mathcal{X} \rightarrow \mathbb{K}$ is linear, continuous, and

$$|\phi(x)| \leq 1, \quad \forall x \in (\mathcal{X})_1.$$

In particular, if we define $f = \phi|_{(\mathcal{X})_1}$, then

$$f(x) \in \mathbb{B}, \quad \forall x \in (\mathcal{X})_1,$$

which means that $f \in P$. Using the fact that ϕ is linear, it is obvious that $f \in L$. Using Claim 1, we have

$$\psi_f(x) = f(x) = \phi(x), \quad \forall x \in (\mathcal{X})_1.$$

Now, since $\psi_f|_{(\mathcal{X})_1} = \phi|_{(\mathcal{X})_1}$, and both ψ_f and ϕ are linear, we immediately get $\psi_f = \phi$. $\square$

REMARKS 4.2. Using the notations from the above proof, the continuous map $\Psi : L \rightarrow (\mathcal{X}^*)_1$ is in fact bijective. The only thing we need to prove is the injectivity. Suppose $\psi_f = \psi_g$, for some $f, g \in L$. Then

$$f = \psi_f|_{(\mathcal{X})_1} = \psi_g|_{(\mathcal{X})_1} = g.$$

Since $\Psi : (\mathcal{X}^*)_1 \rightarrow L$ is bijective, continuous, and the spaces $(\mathcal{X}^*)_1$ and L are compact Hausdorff, it follows that Ψ is in fact a *homeomorphism*. The inverse map $\Psi^{-1} : (\mathcal{X}^*)_1 \rightarrow L$ is simply defined by

$$\Psi^{-1}(\phi) = \phi|_{(\mathcal{X})_1}, \quad \forall \phi \in (\mathcal{X}^*)_1.$$

PROPOSITION 4.4. *Suppose $\mathcal{X}$ is a normed vector space, which is separable in the norm topology. When equipped with the w^* topology, the compact space $(\mathcal{X}^*)_1$ is metrizable.*

PROOF. Fix a countable dense subset $\mathcal{M} \subset \mathcal{X}$, and define $(\mathcal{M})_1 = (\mathcal{X})_1 \cap \mathcal{M}$. Notice that $(\mathcal{M})_1$ is dense in $(\mathcal{X})_1$. Indeed, if we start with some $x \in (\mathcal{X})_1$, and some $\varepsilon > 0$, then we set $x_\varepsilon = (1 - \frac{\varepsilon}{2})x$, and we choose $y \in \mathcal{M}$ such that $\|x_\varepsilon - y\| < \frac{\varepsilon}{2}$. On the one hand, we have

$$\|y\| \leq \|x_\varepsilon - y\| + \|x_\varepsilon\| < \frac{\varepsilon}{2} + (1 - \frac{\varepsilon}{2}) \cdot \|x\| \leq \frac{\varepsilon}{2} + 1 - \frac{\varepsilon}{2} = 1,$$

so $y \in (\mathcal{M})_1$. On the other hand, we have

$$\|y - x\| \leq \|y - x_\varepsilon\| + \|x - x_\varepsilon\| < \frac{\varepsilon}{2} + \left\| \frac{\varepsilon}{2} x \right\| \leq \frac{\varepsilon}{2} \cdot (1 + \|x\|) \leq \varepsilon.$$

Let us use the notations from the proof of Theorem 4.1. Let us then define the product space

$$\prod_{x \in (\mathcal{M})_1} \mathbb{B},$$

equipped with the product topology. Define also the map

$$\Upsilon : \prod_{x \in (\mathcal{X})_1} \mathbb{B} \ni f \longmapsto f|_{(\mathcal{M})_1} \in \prod_{x \in (\mathcal{M})_1} \mathbb{B}.$$

It is obvious that Υ is continuous. Let

$$\varkappa : (\mathcal{X}^*)_1 \ni \phi \longmapsto \phi|_{(\mathcal{X})_1} \in \prod_{x \in (\mathcal{X})_1} \mathbb{B}.$$

We know that $\varkappa$ is continuous and injective (being the inverse of $\Psi : L \rightarrow (\mathcal{X}^*)_1$).

Claim: The composition $\Upsilon \circ \varkappa : (\mathcal{X}^)_1 \rightarrow \prod_{x \in (\mathcal{M})_1} \mathbb{B}$ is injective.*

Indeed, if $\phi, \psi \in (\mathcal{X}^*)_1$ satisfy $(\Upsilon \circ \varkappa)(\phi) = (\Upsilon \circ \varkappa)(\psi)$, then we get $\phi|_{(\mathcal{M})_1} = \psi|_{(\mathcal{M})_1}$. Since $(\mathcal{M})_1$ is dense in $(\mathcal{X})_1$, this will force $\phi|_{(\mathcal{X})_1} = \psi|_{(\mathcal{X})_1}$, which finally forces $\phi = \psi$.

Using the above Claim, we see that if we define $Q = (\Upsilon \circ \varkappa)((\mathcal{X}^*)_1)$, then $Q \subset \prod_{x \in (\mathcal{M})_1} \mathbb{B}$ is compact, and $\Upsilon \circ \varkappa : (\mathcal{X}^*)_1 \rightarrow Q$ is a homeomorphism. Notice that $\prod_{x \in (\mathcal{M})_1} \mathbb{B}$ is a countable product of metric spaces, so it is metrizable. Therefore Q is also metrizable, and so will be $(\mathcal{X}^*)_1$. $\square$

REMARK 4.3. Assuming $\mathcal{X}$ is separable, and $\mathcal{M} \subset \mathcal{X}$ is a countable dense subset. If we enumerate the countable set $(\mathcal{M})_1$ as

$$(\mathcal{M})_1 = \{y_n : n \geq 1\},$$

then a metric d that defines the w^* topology on $(\mathcal{X}^*)_1$ can be constructed as

$$d(\phi, \psi) = \sum_{n=1}^{\infty} \frac{|\phi(y_n) - \psi(y_n)|}{2^n}, \quad \forall \phi, \psi \in (\mathcal{X}^*)_1.$$

COMMENTS. Let $\mathcal{X}$ be a normed vector space. One can extend the map $\varkappa$ to a map

$$\tilde{\varkappa} : \mathcal{X}^* \ni \phi \longmapsto \phi|_{(\mathcal{X})_1} \in \prod_{x \in (\mathcal{X})_1} \mathbb{K}.$$

This map will still be injective and continuous, and one can show that

$$\tilde{\varkappa} : \mathcal{X}^* \rightarrow \tilde{\varkappa}(\mathcal{X}^*)$$

is a homeomorphism, when $\tilde{\mathcal{X}}(\mathcal{X}^*)$ is equipped with the induced topology from the product space $\prod_{x \in (\mathcal{X})_1} \mathbb{K}$. In general however, the set $\tilde{\mathcal{X}}(\mathcal{X}^*)$ is *not* closed in the product space $\prod_{x \in (\mathcal{X})_1} \mathbb{K}$.

If $\mathcal{X}$ is separable, and if one takes a countable dense set $\mathcal{M} \subset \mathcal{X}$, then as before, one also still has a continuous map

$$\tilde{\Upsilon} : \prod_{x \in (\mathcal{X})_1} \mathbb{K} \ni f \longmapsto f|_{(\mathcal{M})_1} \in \prod_{x \in (\mathcal{M})_1} \mathbb{K},$$

and the composition

$$\tilde{\Upsilon} \circ \tilde{\mathcal{X}} : \mathcal{X}^* \rightarrow \prod_{x \in (\mathcal{M})_1} \mathbb{B}$$

will still be continuous and injective. In general however, it turns out that the map

$$\tilde{\Upsilon} \circ \tilde{\mathcal{X}} : \mathcal{X}^* \rightarrow \tilde{\Upsilon} \circ \tilde{\mathcal{X}}(\mathcal{X}^*)$$

is *not* a homeomorphism. The exercise below explains exactly when this is the case.

*Exercise 1**. Let $\mathcal{X}$ be a normed vector space, which is of *uncountable* dimension (for example, a Banach space). Prove that the topological space $(\mathcal{X}^*, w^*)$ is not metrizable.

HINT: Assume $(\mathcal{X}^*, w^*)$ is metrizable. Let d be a metric which gives the w^* -topology. Then $0 \in \mathcal{X}^*$ will have a countable basic system of neighborhoods. In particular, there exist sequences $(x_n)_{n \geq 1} \subset \mathcal{X}$, and $(\varepsilon_n)_{n \geq 1} \in (0, \infty)$, such that the sets

$$B_n = \bigcap_{k=1}^n W(0; \varepsilon_n, x_k)$$

satisfy $B_n \subset \mathcal{B}_{1/n}(0)$, $\forall n \geq 1$, where $\mathcal{B}_{1/n}(0)$ denotes the d -open ball of center 0 and radius $1/n$. Consider the set $\mathcal{M} = \{x_n : n \in \mathbb{N}\}$. We know that $\text{Span } \mathcal{M} \subsetneq \mathcal{X}$. Choose some vector $y \in \mathcal{X} \setminus \text{Span } \mathcal{M}$. For every $n \geq 1$, choose a linear map $\psi_n : \text{Span}\{y, x_1, \dots, x_n\} \rightarrow \mathbb{K}$, such that $\psi_n(y) = 1$, and $\psi_n(x_k) = 0$, $\forall k \in \{1, \dots, n\}$. Extend (use Hahn-Banach) ψ_n to a linear continuous map $\phi_n : \mathcal{X} \rightarrow \mathbb{K}$. Notice now that $\phi_n \in B_n$, for all $n \geq 1$, which would then force $d\text{-}\lim_{n \rightarrow \infty} \phi_n = 0$. In particular, this would force $\lim_{n \rightarrow \infty} \phi_n(x) = 0$, $\forall x \in \mathcal{X}$. But this is impossible, since $\phi_n(y) = 1$, $\forall n \geq 1$.

COMMENT. If $\mathcal{X}$ is a normed vector space of *countable* dimension, then $(\mathcal{X}^*, w^*)$ is metrizable. Indeed, if we take a linear basis $\{b_n : n \in \mathbb{N}\}$ for $\mathcal{X}$, then the w^* topology on $\mathcal{X}^*$ is clearly defined by the metric

$$d(\phi, \psi) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|\phi(b_n) - \psi(b_n)|}{1 + |\phi(b_n) - \psi(b_n)|}, \quad \phi, \psi \in \mathcal{X}^*.$$

LECTURES 14-15

5. Banach spaces of continuous functions

In this section we discuss a examples of Banach spaces coming from topology.

NOTATION. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let Ω be a topological space. We define

$$C_b^{\mathbb{K}}(\Omega) = \{f : \Omega \rightarrow \mathbb{K} : f \text{ bounded and continuous}\}.$$

In the case when $\mathbb{K} = \mathbb{C}$ we use the notation $C_b(\Omega)$.

PROPOSITION 5.1. *With the notations above, if we define*

$$\|f\| = \sup_{p \in \Omega} |f(p)|, \quad \forall f \in C_b^{\mathbb{K}}(\Omega),$$

then $C_b^{\mathbb{K}}(\Omega)$ is a Banach space.

PROOF. It is obvious that $C_b^{\mathbb{K}}(\Omega)$ is a linear subspace of $\ell_{\mathbb{K}}^{\infty}(\Omega)$, and the norm is precisely the one coming from $\ell_{\mathbb{K}}^{\infty}(\Omega)$. Therefore, it suffices to prove that $C_b^{\mathbb{K}}(\Omega)$ is closed in $\ell_{\mathbb{K}}^{\infty}(\Omega)$.

Start with some sequence $(f_n)_{n \geq 1} \subset C_b^{\mathbb{K}}(\Omega)$, which convergens in norm to some $f \in \ell_{\mathbb{K}}^{\infty}(\Omega)$, and let us prove that $f : \Omega \rightarrow \mathbb{K}$ is continuous (the fact that f is bounded is automatic).

Fix some point $p_0 \in \Omega$, and some $\varepsilon > 0$. We need to find some neighborhood V of p_0 , such that

$$|f(p) - f(p_0)| < \varepsilon, \quad \forall p \in V.$$

Start by choosing n such that $\|f_n - f\| < \frac{\varepsilon}{3}$. Use the fact that f_n is continuous, to find a neighborhood V of p_0 , such that

$$|f_n(p) - f_n(p_0)| < \frac{\varepsilon}{3}, \quad \forall p \in V.$$

Suppose now $p \in V$. We have

$$\begin{aligned} |f(p) - f(p_0)| &\leq |f_n(p) - f(p)| + |f_n(p) - f_n(p_0)| + |f_n(p_0) - f(p_0)| \leq \\ &|f_n(p) - f_n(p_0)| + 2 \left[\sup_{q \in \Omega} |f_n(q) - f(q)| \right] < 2 \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

□

A first application of Banach space techniques is the following:

LEMMA 5.1 (Urysohn type density). *Let Ω be a topological space, let $\mathcal{C} \subset C_b^{\mathbb{R}}(\Omega)$ be a linear subspace, which contains the constant function 1. Assume*

- (U) *for any two closed sets $A, B \subset \Omega$, with $A \cap B = \emptyset$, there exists a function $h \in \mathcal{C}$, such that $h|_A = 0$, $h|_B = 1$, and $h(\Omega) \in [0, 1]$, for all $\Omega \in \Omega$.*

Then $\mathcal{C}$ is dense in $C_b^{\mathbb{R}}(\Omega)$, in the norm topology.

PROOF. The key step in the proof will be the following:

Claim: For any $f \in C_b^{\mathbb{R}}(\Omega)$, there exists $g \in \mathcal{C}$, such that

$$\|g - f\| \leq \frac{2}{3}\|f\|.$$

To prove this claim we define

$$\alpha = \inf_{p \in \Omega} f(p) \text{ and } \beta = \sup_{p \in \Omega} f(p),$$

so that $f(p) \in [\alpha, \beta]$, and $\|f\| = \max\{|\alpha|, |\beta|\}$. Define the sets

$$A = f^{-1}\left(\left[\alpha, \frac{2\alpha + \beta}{3}\right]\right) \text{ and } B = f^{-1}\left(\left[\frac{\alpha + 2\beta}{3}, \beta\right]\right).$$

so that both A and B are closed, and $A \cap B = \emptyset$. Use the hypothesis, to find a function $h \in \mathcal{C}$, such that $h|_A = 0$, $h|_B = 1$, and $h(p) \in [0, 1]$, for all $p \in \Omega$. Define the function $g \in \mathcal{C}$ by

$$g = \frac{1}{3}[\alpha 1 + (\beta - \alpha)h].$$

Let us examine the difference $g - f$. Start with some arbitrary point $p \in \Omega$. There are three cases to examine:

CASE I: $p \in A$. In this case we have $h(p) = 0$, so we get $g(p) = \frac{\alpha}{3}$. By the construction of A we also have $\alpha \leq f(p) \leq \frac{2\alpha + \beta}{3}$, so we get

$$\frac{2\alpha}{3} \leq f(p) - g(p) \leq \frac{\alpha + \beta}{3}.$$

CASE II: $p \in B$. In this case we have $h(p) = 1$, so we get $g(p) = \frac{\beta}{3}$. We also have $\frac{2\beta + \alpha}{3} \leq f(p) \leq \beta$, so we get

$$\frac{\alpha + \beta}{3} \leq f(p) - g(p) \leq \frac{2\beta}{3}.$$

CASE III: $p \in \Omega \setminus (A \cup B)$. In this case we have $0 \leq h(p) \leq 1$, so we get $\frac{\alpha}{3} \leq g(p) \leq \frac{\beta}{3}$, and $\frac{2\alpha + \beta}{3} < f(p) < \frac{\alpha + 2\beta}{3}$. In particular we get

$$f(p) - g(p) > \frac{2\alpha + \beta}{3} - \frac{\beta}{3} = \frac{2\alpha}{3};$$

$$f(p) - g(p) < \frac{\alpha + 2\beta}{3} - \frac{\alpha}{3} = \frac{2\beta}{3}.$$

Since $\frac{2\alpha}{3} \leq \frac{\alpha + \beta}{3} \leq \frac{2\beta}{3}$, we see that in all three cases we have

$$\frac{2\alpha}{3} \leq f(p) - g(p) \leq \frac{2\beta}{3},$$

so we get

$$\frac{2\alpha}{3} \leq \inf_{p \in \Omega} [f(p) - g(p)] \leq \sup_{p \in \Omega} [f(p) - g(p)] \leq \frac{2\beta}{3},$$

so we indeed get the desired inequality

$$\|g - f\| \leq \frac{2}{3}\|f\|.$$

Having proven the Claim, we now prove the density of $\mathcal{C}$ in $C_b^{\mathbb{R}}(\Omega)$. Start with some $f \in C_b^{\mathbb{R}}(\Omega)$, and we construct recursively two sequences $(g_n)_{n \geq 1} \subset \mathcal{C}$ and $(f_n)_{n \geq 1} \subset C_b^{\mathbb{R}}(\Omega)$, as follows. Set $f_1 = f$. Apply the Claim to find $g_1 \in \mathcal{C}$ such that

$$\|g_1 - f\| \leq \frac{2}{3}\|f_1\|.$$

Once $f_1, f_2, \dots, f_n$ and $g_1, g_2, \dots, g_n$ have been constructed, we set

$$f_{n+1} = g_n - f_n,$$

and we choose $g_{n+1} \in \mathcal{C}$ such that

$$\|g_{n+1} - f_{n+1}\| \leq \frac{2}{3}\|f_{n+1}\|.$$

It is clear, by construction, that

$$\|f_n\| \leq \left(\frac{2}{3}\right)^{n-1} \|f\|, \quad \forall n \geq 1.$$

Consider the sequence $(s_n)_{n \geq 1} \subset \mathcal{C}$ of partial sums, defined by

$$s_n = g_1 + g_2 + \dots + g_n, \quad \forall n \geq 1.$$

Using the equalities

$$g_n = f_n - f_{n+1}, \quad \forall n \geq 1,$$

we get

$$s_n - f = g_1 + g_2 + \dots + g_n - f_1 = f_{n+1},$$

so we have

$$\|s_n - f\| \leq \left(\frac{2}{3}\right)^n \|f\|, \quad \forall n \geq 1,$$

which clearly give $f = \lim_{n \rightarrow \infty} s_n$, so f indeed belongs to the closure $\bar{\mathcal{C}}$. $\square$

We are now in position to prove the following

THEOREM 5.1 (Tietze Extension Theorem). *Let Ω be a normal topological space, let $T \subset \Omega$ be a closed subset. Let $f : T \rightarrow [0, 1]$ be a continuous function. (Here Y is equipped with the induced topology.) There there exists a continuous function $g : \Omega \rightarrow [0, 1]$ such that $g|_T = f$.*

PROOF. Let us introduce the Banach space setting that will make the proof clearer. We consider the Banach spaces $C^{\mathbb{R}}(\Omega)$ and $C_b^{\mathbb{R}}(T)$. To avoid any confusion, the norms on these Banach spaces will be denoted by $\|\cdot\|_{\Omega}$ and $\|\cdot\|_T$. If we define the restriction map

$$R : C_b^{\mathbb{R}}(\Omega) \ni g \longmapsto g|_T \in C_b^{\mathbb{R}}(T),$$

then R is obviously linear and continuous.

We define the subspace $\mathcal{C} = R(C_b^{\mathbb{R}}(\Omega)) \subset C_b^{\mathbb{R}}(T)$.

Claim: *For every $f \in \mathcal{C}$, there exists some $g \in C_b^{\mathbb{R}}(\Omega)$ such that $f = Rg$, and*

$$\inf_{q \in T} f(q) \leq g(p) \leq \sup_{q \in T} f(q), \quad \forall p \in \Omega.$$

To prove this fact, we start first with some arbitrary $g_0 \in C_b^{\mathbb{R}}(\Omega)$, such that $f = Rg_0 = g_0|_Y$. Put

$$\alpha = \inf_{q \in T} f(q) \text{ and } \beta = \sup_{q \in T} f(q),$$

so that $\|f\|_T = \max\{|\alpha|, |\beta|\}$. Define the function $\theta : \mathbb{R} \rightarrow [\alpha, \beta]$ by

$$\theta(t) = \begin{cases} \alpha & \text{if } t < \alpha \\ t & \text{if } \alpha \leq t \leq \beta \\ \beta & \text{if } t > \beta \end{cases}$$

Then obviously θ is continuous, and the composition $g = \theta \circ g_0 : \Omega \rightarrow [\alpha, \beta]$ will still satisfy $g|_T = f$, and we will clearly have

$$\alpha \leq g(p) \leq \beta, \quad \forall p \in \Omega.$$

Having proven the Claim, we are going to prove that $\mathcal{C}$ is *closed*. We do this by showing that $\mathcal{C}$ is a *Banach space*, in the norm $\|\cdot\|_Y$. To get this, we use Remark ???. Start with some sequence $(f_n)_{n \geq 1} \subset \mathcal{C}$, with $\sum_{n=1}^{\infty} \|f_n\|_T < \infty$. Apply the Claim, to construct a sequence $(g_n)_{n \geq 1} \subset C_b^{\mathbb{R}}(\Omega)$, such that $Rg_n = f_n$, and

$$\inf_{q \in T} f_n(q) \leq g_n(p) \leq \sup_{q \in T} f_n(q), \quad \forall p \in \Omega,$$

for each $n \geq 1$. Notice that this forces

$$\|g_n\|_{\Omega} \leq \|f_n\|_T, \quad \forall n \geq 1.$$

Define the sequences of partial sums $(h_n)_{n \geq 1} \subset \mathcal{C}$ and $(s_n)_{n \geq 1} \subset C_b^{\mathbb{R}}(\Omega)$, by

$$h_n = f_1 + \cdots + f_n \text{ and } s_n = g_1 + \cdots + g_n, \quad \forall n \geq 1.$$

Since

$$\sum_{n=1}^{\infty} \|g_n\|_{\Omega} \leq \sum_{n=1}^{\infty} \|f_n\|_T < \infty,$$

and $C_b^{\mathbb{R}}(\Omega)$ is a Banach space, it follows that the sequence $(s_n)_{n \geq 1}$ is convergent to some point $g \in C_b^{\mathbb{R}}(\Omega)$. Since $R : C_b^{\mathbb{R}}(\Omega) \rightarrow C_b^{\mathbb{R}}(T)$ is linear and continuous, we will have

$$Rs = \lim_{n \rightarrow \infty} [Rg_1 + \cdots + Rg_n] = \lim_{n \rightarrow \infty} [f_1 + \cdots + f_n] = \lim_{n \rightarrow \infty} h_n,$$

which proves that the sequence of partial sums $(h_n)_{n \geq 1} \subset \mathcal{C}$ is indeed convergent to $Rs \in \mathcal{C}$.

Let us remark now that obviously $\mathcal{C}$ contains the constant function $1 = R1$. Using Urysohn Lemma (applied to T) it is clear that $\mathcal{C}$ satisfies the condition (U) in the above lemma. Using the Lemma ??, it follows that $\mathcal{C} = C_b^{\mathbb{R}}(T)$, i.e. R is surjective.

To finish the proof, start with some arbitrary continuous function $f : Y \rightarrow [0, 1]$. Use surjectivity of R , combined with the Claim, to find $g \in C_b^{\mathbb{R}}(\Omega)$, such that $Rg = f$, and

$$\inf_{q \in T} f(q) \leq g(p) \leq \sup_{q \in T} f(q), \quad \forall p \in \Omega.$$

This clearly forces g to take values in $[0, 1]$. □

Next we concentrate on the case when Ω is a compact Hausdorff space. In this case, every continuous function $F : \Omega \rightarrow \mathbb{K}$ is automatically bounded, and the Banach space $C_b^{\mathbb{K}}(\Omega)$ will be denoted simply by $C^{\mathbb{K}}(\Omega)$. (When $\mathbb{K} = \mathbb{C}$ this space will be denoted simply by $C(\Omega)$.)

THEOREM 5.2 (Dini). *Let K be a compact Hausdorff space, let $(f_n)_{n \geq 1} \subset C^{\mathbb{R}}(K)$ be a monotone sequence. Assume there is some $f \in C^{\mathbb{R}}(K)$, such that*

$$\lim_{n \rightarrow \infty} f_n(p) = f(p), \quad \forall p \in K.$$

Then $\lim_{n \rightarrow \infty} f_n = f$, in the norm topology.

PROOF. Replacing f_n with $f_n - f$, we can assume that $\lim_{n \rightarrow \infty} f_n(p) = 0$, $\forall p \in K$. Replacing (if necessary) f_n with $-f_n$, we can also assume that the sequence $(f_n)_{n \geq 1}$ is *decreasing*. In particular, each f_n is non-negative.

We need to prove that $\lim_{n \rightarrow \infty} \|f_n\| = 0$. Assume this is not true, so there exists some $\varepsilon > 0$, such that the set

$$M = \{m \in \mathbb{N} : \|f_m\| \geq \varepsilon\}$$

is infinite. For each integer $n \geq 1$, let us define the set

$$F_n = \{p \in K : f_n(p) \geq \varepsilon\}.$$

Then by the definition of M , we have

$$F_m \neq \emptyset, \quad \forall m \in M.$$

Claim: One has the inclusion $F_n \supset F_{n+1}$, $\forall n \geq 1$.

Indeed, if $p \in F_{n+1}$, then

$$\varepsilon \leq f_{n+1}(p) \leq f_n(p),$$

which proves that $p \in F_n$.

Using the claim, plus the fact that the set M is infinite, it follows that, $F_n \neq \emptyset$, $\forall n \geq 1$. (Indeed, if we start with some arbitrary n , then since M is infinite, we can find $m \in M$, with $m \geq n$, and then using the Claim we have $\emptyset \neq F_m \subset F_n$.)

Since K is compact, and the sets $F_1 \supset F_2 \supset \dots$ are closed and non-empty, by the finite intersection property, it follows that

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

But this leads to a contradiction, because if we pick an element $p \in \bigcap_{n=1}^{\infty} F_n$, then we will have $f_n(p) \geq \varepsilon$, $\forall n \geq 1$, and then the equality $\lim_{n \rightarrow \infty} f_n(p) = 0$ is impossible. $\square$

Exercise 1. Define the sequence $(P_n)_{n \geq 1}$ of polynomials, by $P_1(t) = 0$, and

$$P_{n+1}(t) = \frac{1}{2}[t - P_n(t)^2] + P_n(t), \quad \forall n \geq 1.$$

Prove that

$$\lim_{n \rightarrow \infty} \left(\max_{t \in [0,1]} |P_n(t) - \sqrt{t}| \right) = 0.$$

HINT: Define the functions $f_n, f : [0, 1] \rightarrow \mathbb{R}$ by $f_n(t) = P_n(t)$ and $f(t) = \sqrt{t}$. Prove that, for every $t \in [0, 1]$, the sequence $(f_n(t))_{n \geq 1}$ is increasing, bounded, and $\lim_{n \rightarrow \infty} f_n(t) = f(t)$. Then apply Dini's Theorem.

THEOREM 5.3 (Stone-Weierstrass). *Let K be a compact Hausdorff space. Let $\mathcal{A} \subset C^{\mathbb{R}}(K)$ be a unital subalgebra, i.e.*

- $\mathcal{A} \ni 1$ - the constant function 1;
- $\mathcal{A}$ is a linear subspace;
- if $f, g \in \mathcal{A}$, then $fg \in \mathcal{A}$.

Assume $\mathcal{A}$ separates the points of K , i.e. for any $p, q \in K$, with $p \neq q$, there exists $f \in \mathcal{A}$ such that $f(p) \neq f(q)$.

Then $\mathcal{A}$ is dense in $C^{\mathbb{R}}(K)$, in the norm topology.

PROOF. Let $\mathcal{C}$ denote the closure of $\mathcal{A}$. Remark that $\mathcal{C}$ is again a unital subalgebra and it still separates the points.

The proof will eventually use the Urysohn density Lemma. Before we get to that point, we need several preparations.

STEP 1. If $f \in \mathcal{C}$, then $|f| \in \mathcal{C}$.

To prove this fact, we define $g = f^2 \in \mathcal{C}$, and we set $h = \|g\|^{-1}g$, so that $h \in \mathcal{C}$, and $h(p) \in [0, 1]$, for all $p \in K$. Let $P_n(t)$, $n \geq 1$ be the polynomials defined in the above exercise. The functions $h_n = P_n \circ h$, $n \geq 1$ are clearly all in $\mathcal{C}$. By the above Exercise, we clearly get

$$\lim_{n \rightarrow \infty} \left(\max_{p \in K} |h_n(p) - \sqrt{h(p)}| \right) = 0,$$

which means that $\lim_{n \rightarrow \infty} h_n = \sqrt{h}$, in the norm topology. In particular, $\sqrt{h}$ belongs to $\mathcal{C}$. Obviously we have

$$\sqrt{h} = \|f\|^{-1} \cdot |f|,$$

so $|f|$ indeed belongs to $\mathcal{C}$.

STEP 2: Given two functions $f, g \in \mathcal{C}$, the continuous functions $\max\{f, g\}$ and $\min\{f, g\}$ both belong to $\mathcal{C}$.

This follows immediately from Step 1, and the equalities

$$\max\{f, g\} = \frac{1}{2}(f + g + |f - g|) \text{ and } \min\{f, g\} = \frac{1}{2}(f + g - |f - g|).$$

STEP 3: For any two points $p, q \in K$, $p \neq q$, there exists $h \in \mathcal{C}$, such that $h(p) = 0$, $h(q) = 1$, and $h(s) \in [0, 1]$, $\forall s \in K$.

Use the assumption on $\mathcal{A}$, to find first a function $f \in \mathcal{A}$, such that $f(p) \neq f(q)$. Put $\alpha = f(p)$ and $\beta = f(q)$, and define

$$g = \frac{1}{\beta - \alpha}(f - \alpha 1).$$

The function g still belongs to $\mathcal{A}$, but now we have $g(p) = 0$ and $g(q) = 1$. Define the function $h = \min\{g^2, 1\}$. By Step 3, $h \in \mathcal{C}$, and it clearly satisfies the required properties.

STEP 4: Given a closed subset $A \subset K$, and a point $p \in K \setminus A$, there exists a function $h \in \mathcal{C}$, such that $h(p) = 0$, $h|_A = 1$, and $h(q) \in [0, 1]$, $\forall q \in K$.

For every $q \in A$, we use Step 3 to find a function $h_q \in \mathcal{C}$, such that $h_q(p) = 0$, $h_q(q) = 1$, and $h_q(s) \in [0, 1]$, $\forall s \in K$, and we define the open set

$$D_q = \{s \in K : h_q(s) > 0\}.$$

Using the compactness of A , we find points $q_1, \dots, q_n \in A$, such that

$$A \subset D_{q_1} \cup \dots \cup D_{q_n}.$$

Define the function $f = h_{q_1} + \dots + h_{q_n} \in \mathcal{C}$, so that $f(p) = 0$, $f(q) > 0$, for all $q \in A$, and $f(s) \geq 0$, $\forall s \in K$. If we define

$$m = \min_{q \in A} f(q),$$

then the function $g = m^{-1}f$ again belongs to $\mathcal{C}$, and it satisfies $g(p) = 0$, $g(q) \geq 1$, $\forall q \in A$, and $g(s) \geq 0$, $\forall s \in K$. Finally, the function

$$h = \min\{g, 1\}$$

will satisfy the required properties.

STEP 5: *Given closed sets $A, B \subset K$ with $A \cap B = \emptyset$, there exists $h \in \mathcal{C}$, such that $h|_A = 1$, $h|_B = 0$, and $h(q) \in [0, 1]$, $\forall q \in K$.*

Use Step 4, to find for every $p \in B$, a function $h_p \in \mathcal{C}$, such that $h_p|_B = 1$, $h_p(p) = 0$, and $h_p(s) \in [0, 1]$, $\forall s \in K$. Put $g_p = 1 - h_p$, so that $g_p(p) = 1$, $g_p|_B = 0$, and $g_p(s) \in [0, 1]$, $\forall s \in K$. We then proceed as above. For each $p \in A$ we define the open set

$$D_p = \{s \in K : g_p(s) > 0\}.$$

Using the compactness of A , we find points $p_1, \dots, p_n \in A$, such that

$$A \subset D_{p_1} \cup \dots \cup D_{p_n}.$$

Define the function $f = g_{p_1} + \dots + g_{p_n} \in \mathcal{C}$, so that $f|_B = 0$, $f(q) > 0$, for all $q \in A$, and $f(s) \geq 0$, $\forall s \in K$. If we define

$$m = \min_{q \in A} f(q),$$

then the function $g = m^{-1}f$ again belongs to $\mathcal{C}$, and it satisfies $g|_B = 0$, $g(q) \geq 1$, $\forall q \in A$, and $g(s) \geq 0$, $\forall s \in K$. Finally, the function

$$h = \min\{g, 1\}$$

will satisfy the required properties.

We now apply the Urysohn density Lemma, to conclude that $\mathcal{C}$ is dense in $C^{\mathbb{R}}(K)$. Since $\mathcal{C}$ is already closed, this forces $\mathcal{C} = C^{\mathbb{R}}(K)$, i.e. $\mathcal{A}$ is dense in $C^{\mathbb{R}}(K)$. $\square$

COROLLARY 5.1 (Complex version of Stone-Weierstrass Theorem). *Let K be a compact Hausdorff space. Let $\mathcal{A} \subset C(K)$ be a unital subalgebra, which satisfies;*

- *if $f \in \mathcal{A}$, then $\overline{f} \in \mathcal{A}$.*

Assume $\mathcal{A}$ separates the points of K . Then $\mathcal{A}$ is dense in $C(K)$, in the norm topology.

PROOF. Consider the sub-algebra

$$\mathcal{A}_{\mathbb{R}} = \{f \in \mathcal{A} : f = \overline{f}\}.$$

It is clear that

$$\mathcal{A} = \mathcal{A}_{\mathbb{R}} + i\mathcal{A}_{\mathbb{R}},$$

and $\mathcal{A}_{\mathbb{R}}$ is a unital sub-algebra of $C^{\mathbb{R}}(K)$, which separates the points of K . Using the real version, we know that $\mathcal{A}_{\mathbb{R}}$ is dense in $C^{\mathbb{R}}(K)$. Then $\mathcal{A}$ is clearly dense in $C(K)$. $\square$

EXAMPLE 5.1. Consider the unit disk

$$\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\},$$

and let $\overline{\mathbb{D}}$ denote its closure. Consider the algebra $\mathcal{A} \subset C(\overline{\mathbb{D}})$ consisting of all polynomial functions. Notice that, although $\mathcal{A}$ is unital and separates the points of $\mathbb{D}$, it does not have the property

$$f \in \mathcal{A} \Rightarrow \overline{f} \in \mathcal{A}.$$

In fact, one way to see that this property fails is by inspecting the closure of $\mathcal{A}$ in $C(\overline{\mathbb{D}})$. This closure is denoted by $\mathbf{A}(\mathbb{D})$ and is called the *disk algebra*. The main feature of $\mathbf{A}(\mathbb{D})$ is the following:

*Exercise 2**. Prove that

$$\mathbf{A}(\mathbb{D}) = \{f : \overline{\mathbb{D}} \rightarrow \mathbb{C} : f \text{ continuous, and } f|_{\mathbb{D}} \text{ holomorphic}\}.$$

We now examine the topological dual of $C(K)$.

NOTATIONS. Let K be a compact Hausdorff space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. We define the space

$$\mathcal{M}^{\mathbb{K}}(K) = C^{\mathbb{K}}(K)^* = \{\phi : C^{\mathbb{K}}(K) \rightarrow \mathbb{K} : \phi \text{ } \mathbb{K}\text{-linear continuous}\}.$$

The unit ball will be denoted by $\mathcal{M}^{\mathbb{K}}(K)_1$. When $\mathbb{K} = \mathbb{C}$, the superscript $\mathbb{C}$ will be omitted from the notation.

REMARKS 5.1. Let K be a compact Hausdorff space. The space $\mathcal{M}(K) = C(K)^*$ carries a natural involution, defined as follows. For $\phi \in \mathcal{M}(K)$, we define the map $\phi^* : C(K) \rightarrow \mathbb{C}$ by

$$\phi^*(f) = \overline{\phi(f)}, \quad \forall f \in C(K).$$

For every $\phi \in \mathcal{M}(K)$, the map $\phi^* : C(K) \rightarrow \mathbb{C}$ is again linear, continuous, and has

$$\|\phi^*\| = \|\phi\|.$$

The map ϕ^* will be called the *adjoint of ϕ* . We used the term *involution*, because the map

$$\mathcal{M}(K) \ni \phi \longmapsto \phi^* \in \mathcal{M}(K)$$

has the following properties:

- $(\phi^*)^* = \phi, \forall \phi \in \mathcal{M}(K)$;
- $(\phi + \psi)^* = \phi^* + \psi^*, \forall \phi, \psi \in \mathcal{M}(K)$;
- $(\lambda\phi)^* = \overline{\lambda}\phi^*, \forall \phi \in \mathcal{M}(K), \lambda \in \mathbb{C}$.

If we define the space of self-adjoint maps

$$\mathcal{M}^{sa}(K) = \{\phi \in \mathcal{M}(K) : \phi^* = \phi\},$$

then is clear that, for any $\phi \in \mathcal{M}^{sa}(K)$, the restriction $\phi|_{C^{\mathbb{R}}(K)}$ is real-valued. In fact, for $\phi \in \mathcal{M}(K)$, one has

$$\phi^* = \phi \iff \phi|_{C^{\mathbb{R}}(K)} \text{ is real-valued.}$$

Moreover, one has a map

$$(1) \quad \mathcal{M}^{sa}(K) \ni \phi \longmapsto \phi|_{C^{\mathbb{R}}(K)} \in \mathcal{M}^{\mathbb{R}}(K),$$

which is an *isomorphism of $\mathbb{R}$ -vector spaces*. The inverse of this map is defined as follows. Start with some $\phi \in \mathcal{M}^{\mathbb{R}}(K)$, i.e. $\phi : C^{\mathbb{R}}(K) \rightarrow \mathbb{R}$ is $\mathbb{R}$ -linear and continuous, and we define $\hat{\phi} : C(K) \rightarrow \mathbb{C}$ by

$$\hat{\phi}(f) = \phi(\operatorname{Re} f) + i\phi(\operatorname{Im} f), \quad \forall f \in C(K).$$

It turns out that $\hat{\phi}$ is again linear, continuous, and self-adjoint. Moreover, the correspondence

$$\mathcal{M}^{\mathbb{R}}(K) \ni \phi \longmapsto \hat{\phi} \in \mathcal{M}^{sa}(K)$$

is the inverse of (1).

PROPOSITION 5.2. *Let K be a compact Hausdorff space. Then the map*

$$\mathcal{M}^{sa}(K) \ni \phi \longmapsto \phi|_{C(K)} \in \mathcal{M}^{\mathbb{R}}(K)$$

is isometric. Moreover, when the two spaces are equipped with the w^ topology, this map is a homeomorphism.*

PROOF. To prove the first statement, fix $\phi \in \mathcal{M}^{sa}(K)$. It is obvious that $\|\phi|_{C(K)}\| \leq \|\phi\|$. To prove the other inequality, fix for the moment $\varepsilon > 0$, and choose $f \in C(K)$ such that $\|f\| \leq 1$, and

$$|\phi(f)| \geq \|\phi\| - \varepsilon.$$

Choose a complex number λ with $|\lambda| = 1$, such that

$$|\phi(f)| = \lambda \phi(f) = \phi(\lambda f).$$

If we write $\lambda f = g + ih$, with $g, h \in C(K)$, then using the fact that ϕ is self-adjoint, we will have

$$|\phi(f)| = \phi(g).$$

Since $\|g\| \leq \|\lambda f\| = \|f\| \leq 1$, we will get

$$|\phi(f)| \leq \|\phi|_{C(K)}\|,$$

so our choice of f will give

$$\|\phi\| - \varepsilon \leq \|\phi|_{C(K)}\|.$$

Since this holds for all $\varepsilon > 0$, we get

$$\|\phi\| \leq \|\phi|_{C(K)}\|.$$

The w^* continuity (both ways) is obvious. $\square$

CONVENTION. From now on, we will identify the space $\mathcal{M}^{\mathbb{R}}(K)$ with $\mathcal{M}^{sa}(K)$.

PROPOSITION 5.3. *Let K be a compact Hausdorff space. For every $p \in K$, let $\gamma_p : C(K) \rightarrow \mathbb{C}$ be the map*

$$\gamma_p : C(K) \ni f \longmapsto f(p) \in \mathbb{C}.$$

- (i) *For every $p \in K$, the maps γ_p and $\gamma_p^{\mathbb{R}} = \gamma_p|_{C(K)} : C(K) \rightarrow \mathbb{R}$ are linear and continuous.*
- (ii) *For every $p \in K$, one has $\|\gamma_p\| = \|\gamma_p^{\mathbb{R}}\| = 1$.*
- (ii) *The maps*

$$\Gamma_K : K \ni p \longmapsto \gamma_p \in \mathcal{M}(K)_1$$

$$\Gamma_K^{\mathbb{R}} : K \ni p \longmapsto \gamma_p^{\mathbb{R}} \in \mathcal{M}^{\mathbb{R}}(K)_1$$

are injective and continuous, when the target spaces $\mathcal{M}(K)_1$ and $\mathcal{M}^{\mathbb{R}}(K)_1$ are equipped with the w^ topology.*

PROOF. (i)-(ii). The fact that γ_p is $\mathbb{C}$ -linear is obvious. This will also give the $\mathbb{R}$ -linearity of $\gamma_p^{\mathbb{R}}$. The continuity follows from the obvious inequality

$$|\gamma_p(f)| = |f(p)| \leq \max_{q \in K} |f(q)| = \|f\|, \quad \forall f \in C(K).$$

Among other things, the above inequality also proves

$$\|\gamma_p\| \leq 1 \text{ and } \|\gamma_p^{\mathbb{R}}\| \leq 1.$$

The fact that we have in fact equalities follows from $\gamma_p(1) = 1$.

(iii) Let us first prove the injectivity. Assume we have two point $p, q \in K$, with $p \neq q$. Use Urysohn Lemma to find $f : K \rightarrow [0, 1]$ continuous, such that $f(p) = 0$ and $f(q) = 1$. Then $f \in C^{\mathbb{R}}(K)$ and $\gamma_p^{\mathbb{R}}(f) = f(p) = 0$, and $\gamma_q^{\mathbb{R}}(f) = f(q) = 1$, so we indeed have $\gamma_p^{\mathbb{R}} \neq \gamma_q^{\mathbb{R}}$. (This will also imply $\gamma_p \neq \gamma_q$.)

To prove the continuity of the maps $\Gamma_K : K \rightarrow \mathcal{M}(K)_1$ and $\Gamma_K^{\mathbb{R}} : K \rightarrow \mathcal{M}^{\mathbb{R}}(K)_1$, we need to prove the continuity of the maps $\epsilon_f \circ \Gamma_K : K \rightarrow \mathbb{C}$, $f \in C(K)$, and of the maps $\epsilon_f \circ \Gamma_K^{\mathbb{R}} : K \rightarrow \mathbb{R}$, $f \in C^{\mathbb{R}}(K)$. (Recall that $\epsilon_f(\phi) = \phi(f)$, $\forall \phi \in C^{\mathbb{K}}(K)^*$.) Notice however that we have in fact equalities

$$\begin{aligned}\epsilon_f \circ \Gamma_K &= f, \quad \forall f \in C(K), \\ \epsilon_f \circ \Gamma_K^{\mathbb{R}} &= f, \quad \forall f \in C^{\mathbb{R}}(K),\end{aligned}$$

so the desired continuity is automatic. $\square$

COROLLARY 5.2. *With the above notations, the spaces*

$$\Gamma(K) = \{\gamma_p : p \in K\} \subset \mathcal{M}(K)_1 \text{ and } \Gamma^{\mathbb{R}}(K) = \{\gamma_p^{\mathbb{R}} : p \in K\} \subset \mathcal{M}^{\mathbb{R}}(K)_1$$

are w^ compact, and the maps*

$$\Gamma_K : K \rightarrow \Gamma(K) \text{ and } \Gamma_K^{\mathbb{R}} : K \rightarrow \Gamma^{\mathbb{R}}(K)$$

are homeomorphisms.

Here is an interesting application of the above result to topology.

THEOREM 5.4 (Urysohn Metrization Theorem). *Let K be a compact Hausdorff space. The following are equivalent:*

- (i) *K is metrizable;*
- (ii) *K is second countable, i.e. the topology has a countable base;*
- (iii) $_{\mathbb{R}}$ *the Banach space $C^{\mathbb{R}}(K)$ is separable;*
- (iii) $_{\mathbb{C}}$ *the Banach space $C(K)$ is separable.*

PROOF. (i) $\Rightarrow$ (ii). We already know this fact. (See the section on metric spaces).

(ii) $\Rightarrow$ (iii) $_{\mathbb{R}}$. Assume K is second countable. Fix a countable base $\{D_n : n \in \mathbb{N}\}$ for the topology. Consider the countable set

$$\Delta = \{(m, n) \in \mathbb{N}^2 : \overline{D_m} \cap \overline{D_n} = \emptyset\}.$$

Claim: *For any two points $p, q \in K$, with $p \neq q$, there exists a pair $(m, n) \in \Delta$ with $p \in D_m$ and $q \in D_n$.*

Indeed, since K is Hausdorff, there exist open sets $U_0, V_0 \subset K$ with $p \in U_0$, $q \in V_0$, and $U_0 \cap V_0 = \emptyset$. Since K is (locally) compact, there exist open sets $U, V \subset K$, such that $p \in U \subset \overline{U} \subset U_0$ and $q \in V \subset \overline{V} \subset V_0$. Finally, since $\{D_n : n \in \mathbb{N}\}$ is a basis for the topology, there exist $m, n \in \mathbb{N}$ such that $p \in D_m \subset U$ and $q \in D_n \subset V$. Then clearly we have $\overline{D_m} \subset \overline{U} \subset U_0$, and $\overline{D_n} \subset \overline{V} \subset V_0$, which forces $\overline{D_m} \cap \overline{D_n} = \emptyset$.

Having proven the Claim, for every pair $(m, n) \in \Delta$ we choose (use Urysohn Lemma) a continuous function $h_{mn} : K \rightarrow [0, 1]$ such that $h_{mn}|_{\overline{D_m}} = 0$ and $h_{mn}|_{\overline{D_n}} = 1$, and we define the countable family

$$\mathcal{F} = \{h_{mn} : (m, n) \in \Delta\}.$$

Using the Claim, we know that $\mathcal{F}$ separates the points of K . We set

$$\mathcal{P} = \{h \in C^{\mathbb{R}}(K) : h \text{ is a finite product of functions in } \mathcal{F}\}.$$

Notice that $\mathcal{P}$ is still countable, it also separates the points of K , but also has the property:

$$f, g \in \mathcal{P} \Rightarrow fg \in \mathcal{P}.$$

If we define

$$\mathcal{A} = \text{Span}(\{1\} \cup \mathcal{P}),$$

then $\mathcal{A} \subset C^{\mathbb{R}}(K)$ satisfies the hypothesis of the Stone-Weierstrass Theorem, hence $\mathcal{A}$ is dense in $C^{\mathbb{R}}(K)$. Notice that if we define

$$\mathcal{A}_{\mathbb{Q}} = \text{Span}_{\mathbb{Q}}(\{1\} \cup \mathcal{P}),$$

i.e. the set of linear combinations of elements in $\{1\} \cup \mathcal{P}$ with *rational* coefficients, then clearly $\mathcal{A}_{\mathbb{Q}}$ is dense in $\mathcal{A}$, and so $\mathcal{A}_{\mathbb{Q}}$ is dense in $C^{\mathbb{R}}(K)$. But now we are done, since $\mathcal{A}_{\mathbb{Q}}$ is obviously countable.

(iii $_{\mathbb{R}}$) $\Rightarrow$ (iii $_{\mathbb{C}}$). Assume $C^{\mathbb{R}}(K)$ is separable. Let $\mathcal{S} \subset C^{\mathbb{R}}(K)$ be a countable dense set. Then the set

$$\mathcal{S} + i\mathcal{S} = \{f + ig : f, g \in \mathcal{S}\}$$

is clearly countable, and dense in $C(K)$.

(iii $_{\mathbb{C}}$) $\Rightarrow$ (i). Assume $C(K)$ is separable. By the results from the previous section, it follows that, when equipped with the w^* topology, the compact space $\mathcal{M}(K)_1$ is metrizable. Then the compact subset $\Gamma(K) \subset \mathcal{M}(K)_1$ is also metrizable. Since K is homeomorphic to $\Gamma(K)$, it follows that K itself is metrizable. $\square$

DEFINITION. Let K be a compact Hausdorff space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. A $\mathbb{K}$ -linear map $\phi : C^{\mathbb{K}}(K) \rightarrow \mathbb{K}$ is said to be *positive*, if it has the property

$$f \in C^{\mathbb{R}}(K), f \geq 0 \Rightarrow \phi(f) \geq 0.$$

PROPOSITION 5.4 (Automatic continuity for positive linear maps). *Let K be a compact Hausdorff space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. Any positive $\mathbb{K}$ -linear map $\phi : C^{\mathbb{K}}(K) \rightarrow \mathbb{K}$ is continuous. Moreover, one has the equality $\|\phi\| = \phi(1)$.*

PROOF. In the case when $\mathbb{K} = \mathbb{C}$, it suffices to prove that $\phi|_{C^{\mathbb{R}}(K)}$ is continuous. Therefore, it suffices to prove the statement for $\mathbb{K} = \mathbb{R}$. Start with some arbitrary $f \in C^{\mathbb{R}}(K)$, and define the function $f_{\pm} \in C^{\mathbb{R}}(K)$ by

$$f_+ = \max\{f, 0\} \text{ and } f_- = \max\{-f, 0\},$$

so that $f_{\pm} \geq 0$, $f = f_+ - f_-$, and $\|f\| = \max\{\|f_+\|, \|f_-\|\}$. On the one hand, by positivity, we have the inequalities $\phi(f_{\pm}) \geq 0$, so we get

$$-\phi(f_-) \leq \phi(f_+) - \phi(f_-) \leq \phi(f_+),$$

which give

$$(2) \quad |\phi(f)| = |\phi(f_+) - \phi(f_-)| \leq \max\{\phi(f_+), \phi(f_-)\}.$$

On the other hand, we have

$$\|f_{\pm}\| \cdot 1 - f_{\pm} \geq 0,$$

so by positivity we get

$$\|f_{\pm}\| \cdot \phi(1) \geq \phi(f_{\pm}).$$

Using this in (2) gives

$$|\phi(f)| \leq \phi(1) \cdot \max\{\|f_+\|, \|f_-\|\} = \phi(1) \cdot \|f\|.$$

Since this holds for all $f \in C^{\mathbb{R}}(K)$, the continuity of ϕ follows, together with the estimate

$$\|\phi\| \leq \phi(1).$$

Since $\phi(1) \leq \|\phi\| \cdot \|1\| = \|\phi\|$, the desired norm equality follows. $\square$

NOTATIONS. Let K be a compact Hausdorff space. We define

$$\begin{aligned}\mathcal{M}_+^{\mathbb{K}}(K) &= \{\phi : C^{\mathbb{K}}(K) \rightarrow \mathbb{K} : \phi \text{ } \mathbb{K}\text{-linear, positive}\}; \\ \mathcal{M}_+^{\mathbb{K}}(K)_1 &= \{\phi \in \mathcal{M}_+^{\mathbb{K}}(K) : \|\phi\| \leq 1\} = \mathcal{M}_+^{\mathbb{K}}(K) \cap \mathcal{M}^{\mathbb{K}}(K)_1.\end{aligned}$$

When $\mathbb{K} = \mathbb{C}$, the superscript $\mathbb{C}$ will be omitted.

REMARKS 5.2. Let K be a compact Hausdorff space. We have the inclusion $\mathcal{M}_+(K) \subset \mathcal{M}^{sa}(K)$. Indeed, if we start with $\phi \in \mathcal{M}_+(K)$, then using the fact that every real-valued continuous function $f \in C(K)$ is a difference of non-negative continuous functions $f = f_+ - f_-$, it follows that $\phi(f) = \phi(f_+) - \phi(f_-)$ is a difference of two non-negative (hence real) numbers, so $\phi(f) \in \mathbb{R}$. This implies $\phi^* = \phi$.

The set $\mathcal{M}_+^{\mathbb{R}}(K)$ is w^* -closed in $\mathcal{M}^{\mathbb{R}}(K)$, and the set $\mathcal{M}_+(K)$ is w^* -closed in $\mathcal{M}(K)$. This follows from the fact that, for each $f \in C^{\mathbb{R}}(K)$, the set

$$\mathcal{A}_f^{\mathbb{K}} = \{f \in \mathcal{M}^{\mathbb{K}}(K) : \phi(f) \geq 0\} = \epsilon_f^{-1}([0, \infty))$$

is w^* -closed, being the preimage of a closed set, under a w^* -continuous map. Then everything is a consequence of the equality

$$\mathcal{M}_+^{\mathbb{K}}(K) = \bigcap_{\substack{f \in C^{\mathbb{R}}(K) \\ f \geq 0}} \mathcal{A}_f^{\mathbb{K}}.$$

In particular, the sets $\mathcal{M}_+^{\mathbb{R}}(K)_1$ and $\mathcal{M}_+(K)_1$ are w^* -compact.

The sets $\mathcal{M}_+^{\mathbb{R}}(K)_1$ and $\mathcal{M}_+(K)_1$ are convex.

Using the identification $\mathcal{M}^{\mathbb{R}}(K) \simeq \mathcal{M}^{sa}(K)$, we have the following hierarchies:

$$\begin{array}{ccc} \mathcal{M}_+^{\mathbb{R}}(K) & \simeq & \mathcal{M}_+(K) & \mathcal{M}_+^{\mathbb{R}}(K)_1 & \simeq & \mathcal{M}_+(K)_1 \\ \cap & & \cap & \cap & & \cap \\ \mathcal{M}^{\mathbb{R}}(K) & \simeq & \mathcal{M}^{sa}(K) & \mathcal{M}^{\mathbb{R}}(K)_1 & \simeq & \mathcal{M}^{sa}(K)_1 \\ & & \cap & & & \cap \\ & & \mathcal{M}(K) & & & \mathcal{M}(K)_1 \end{array}$$

with $\simeq$ isometric and w^* -homeomorphism.

PROPOSITION 5.5. *Let K be a compact Hausdorff space. Then one has the equality*

$$\mathcal{M}^{sa}(K)_1 = \text{conv}(\mathcal{M}_+(K)_1 \cup -\mathcal{M}_+(K)_1).$$

(Here conv denotes the convex cover.)

PROOF. Denote the set $\text{conv}(\mathcal{M}_+(K)_1 \cup -\mathcal{M}_+(K)_1)$ simply by $\mathcal{C}$.

Claim: One has the equality:

$$(3) \quad \mathcal{C} = \{t\phi - (1-t)\psi : \phi, \psi \in \mathcal{M}_+(K)_1, t \in [0, 1]\}.$$

In particular, the set $\mathcal{C}$ is w^* -compact.

Denote the set on the right hand side of (3) simply by $\mathcal{D}$. The inclusion $\mathcal{C} \supset \mathcal{D}$ is clear. To prove the inclusion $\mathcal{C} \subset \mathcal{D}$, we only need to prove that $\mathcal{D}$ is convex and it contains $\mathcal{M}_+(K)_1 \cup -\mathcal{M}_+(K)_1$. The second property is clear. The convexity of $\mathcal{D}$ is also clear, being a consequence of the convexity of $\pm\mathcal{M}_+(K)_1$.

The w^* -compactness of $\mathcal{C}$ is then a consequence of the compactness of the product space

$$\mathcal{M}_+(K)_1 \times \mathcal{M}_+(K)_1 \times [0, 1],$$

and of the fact that $\mathcal{C}$ is the range of the continuous map

$$\mathcal{M}_+(K)_1 \times \mathcal{M}_+(K)_1 \times [0, 1] \ni (\phi, \psi, t) \mapsto t\phi - (1-t)\psi \in \mathcal{M}^{sa}(K).$$

Having proven the Claim, we now proceed with the equality

$$\mathcal{M}^{sa}(K)_1 = \mathcal{C}.$$

The inclusion $\supset$ is clear, since $\mathcal{M}^{sa}(K)_1$ is convex, and it contains both $\mathcal{M}_+(K)_1$ and $-\mathcal{M}_+(K)_1$.

We prove the other inclusion by contradiction. Assume there is some $\phi \in \mathcal{M}^{sa}(K)_1 \setminus \mathcal{C}$. Apply Corollary II.4.2 to find some $f \in C(K)$ and a real number α , such that

$$\operatorname{Re} \phi(f) < \alpha \leq \operatorname{Re} \sigma(f), \quad \forall \sigma \in \mathcal{C}.$$

If we take $g = \operatorname{Re} f$, then this gives

$$\phi(g) < \alpha \leq \sigma(g), \quad \forall \sigma \in \mathcal{C}.$$

Notice that $0 \in \mathcal{C}$, so we get $\alpha \leq 0$. If we define $\beta = -\alpha (\geq 0)$, and $h = -g$, the above inequality gives

$$\phi(h) > \beta \geq \sigma(h), \quad \forall \sigma \in \mathcal{C}.$$

Using the obvious inclusions $\pm\Gamma(K) \subset \mathcal{C}$, we get

$$\beta \geq \pm\gamma_p(h) = \pm h(p), \quad \forall p \in K.$$

Since h is real-valued, this will force $\|h\| \leq \beta$. But then we get a contradiction, because we also have

$$\beta < \phi(h) \leq \|\phi\| \cdot \|h\| \leq \|h\|.$$

□

COROLLARY 5.3. *Let K be a compact Hausdorff space, and let $\phi \in \mathcal{M}^{sa}(K)$. Then there exist $\phi_1, \phi_2 \in \mathcal{M}_+(K)$, such that $\phi = \phi_1 - \phi_2$, and $\|\phi\| = \|\phi_1\| + \|\phi_2\|$.*

PROOF. If $\phi \in \mathcal{M}_+(K) \cup -\mathcal{M}_+(K)$, there is nothing to prove. Assume $\phi \notin \mathcal{M}_+(K) \cup -\mathcal{M}_+(K)$, in particular $\phi \neq 0$. We define $\psi = \frac{\phi}{\|\phi\|}$, so that $\psi \in \mathcal{M}^{sa}(K)_1$. Find $\psi_1, \psi_2 \in \mathcal{M}_+(K)_1$ and $t \in [0, 1]$, such that

$$\psi = t\psi_1 - (1-t)\psi_2.$$

Since $\psi \notin \mathcal{M}_+(K) \cup -\mathcal{M}_+(K)$, it follows that $0 < t < 1$. Notice that

$$1 = \|\psi\| = \|t\psi_1 - (1-t)\psi_2\| \leq t\|\psi_1\| + (1-t)\|\psi_2\|.$$

If $\|\psi_1\| < 1$, or $\|\psi_2\| < 1$, then this would imply $t\|\psi_1\| + (1-t)\|\psi_2\| < 1$, which is impossible by the above estimate. This argument proves that we must have $\|\psi_1\| = \|\psi_2\| = 1$. If we define

$$\phi_1 = t\|\phi\|\psi_1 \text{ and } \phi_2 = (1-t)\|\phi\|\psi_2,$$

then $\|\phi_1\| = t\|\phi\|$ and $\|\phi_2\| = (1-t)\|\phi\|$, so we indeed have $\|\phi_1\| + \|\phi_2\| = \|\phi\|$. Obviously ϕ_1 and ϕ_2 are positive, and

$$\phi_1 - \phi_2 = \|\phi\| \cdot [t\psi_1 - (1-t)\psi_2] = \|\phi\| \cdot \psi = \phi.$$

□

PROPOSITION 5.6. *Let K be a compact Hausdorff space. The set*

$$\text{conv}(\Gamma(K) \cup \{0\})$$

is w^ -dense in $\mathcal{M}_+(K)_1$.*

PROOF. Let $\mathcal{C}$ be the w^* -closure of $\text{conv}(\Gamma(K) \cup \{0\})$. It is obvious that $\mathcal{C} \subset \mathcal{M}_+(K)_1$, so we only need to prove the inclusion $\mathcal{M}_+(K)_1 \subset \mathcal{C}$. We do this by contradiction. Assume there exists some $\phi \in \mathcal{M}_+(K)_1 \setminus \mathcal{C}$. Since $\mathcal{C}$ is w^* -closed and convex, there exists some $f \in C(K)$ and a real number α , such that

$$\text{Re } \phi(f) < \alpha \leq \text{Re } \sigma(f), \quad \forall \sigma \in \mathcal{C}.$$

In particular, if we take $h = -\text{Re } f$, and $\beta = -\alpha$, we get

$$(4) \quad \phi(h) > \beta \geq \sigma(h), \quad \forall \sigma \in \mathcal{C}.$$

Since $0 \in \mathcal{C}$, we have $\beta \geq 0$. Since $\Gamma(K) \subset \mathcal{C}$, we also get

$$\beta \geq \gamma_p(h) = h(p), \quad \forall p \in K,$$

which means that $\beta 1 - h \geq 0$. Since ϕ is positive, this will force $\phi(\beta 1 - h) \geq 0$, which gives

$$\phi(h) \leq \phi(\beta 1) = \beta \phi(1) = \beta \|\phi\|.$$

Finally, since $\|\phi\| \leq 1$, this gives

$$\phi(h) \leq \beta,$$

thus contradicting (4). □

The results for the Banach spaces of the form $C(K)$, with K compact Hausdorff space, can be generalized, with suitable modifications, to the situation when K is replaced with a locally compact space. The following result in fact reduces the analysis to the compact case.

THEOREM 5.5. *Let Ω be a locally compact space, and let Ω^β be the Stone-Cech compactification of Ω . Then the restriction map*

$$R : C^\mathbb{K}(\Omega^\beta) \ni f \longmapsto f|_\Omega \in C_b^\mathbb{K}(\Omega)$$

is an isometric linear isomorphism.

PROOF. The linearity is obvious.

Let us show that R is surjective. We show that R is bijective, by exhibiting an inverse for it. For every $h \in C_b^\mathbb{K}(\Omega)$, we consider the compact set

$$K_h = \{z \in \mathbb{K} : |z| \leq \|h\|\},$$

so that we can regard h as a continuous map $\Omega \rightarrow K_h$. We know from the functoriality of the Stone-Cech compactification that there exists a unique continuous map $h^\beta : \Omega^\beta \rightarrow K_h^\beta$, with $h^\beta|_\Omega = h$. Since K_h is compact, we have $K_h^\beta = K_h$. In particular, this gives the inequality

$$(5) \quad |h^\beta(x)| \leq \|h\|, \quad \forall x \in \Omega^\beta.$$

Define the map $T : C_b^{\mathbb{K}}(\Omega) \ni h \mapsto h^\beta \in C^{\mathbb{K}}(\Omega^\beta)$, and let us show that T is an inverse for R . The equality $R \circ T = \text{Id}$ is trivial, by construction. To prove the equality $T \circ R = \text{Id}$, we start with some $f \in C_b^{\mathbb{K}}(\Omega)$, and we consider $h = Rf$. Then $Th = h^\beta$, and since $h^\beta|_\Omega = h = f|_\Omega$, the density of Ω in Ω^β clearly forces $f = h^\beta = Th = T(Rf)$.

The fact that R is isometric is now clear, because on the one hand we clearly have $\|Rf\| \leq \|f\|$, $\forall f \in C^{\mathbb{K}}(\Omega^\beta)$, and on the other hand, by (5), we also have $\|Th\| \leq \|h\|$, $\forall h \in C_b^{\mathbb{K}}(\Omega)$. $\square$

If Ω is a locally compact space, the above result suggests that the space $C_b^{\mathbb{K}}(\Omega)$ is quite “large.” It is then natural to look at smaller spaces.

DEFINITIONS. Let Ω be a locally compact space. If $\mathbb{K}$ is one of the fields $\mathbb{R}$ or $\mathbb{C}$, and $f : \Omega \rightarrow \mathbb{K}$ is a continuous function, we define the *support* of f by

$$\text{supp } f = \overline{\{\omega \in \Omega : f(\omega) \neq 0\}}.$$

We define the space

$$C_c^{\mathbb{K}}(\Omega) = \{f : \Omega \rightarrow \mathbb{K} : f \text{ continuous, with compact support}\}.$$

When $\mathbb{K} = \mathbb{C}$, this space will be denoted simply by $C_c(\Omega)$. Remark that, when equipped with pointwise addition and multiplication, the space $C_c^{\mathbb{K}}(\Omega)$ becomes a $\mathbb{K}$ -algebra. One has obviously the inclusion $C_c^{\mathbb{K}}(\Omega) \subset C_b^{\mathbb{K}}(\Omega)$.

We define $C_0^{\mathbb{K}}(\Omega) = \overline{C_c^{\mathbb{K}}(\Omega)}$, the closure of $C_c^{\mathbb{K}}(\Omega)$ in $C_b^{\mathbb{K}}(\Omega)$. (When $\mathbb{K} = \mathbb{C}$, we will denote this space simply by $C_0(\Omega)$.) The Banach space $C_0^{\mathbb{K}}(\Omega)$ can be regarded as the completion of $C_c^{\mathbb{K}}(\Omega)$. Of course, when Ω is compact, we have the equality $C_0^{\mathbb{K}}(\Omega) = C^{\mathbb{K}}(\Omega)$.

The following result characterizes the Banach space $C_0^{\mathbb{K}}(\Omega)$.

PROPOSITION 5.7. *Let Ω be a locally compact space. For a function $f \in C_b^{\mathbb{K}}(\Omega)$, the following are equivalent:*

- (i) $f \in C_0^{\mathbb{K}}(\Omega)$;
- (ii) for every $\varepsilon > 0$, there exists some compact subset $K_\varepsilon \subset \Omega$, such that

$$\sup_{\omega \in \Omega \setminus K_\varepsilon} |f(\omega)| \leq \varepsilon.$$

PROOF. (i) $\Rightarrow$ (ii). Suppose $f \in C_0^{\mathbb{K}}(\Omega)$, which means that there exists some sequence $(f_n)_{n=1}^\infty \subset C_c^{\mathbb{K}}(\Omega)$, such that $\lim_{n \rightarrow \infty} f_n = f$, in the norm topology in $C_b^{\mathbb{K}}(\Omega)$. Fix some $\varepsilon > 0$, and choose $k \geq 1$, such that $\|f - f_k\| \leq \varepsilon$. If we define $K_\varepsilon = \text{supp } f_k$, then, for every $\omega \in \Omega \setminus K_\varepsilon$, we have $f_k(\omega) = 0$, so the inequality $\|f - f_k\| \leq \varepsilon$ forces $|f(\omega)| \leq \varepsilon$.

(ii) $\Rightarrow$ (i). Suppose f satisfies property (ii). Fix for the moment an integer $n \geq 1$. Use condition (ii) to find a compact subset $K_n \subset \Omega$, such that

$$|f(\omega)| \leq \frac{1}{n}, \quad \forall \omega \in \Omega \setminus K_n.$$

Use Urysohn Lemma to choose some continuous function $h_n : \Omega \rightarrow [0, 1]$, with compact support, such that $h_n|_{K_n} = 1$. Define the function $f_n = h_n f$, so that $f_n \in C_c^{\mathbb{K}}(\Omega)$. If $\omega \in \Omega \setminus K_n$, then, using the inequality $0 \leq h_n \leq 1$, and the choice of K_n , we have

$$|f(\omega) - f_n(\omega)| = |f(\omega)| \cdot [1 - h_n(\omega)] \leq |f(\omega)| \leq \frac{1}{n}.$$

Using the fact that $f_n|_{K_n} = f|_{K_n}$, the above equality proves that $\|f - f_n\| \leq \frac{1}{n}$. This way we have constructed a sequence $(f_n)_{n=1}^\infty \subset C_c^\mathbb{K}(\Omega)$, such that $\lim_{n \rightarrow \infty} f_n = f$, in $C_b^\mathbb{K}(\Omega)$, so by the definition it follows that $f \in C_0^\mathbb{K}(\Omega)$. $\square$

The following establishes an interesting connection with the Alexandrov compactification.

PROPOSITION 5.8. *Let Ω be a locally compact space, which is non-compact, and let $\Omega^\alpha = \Omega \sqcup \{\infty\}$ denote the Alexandrov compactification.*

- (i) *For every function $f \in C_0^\mathbb{K}(\Omega)$, the function $f^\alpha : \Omega^\alpha \rightarrow \mathbb{K}$, defined by $f^\alpha|_\Omega = f$, and $f^\alpha(\infty) = 0$, is continuous.*
- (ii) *The correspondence $U : C_0^\mathbb{K}(\Omega) \ni f \mapsto f^\alpha \in C^\mathbb{K}(\Omega^\alpha)$ is an isometric linear map.*
- (iii) *One has the equality*

$$(6) \quad \text{Ran } U = \{g \in C^\mathbb{K}(\Omega^\alpha) : g(\infty) = 0\}.$$

PROOF. (i). We know that Ω is open in Ω^α , which immediately gives the fact that f^α is continuous at every point $\omega \in \Omega$. So all we need to show is the continuity of f^α at ∞ . This amounts to showing that for every neighborhood N of $f^\alpha(\infty) = 0$ in $\mathbb{K}$, there exists a neighborhood V of ∞ in Ω^α , such that $f^\alpha(V) \subset N$. Start with a neighborhood N of 0, and choose $\varepsilon > 0$, such that the set $B_\varepsilon = \{z \in \mathbb{K} : |z| \leq \varepsilon\}$ is contained in N . Choose some compact set $K_\varepsilon \subset \Omega$, such that

$$\sup_{\omega \in \Omega \setminus K_\varepsilon} |f(\omega)| \leq \varepsilon.$$

Define the set $D = (\Omega \setminus K_\varepsilon) \cup \{\infty\}$. By the definition of the topology on Ω^α , the set D is an open neighborhood of ∞ . We are now done, because we clearly have

$$|f^\alpha(x)| \leq \varepsilon, \quad \forall x \in D,$$

which gives the inclusion $f^\alpha(D) \subset B_\varepsilon \subset N$.

(ii). This part is trivial.

(iii). Denote the right hand side of (6) by $\mathcal{A}$. The inclusion $\text{Ran } U \subset \mathcal{A}$ is trivial, by definition. Conversely, let us start with some $g \in \mathcal{A}$, and let us consider the function $f = g|_\Omega$. Let us show that $f \in C_0^\mathbb{K}(\Omega)$, using Proposition 5.7. Start with some $\varepsilon > 0$, and choose some open neighborhood D_ε of ∞ , in Ω^α , such that

$$|g(x)| \leq \varepsilon, \quad \forall x \in D_\varepsilon.$$

By definition, there exists a compact subset $K_\varepsilon \subset \Omega$, such that $D_\varepsilon = \Omega^\alpha \setminus K_\varepsilon$, so it is immediate that f satisfies condition (ii) from Proposition 5.7. Notice now that, by construction we have $f^\alpha|_\Omega = g|_\Omega$, and $f^\alpha(\infty) = 0 = g(\infty)$, so we indeed get $g = Uf$. $\square$

REMARK 5.3. Let Ω be a locally compact space, which is non-compact. Use the map U defined above, to identify $C_0^\mathbb{K}(\Omega)$ with the subspace $\text{Ran } U \subset C^\mathbb{K}(\Omega^\alpha)$. With this identification, we have the equality

$$C^\mathbb{K}(\Omega^\alpha) = \mathbb{K}1 + C_0^\mathbb{K}(\Omega) = \{\lambda 1 + f : \lambda \in \mathbb{K}, f \in C_0^\mathbb{K}(\Omega)\}.$$

Indeed, if we start with some function $g \in C^\mathbb{K}(\Omega^\alpha)$ and we take $\lambda = g(\infty)$ and $f = g - \lambda 1$, then $f(\infty) = 0$. Note that this argument proves that in fact every $g \in C^\mathbb{K}(\Omega^\alpha)$, can be *uniquely* represented as $g = \lambda 1 + f$, with $\lambda \in \mathbb{K}$, and $f \in C_0^\mathbb{K}(\Omega)$.

We conclude with a couple of generalizations of the various results in this section. The first two ones are proven, the rest are stated as exercises. The following result is a generalization of Proposition 5.4.

PROPOSITION 5.9. *Let Ω be a locally compact space, and let $\phi : C_0^{\mathbb{R}}(\Omega) \rightarrow \mathbb{R}$ be a positive linear map. Then ϕ is continuous, and one has the equality*

$$(7) \quad \|\phi\| = \sup\{\phi(f) : f \in C_0^{\mathbb{R}}(\Omega), 0 \leq f \leq 1\}.$$

PROOF. Let us denote the right hand side of (7) by M . First we show that $M < \infty$. If $M = \infty$, there exists a sequence $(f_n)_{n=1}^{\infty} \subset C_0^{\mathbb{R}}(\Omega)$, such that

$$0 \leq f_n \leq 1 \text{ and } \phi(f_n) \geq 4^n, \quad \forall n \geq 1.$$

Consider then the function $f = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n$. Since $\sum_{n=1}^{\infty} \left\| \frac{1}{2^n} f_n \right\| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, it follows that $f \in C_0^{\mathbb{R}}(\Omega)$. Notice however that, since we obviously have $\frac{1}{2^n} f_n \leq f$, by the positivity of ϕ , we get

$$\phi(f) \geq \phi\left(\frac{1}{2^n} f_n\right) = \frac{1}{2^n} \phi(f_n) \geq 2^n, \quad \forall n \geq 1,$$

which is clearly impossible. Let us show now that ϕ is continuous, by proving the inequality

$$(8) \quad |\phi(f)| \leq M, \quad \forall f \in C_0^{\mathbb{R}}(\Omega), \text{ with } \|f\| \leq 1.$$

Start with some arbitrary function $f \in C_0^{\mathbb{R}}(\Omega)$. The functions $g^{\pm} = |f| \pm f \in C_0^{\mathbb{R}}(\Omega)$, clearly satisfy $g \geq 0$, so we get $\phi(|f| \pm f) \geq 0$, so we get $\phi(|f|) \geq \pm \phi(f)$. This gives $|\phi(f)| \leq \phi(|f|)$, and since $0 \leq |f| \leq 1$, we immediately get (8).

The inequality (8) proves the inequality $\|\phi\| \leq M$. Since we obviously have $M \leq \|\phi\|$, we get in fact the equality (7). $\square$

COROLLARY 5.4. *Let Ω be a locally compact space, which is non-compact, and let Ω^{α} be the Alexandrov compactification of Ω . Using the inclusion $C_0^{\mathbb{R}}(\Omega) \subset C^{\mathbb{R}}(\Omega^{\alpha})$, given by Proposition 5.8, every positive linear map $\phi : C_0^{\mathbb{R}}(\Omega) \rightarrow \mathbb{R}$ can be uniquely extended to a positive linear map $\psi : C^{\mathbb{R}}(\Omega^{\alpha}) \rightarrow \mathbb{R}$, such that $\|\psi\| = \|\phi\|$.*

PROOF. For every $g \in C^{\mathbb{R}}(\Omega^{\alpha})$, we know that there exists a unique $\lambda \in \mathbb{R}$ and $f \in C_0^{\mathbb{R}}(\Omega)$, such that $g = \lambda 1 + f$ (namely $\lambda = g(\infty)$ and $f = g - \lambda 1$). We then define $\psi(g) = \lambda \|\phi\| + \phi(f)$. Notice that $\psi(1) = \|\phi\|$. It is obvious that $\psi : C^{\mathbb{R}}(\Omega^{\alpha}) \rightarrow \mathbb{R}$ is linear, and $\psi|_{C_0^{\mathbb{R}}(\Omega)} = \phi$. Let us show that ψ is positive. Start with some $g \in C^{\mathbb{R}}(\Omega^{\alpha})$ with $g \geq 0$, and let us prove that $\psi(g) \geq 0$. Write $g = \lambda 1 + f$ with $\lambda \in \mathbb{R}$ and $f \in C_0^{\mathbb{R}}(\Omega)$. We know that $\lambda = g(\infty) \geq 0$. If $\lambda = 0$, there is nothing to prove. If $\lambda > 0$, we define the function $h = \lambda^{-1} f \in C_0^{\mathbb{R}}(\Omega)$, so that $g = \lambda(1 + h)$. The positivity of g forces $1 + h \geq 0$, which means if we consider the function $h^- = \max\{-h, 0\} \in C_0^{\mathbb{R}}(\Omega)$, then we have $0 \leq h^- \leq 1$, as well as $h^- + h \geq 0$. Using the above result, this will then give

$$\|\phi\| + \phi(h) \geq \phi(h^-) + \phi(h) = \phi(h^- + h) \geq 0,$$

which means that $\psi(1 + h) \geq 0$. Consequently we also get

$$\psi(g) = \psi(\lambda(1 + h)) = \lambda \psi(h) \geq 0.$$

Having shown the positivity of ψ , we know that

$$\|\psi\| = \psi(1) = \|\phi\|.$$

To prove uniqueness, start with another positive linear map $\xi : C_0^{\mathbb{R}}(\Omega) \rightarrow \mathbb{R}$, such that $\|\xi\| = \|\phi\|$, with $\xi|_{C_0^{\mathbb{R}}(\Omega)} = \phi$. Since ξ is positive, this forces $\xi(1) = \|\xi\| = \|\phi\| = \psi(1)$. But then we have

$$\xi(\lambda 1 + f) = \lambda \|\phi\| + \phi(f) = \psi(\lambda 1 + f), \quad \forall \lambda \in \mathbb{R}, f \in C_0^{\mathbb{R}}(\Omega),$$

which proves that $\xi = \psi$. $\square$

REMARK 5.4. Let Ω be a locally compact space, which is not compact, and let $\phi : C_c^{\mathbb{R}}(\Omega) \rightarrow \mathbb{R}$ be a positive linear map. Then the following are equivalent:

- (i) ϕ is continuous;
- (ii) $\sup \{ \phi(f) : f \in C_c^{\mathbb{R}}(\Omega), 0 \leq f \leq 1 \} < \infty$.

The implication (i) $\Rightarrow$ (ii) is trivial. To prove the implication (ii) $\Rightarrow$ (i) we follow the exact same steps as in the proof of the equality (7) in Proposition 5.9. Denote the quantity in (ii) by M , and using the inequality $|\phi(f)| \leq \phi(|f|)$, we immediately get $|\phi(f)| \leq M$, $\forall f \in C_c^{\mathbb{R}}(\Omega)$, with $\|f\| \leq 1$.

Remark also that if ϕ is as above, then we have in fact the equality

$$\|\phi\| = \sup \{ \phi(f) : f \in C_c^{\mathbb{R}}(\Omega), 0 \leq f \leq 1 \}.$$

The following is a generalization of Corollary 5.3.

PROPOSITION 5.10. *Let Ω be a locally compact space, and let $\phi : C_0^{\mathbb{R}}(\Omega) \rightarrow \mathbb{R}$ be a linear continuous map. Then there exist positive linear maps $\phi_1, \phi_2 : C_0^{\mathbb{R}}(\Omega) \rightarrow \mathbb{R}$, such that $\phi = \phi_1 - \phi_2$, and $\|\phi\| = \|\phi_1\| + \|\phi_2\|$.*

PROOF. If Ω is compact there is nothing to prove (this is Corollary 5.3). Assume Ω is non-compact. Use Hahn-Banach Theorem to find a linear continuous map $\psi : C^{\mathbb{R}}(\Omega^{\alpha}) \rightarrow \mathbb{R}$, with $\|\psi\| = 1$ and $\psi|_{C_0^{\mathbb{R}}(\Omega)} = \phi$. Apply Corollary 5.3 to find two positive linear maps $\psi_1, \psi_2 : C^{\mathbb{R}}(\Omega^{\alpha}) \rightarrow \mathbb{R}$ such that $\psi = \psi_1 - \psi_2$ and $\|\psi\| = \|\psi_1\| + \|\psi_2\|$. Define the positive linear maps $\phi_k = \psi_k|_{C_0^{\mathbb{R}}(\Omega)}$, $k = 1, 2$. We clearly have $\phi = \phi_1 - \phi_2$, and

$$\|\phi_1\| + \|\phi_2\| \leq \|\psi_1\| + \|\psi_2\| = \|\psi\| = \|\phi\| = \|\phi_1 - \phi_2\| \leq \|\phi_1\| + \|\phi_2\|,$$

which forces $\|\phi\| = \|\phi_1\| + \|\phi_2\|$. $\square$

Exercise 3. (Dini's Theorem for locally compact spaces) Let Ω be a locally compact space, let $(f_n)_{n \geq 1} \subset C_0^{\mathbb{R}}(\Omega)$ be a monotone sequence. Assume there is some $f \in C_0^{\mathbb{R}}(\Omega)$, such that

$$\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega), \quad \forall \omega \in \Omega.$$

Then $\lim_{n \rightarrow \infty} f_n = f$, in the norm topology.

Exercise 4. (Stone-Weierstrass Theorems) Let Ω be a locally compact space, which is non-compact, and let $\mathcal{A} \subset C_0^{\mathbb{K}}(\Omega)$ be a subalgebra, with the following separation properties

- For any two points $\omega_1, \omega_2 \in \Omega$, with $\omega_1 \neq \omega_2$, there exists $f \in \mathcal{A}$ such that $f(\omega_1) \neq f(\omega_2)$.
- For any $\omega \in \Omega$, there exists $f \in \mathcal{A}$ with $f(\omega) \neq 0$.

A. Prove that, if $\mathbb{K} = \mathbb{R}$, then $\mathcal{A}$ is dense in $C_0^{\mathbb{R}}(\Omega)$, in the norm topology.

B. Prove that, if $\mathbb{K} = \mathbb{C}$, and if $\mathcal{A}$ has the property $f \in \mathcal{A} \Rightarrow \bar{f} \in \mathcal{A}$, then $\mathcal{A}$ is dense in $C_0(\Omega)$.

HINT: Work in Ω^{α} (use Remark 5.3), and prove that $\mathbb{K}1 + \mathcal{A}$ is dense in $C^{\mathbb{K}}(\Omega^{\alpha})$.

LECTURES 16-17

6. Hilbert spaces

In this section we examine a special type of Banach spaces.

DEFINITION. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $\mathcal{X}$ be a vector space over $\mathbb{K}$. An *inner product on $\mathcal{X}$* is a map

$$\mathcal{X} \times \mathcal{X} \ni (\xi, \eta) \longmapsto (\xi | \eta) \in \mathbb{K},$$

with the following properties:

- $(\xi | \xi) \geq 0, \forall \xi \in \mathcal{X}$;
- if $\xi \in \mathcal{X}$ satisfies $(\xi | \xi) = 0$, then $\xi = 0$;
- for any $\xi \in \mathcal{X}$, the map $\mathcal{X} \ni \eta \longmapsto (\xi | \eta) \in \mathbb{K}$ is $\mathbb{K}$ -linear;
- $(\eta | \xi) = \overline{(\xi | \eta)}, \forall \xi, \eta \in \mathcal{X}$.

COMMENTS. Combining the last two properties, one gets

$$\begin{aligned} (\xi | \lambda \eta_1 + \eta_2) &= \lambda (\xi | \eta_1) + (\xi | \eta_2), \quad \forall \xi, \eta_1, \eta_2 \in \mathcal{X}, \lambda \in \mathbb{K}; \\ (\lambda \xi_1 + \xi_2 | \eta) &= \overline{\lambda} (\xi_1 | \eta) + (\xi_2 | \eta), \quad \forall \xi_1, \xi_2, \eta \in \mathcal{X}, \lambda \in \mathbb{K}. \end{aligned}$$

In particular, one has

$$(1) \quad (\lambda \xi | \lambda \xi) = \overline{\lambda} \lambda (\xi | \xi) = |\lambda|^2 \cdot (\xi | \xi), \quad \forall \xi \in \mathcal{X}, \lambda \in \mathbb{K}.$$

PROPOSITION 6.1 (Cauchy-Bunyakowski-Schwartz Inequality). *Let $(\cdot | \cdot)$ be an inner product on the $\mathbb{K}$ -vector space $\mathcal{X}$. Then*

$$(2) \quad |(\xi | \eta)|^2 \leq (\xi | \xi) \cdot (\eta | \eta), \quad \forall \xi, \eta \in \mathcal{X}.$$

Moreover, if equality holds then ξ and η are proportional, in the sense that either $\xi = 0$, or $\eta = 0$, or $\xi = \lambda \eta$.

PROOF. Fix $\xi, \eta \in \mathcal{X}$. Assume $\eta \neq 0$. (In the case when $\eta = 0$, both statements are trivial). Choose a number $\lambda \in \mathbb{K}$, with $|\lambda| = 1$, such that

$$|(\xi | \eta)| = \lambda (\xi | \eta) = (\xi | \lambda \eta).$$

Define the map $F : \mathbb{K} \rightarrow \mathbb{K}$ by

$$F(z) = (z\lambda\eta + \xi | z\lambda\eta + \xi), \quad \forall z \in \mathbb{K}.$$

A simple computation gives

$$\begin{aligned} F(z) &= z\lambda\overline{\lambda}(\eta | \eta) + z\lambda(\xi | \eta) + \overline{z\lambda}(\eta | \xi) + (\xi | \xi) = \\ &= |z|^2|\lambda|^2(\eta | \eta) + z\lambda(\xi | \eta) + \overline{z\lambda}(\overline{(\xi | \eta)}) + (\xi | \xi) = \\ &= |z|^2(\eta | \eta) + z|(\xi | \eta)| + \overline{z}|(\xi | \eta)| + (\xi | \xi), \quad \forall z \in \mathbb{K}. \end{aligned}$$

In particular, when we restrict F to $\mathbb{R}$, it becomes a quadratic function:

$$F(t) = at^2 + bt + c, \quad \forall t \in \mathbb{R},$$

where $a = (\eta | \eta) > 0$, $b = 2|(\xi | \eta)|$, $c = (\xi | \xi)$. Notice that we have

$$F(t) \geq 0, \quad \forall t \in \mathbb{R}.$$

This forces $b^2 - 4ac \leq 0$. This last inequality gives

$$4|(\xi | \eta)|^2 - 4(\xi | \xi) \cdot (\eta | \eta) \leq 0,$$

so we get

$$|(\xi | \eta)|^2 \leq (\xi | \xi) \cdot (\eta | \eta),$$

and the inequality (2) is proven. Let us examine now when we have equality. The equality in (2) gives $b^2 - 4ac = 0$, which in terms of quadratic equations says that the equation

$$F(t) = at^2 + bt + c = 0$$

has a (unique) solution t_0 . This will give

$$(t_0\lambda\eta + \xi | t_0\lambda\eta + \xi) = F(t_0) = 0,$$

which forces $t_0\lambda\eta + \xi = 0$, i.e. $\xi = (-t_0\lambda)\eta$. □

COROLLARY 6.1. *Let $(\cdot | \cdot)$ be an inner product on the $\mathbb{K}$ -vector space $\mathcal{X}$. Then the map*

$$\mathcal{X} \ni \xi \mapsto \sqrt{(\xi | \xi)} \in [0, \infty)$$

is a norm on $\mathcal{X}$.

PROOF. Denote $\sqrt{(\xi | \xi)}$ simply by $\|\xi\|$. The fact that $\|\xi\|$ is non-negative is clear. The implication $\|\xi\| = 0 \Rightarrow \xi = 0$ is also clear. Using (1) we have

$$\|\lambda\xi\| = \sqrt{(\lambda\xi | \lambda\xi)} = \sqrt{|\lambda|^2(\xi | \xi)} = |\lambda| \cdot \sqrt{(\xi | \xi)} = |\lambda| \cdot \|\xi\|, \quad \forall \xi \in \mathcal{X}, \lambda \in \mathbb{K}.$$

Finally, for $\xi, \eta \in \mathcal{X}$, we have

$$\begin{aligned} \|\xi + \eta\|^2 &= (\xi + \eta | \xi + \eta) = (\xi | \xi) + (\eta | \eta) + (\xi | \eta) + (\eta | \xi) = \\ &= \|\xi\|^2 + \|\eta\|^2 + (\xi | \eta) + \overline{(\xi | \eta)} = \|\xi\|^2 + \|\eta\|^2 + 2\operatorname{Re}(\xi | \eta). \end{aligned}$$

We now use the C-B-S inequality, which reads

$$(3) \quad |(\xi | \eta)| \leq \|\xi\| \cdot \|\eta\|,$$

so the above computation gives

$$\begin{aligned} \|\xi + \eta\|^2 &= \|\xi\|^2 + \|\eta\|^2 + 2\operatorname{Re}(\xi | \eta) \leq \|\xi\|^2 + \|\eta\|^2 + 2|(\xi | \eta)| \leq \\ &\leq \|\xi\|^2 + \|\eta\|^2 + 2\|\xi\| \cdot \|\eta\| = (\|\xi\| + \|\eta\|)^2, \end{aligned}$$

so we immediately get $\|\xi + \eta\| \leq \|\xi\| + \|\eta\|$. □

DEFINITION. The norm constructed in the above result is called the *norm defined by the inner product* $(\cdot | \cdot)$.

Exercise 1. Use the above notations, and assume we have two vectors $\xi, \eta \neq 0$, such that $\|\xi + \eta\| = \|\xi\| + \|\eta\|$. Prove that there exists some $\lambda > 0$ such that $\xi = \lambda\eta$.

LEMMA 6.1. *Let $\mathcal{X}$ be a $\mathbb{K}$ -vector space, equipped with an inner product.*

(ii) [Parallelogram Law] $\|\xi + \eta\|^2 + \|\xi - \eta\|^2 = 2(\|\xi\|^2 + \|\eta\|^2)$.

(i) [Polarization Identities]

(a) If $\mathbb{K} = \mathbb{R}$, then

$$(\xi | \eta) = \frac{1}{4} [\|\xi + \eta\|^2 - \|\xi - \eta\|^2], \quad \forall \xi, \eta \in \mathcal{X}.$$

(b) If $\mathbb{K} = \mathbb{R}$, then

$$(\xi | \eta) = \frac{1}{4} \sum_{k=0}^3 i^{-k} \|\xi + i^k \eta\|^2, \quad \forall \xi, \eta \in \mathcal{X}.$$

PROOF. (i). This is obvious, since (since the computations from the proof of Corollary ??)

$$\|\xi \pm \eta\|^2 = \|\xi\|^2 + \|\eta\|^2 \pm 2\operatorname{Re}(\xi | \eta).$$

(ii).(a). In the real case, the above identity gives

$$\|\xi \pm \eta\|^2 = \|\xi\|^2 + \|\eta\|^2 \pm 2(\xi | \eta),$$

so we immediately get

$$\|\xi + \eta\|^2 - \|\xi - \eta\|^2 = 4(\xi | \eta).$$

(b). For every $k \in \{0, 1, 2, 3\}$, we have

$$\|\xi + i^k \eta\|^2 = \|\xi\|^2 + \|\eta\|^2 + 2\operatorname{Re}(\xi | i^k \eta) = \|\xi\|^2 + \|\eta\|^2 + i^k (\xi | \eta) + i^{-k} (\eta | \xi).$$

Then, when we sum up, we have

$$\sum_{k=0}^3 i^{-k} \|\xi + i^k \eta\|^2 = (\|\xi\|^2 + \|\eta\|^2) \sum_{k=0}^3 i^{-k} + 4(\xi | \eta) + (\eta | \xi) \sum_{k=0}^3 i^{-2k}.$$

Since

$$\sum_{k=0}^3 i^{-k} = \sum_{k=0}^3 i^{-2k} = 0,$$

the above computation proves that we indeed have

$$\sum_{k=0}^3 i^{-k} \|\xi + i^k \eta\|^2 = 4(\xi | \eta).$$

□

COROLLARY 6.2. Let $\mathcal{X}$ be a $\mathbb{K}$ -vector space equipped with an inner product $(\cdot | \cdot)$. Then the map

$$\mathcal{X} \times \mathcal{X} \ni (\xi, \eta) \longmapsto (\xi | \eta) \in \mathbb{K}$$

is continuous, with respect to the product topologies.

PROOF. Immediate from the polarization identities. □

COROLLARY 6.3. Let $\mathcal{X}$ and $\mathcal{Y}$ be two $\mathbb{K}$ -vector spaces equipped with inner products $(\cdot | \cdot)_{\mathcal{X}}$ and $(\cdot | \cdot)_{\mathcal{Y}}$. If $T : \mathcal{X} \rightarrow \mathcal{Y}$ is an isometric linear map, then

$$(T\xi | T\eta)_{\mathcal{Y}} = (\xi | \eta)_{\mathcal{X}}, \quad \forall \xi, \eta \in \mathcal{X}.$$

PROOF. Immediate from the polarization identities. □

Exercise 2. Let $\mathcal{X}$ be a normed $\mathbb{K}$ -vector space. Assume the norm satisfies the Parallelogram Law. Prove that there exists an inner product $(\cdot | \cdot)$ on $\mathcal{X}$, such that

$$\|\xi\| = \sqrt{(\xi | \xi)}, \quad \forall \xi \in \mathcal{X}.$$

HINT: Define the inner product by the Polarization Identity, and then prove that it is indeed an inner product.

PROPOSITION 6.2. Let $\mathcal{X}$ be a $\mathbb{K}$ -vector space, equipped with an inner product $(\cdot | \cdot)_{\mathcal{X}}$. Let $\mathcal{Z}$ be the completion of $\mathcal{X}$ with respect to the norm defined by the inner product. Then $\mathcal{Z}$ carries a unique inner product $(\cdot | \cdot)_{\mathcal{Z}}$, so that the norm on $\mathcal{Z}$ is defined by $(\cdot | \cdot)_{\mathcal{Z}}$. Moreover, this inner product extends $(\cdot | \cdot)_{\mathcal{X}}$, in the sense that

$$(\langle \xi | \langle \eta \rangle)_{\mathcal{Z}} = (\xi | \eta)_{\mathcal{X}}, \quad \forall \xi, \eta \in \mathcal{X}.$$

PROOF. It is obvious that the norm on $\mathcal{Z}$ satisfies the Parallelogram Law. We then apply Exercise 2. $\square$

DEFINITIONS. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. A *Hilbert space over $\mathbb{K}$* is a $\mathbb{K}$ -vector space, equipped with an inner product, which is complete with respect to the norm defined by the inner product. Some textbooks use the term *Euclidean* for real Hilbert spaces, and reserve the term *Hilbert* only for the complex case.

EXAMPLES 6.1. For I a non-empty set, the space $\ell_{\mathbb{K}}^2(I)$ is a Hilbert space. We know that this is a Banach space. The inner product defining the norm is

$$(\alpha | \beta) = \sum_{j \in I} \overline{\alpha(j)} \beta(j), \quad \forall \alpha, \beta \in \ell_{\mathbb{K}}^2(I).$$

The fact that the function $\overline{\alpha}\beta : I \rightarrow \mathbb{K}$ is summable is a consequence of Hölder's inequality.

More generally, a Banach space whose norm satisfies the Parallelogram Law is a Hilbert space.

DEFINITIONS. Let $\mathcal{X}$ be a $\mathbb{K}$ -vector space, equipped with an inner product $(\cdot | \cdot)$. Two vectors $\xi, \eta \in \mathcal{X}$ are said to be *orthogonal*, if $(\xi | \eta) = 0$. In this case we write $\xi \perp \eta$. Given a set $\mathcal{M} \subset \mathcal{X}$, and a vector $\xi \in \mathcal{X}$, we write $\xi \perp \mathcal{M}$, if

$$\xi \perp \eta, \quad \forall \eta \in \mathcal{M}.$$

Finally, two subsets $\mathcal{M}, \mathcal{N} \subset \mathcal{X}$ are said to be orthogonal, and we write $\mathcal{M} \perp \mathcal{N}$, if

$$\xi \perp \eta, \quad \forall \xi \in \mathcal{M}, \eta \in \mathcal{N}.$$

NOTATION. Let $\mathcal{X}$ be a vector space equipped with an inner product. For a subset $\mathcal{M} \subset \mathcal{X}$, we define the set

$$\mathcal{M}^{\perp} = \{\xi \in \mathcal{X} : \xi \perp \mathcal{M}\}.$$

REMARKS 6.1. Let $\mathcal{X}$ be a $\mathbb{K}$ -vector space equipped with an inner product.

A. The relation $\perp$ is symmetric.

B. If $\xi, \eta \in \mathcal{X}$ satisfy $\xi \perp \eta$, then one has the Pythagorean Theorem:

$$\|\xi + \eta\|^2 = \|\xi\|^2 + \|\eta\|^2.$$

This is a consequence of the equality $\|\xi + \eta\|^2 = \|\xi\|^2 + \|\eta\|^2 + 2\operatorname{Re}(\xi | \eta)$.

C. If $\mathcal{M} \subset \mathcal{X}$ is an arbitrary subset, then $\mathcal{M}^\perp$ is a closed linear subspace of $\mathcal{X}$. This follows from the linearity of the inner product in the second variable, and from the continuity.

D. For sets $\mathcal{M} \subset \mathcal{N} \subset \mathcal{X}$, one has

$$\mathcal{M}^\perp \supset \mathcal{N}^\perp.$$

E. For any set $\mathcal{M} \subset \mathcal{X}$, one has

$$\mathcal{M}^\perp = (\overline{\text{Span } \mathcal{M}})^\perp,$$

where $\overline{\text{Span } \mathcal{M}}$ denotes the norm closure of the linear span of $\mathcal{M}$. The inclusion

$$\mathcal{M}^\perp \supset (\overline{\text{Span } \mathcal{M}})^\perp$$

is trivial, since we have $\mathcal{M} \subset \overline{\text{Span } \mathcal{M}}$. Conversely, if $\xi \in \mathcal{M}^\perp$, then $\mathcal{M} \subset \{\xi\}^\perp$. But since $\{\xi\}^\perp$ is a closed linear subspace, this gives

$$\overline{\text{Span } \mathcal{M}} \subset \{\xi\}^\perp,$$

i.e. $\xi \in (\overline{\text{Span } \mathcal{M}})^\perp$.

The following result gives a very interesting property of Hilbert spaces.

PROPOSITION 6.3. *Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{C} \subset \mathcal{H}$ be a non-empty closed convex set. For every $\xi \in \mathcal{H}$, there exists a unique vector $\xi' \in \mathcal{C}$, such that*

$$\|\xi - \xi'\| = \text{dist}(\xi, \mathcal{C}).$$

PROOF. Denote $\text{dist}(\xi, \mathcal{C})$ simply by d . By definition, we have

$$d = \inf_{\eta \in \mathcal{C}} \|\xi - \eta\|.$$

Choose a sequence $(\eta_n)_{n \geq 1} \subset \mathcal{C}$, such that $\lim_{n \rightarrow \infty} \|\xi - \eta_n\| = d$.

Claim: One has the inequality

$$\|\eta_m - \eta_n\|^2 \leq 2\|\xi - \eta_m\|^2 + 2\|\xi - \eta_n\|^2 - 4d^2, \quad \forall m, n \geq 1.$$

Use the Parallelogram Law

$$(4) \quad 2\|\xi - \eta_m\|^2 + 2\|\xi - \eta_n\|^2 = \|2\xi - \eta_m - \eta_n\|^2 + \|\eta_m - \eta_n\|^2.$$

We notice that, since $\frac{1}{2}(\eta_m + \eta_n) \in \mathcal{C}$, we have

$$\|\xi - \frac{1}{2}(\eta_m + \eta_n)\| \geq d,$$

so we get

$$\|2\xi - \eta_m - \eta_n\|^2 = 4\|\xi - \frac{1}{2}(\eta_m + \eta_n)\|^2 \geq 4d^2,$$

so if we go back to (4) we get

$$2\|\xi - \eta_m\|^2 + 2\|\xi - \eta_n\|^2 = \|2\xi - \eta_m - \eta_n\|^2 + \|\eta_m - \eta_n\|^2 \geq 4d^2 + \|\eta_m - \eta_n\|^2,$$

and the Claim follows.

Having proven the Claim, we now notice that, since $\lim_{n \rightarrow \infty} \|\xi - \eta_n\| = d$, we immediately get the fact that *the sequence $(\eta_n)_{n \geq 1}$ is Cauchy*. Since $\mathcal{H}$ is complete, it follows that the sequence is convergent to some point ξ' . Since $\mathcal{C}$ is closed, it follows that $\xi' \in \mathcal{C}$. So far we have

$$\|\xi - \xi'\| = \lim_{n \rightarrow \infty} \|\xi - \eta_n\| = d = \text{dist}(\xi, \mathcal{C}),$$

thus proving the existence.

Let us prove now the uniqueness. Assume $\xi'' \in \mathcal{C}$ is another point such that $\|\xi - \xi''\| = \delta$. Using the Parallelogram Law, we have

$$4\delta^2 = 2\|\xi - \xi'\|^2 + \|\xi - \xi''\|^2 = \|2\xi - \xi' - \xi''\|^2 + \|\xi' - \xi''\|^2.$$

If $\xi' \neq \xi''$, then we will have

$$4\delta^2 > \|2\xi - \xi' - \xi''\|^2 = 4\|\xi - \frac{1}{2}(\xi' + \xi'')\|^2,$$

so we have a new vector $\eta = \frac{1}{2}(\xi' + \xi'') \in \mathcal{C}$, such that

$$\|\xi - \eta\| < \delta,$$

thus contracting the definition of δ . $\square$

DEFINITION. Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{X} \subset \mathcal{H}$ be a closed linear subspace. For every $\xi \in \mathcal{H}$, using the above result, we let $P_{\mathcal{X}}\xi \in \mathcal{X}$ denote the unique vector in $\mathcal{X}$ with the property

$$\|\xi - P_{\mathcal{X}}\xi\| = \text{dist}(\xi, \mathcal{X}).$$

This way we have constructed a map $P_{\mathcal{X}} : \mathcal{H} \rightarrow \mathcal{H}$, which is called the *orthogonal projection onto $\mathcal{X}$* .

The properties of the orthogonal projection are summarized in the following result.

PROPOSITION 6.4. *Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{X} \subset \mathcal{H}$ be a closed linear subspace.*

(i) *For vectors $\xi \in \mathcal{H}$ and $\zeta \in \mathcal{X}$ one has the equivalence*

$$\zeta = P_{\mathcal{X}}\xi \iff (\xi - \zeta) \perp \mathcal{X}.$$

(ii) $P_{\mathcal{X}}|_{\mathcal{X}} = \text{Id}_{\mathcal{X}}$.

(iii) *The map $P_{\mathcal{X}} : \mathcal{H} \rightarrow \mathcal{X}$ is linear, continuous. If $\mathcal{X} \neq \{0\}$, then $\|P_{\mathcal{X}}\| = 1$.*

(iv) $\text{Ran } P_{\mathcal{X}} = \mathcal{X}$ and $\text{Ker } P_{\mathcal{X}} = \mathcal{X}^{\perp}$.

PROOF. (i). “ $\Rightarrow$.” Assume $\zeta = P_{\mathcal{X}}\xi$. Fix an arbitrary vector $\eta \in \mathcal{X} \setminus \{0\}$, and choose a number $\lambda \in \mathbb{K}$, with $|\lambda| = 1$, such that

$$\lambda(\xi - \zeta | \eta) = |(\xi - \zeta | \eta)|.$$

In particular, we have

$$|(\xi - \zeta | \eta)| = \text{Re}(\xi - \zeta | \lambda\eta).$$

Define the map $F : \mathbb{R} \rightarrow \mathbb{R}$ by

$$F(t) = \|\xi - \zeta - t\lambda\eta\|^2 - \|\xi - \zeta\|^2.$$

By the definition of $\zeta = P_{\mathcal{X}}\xi$, we have

$$(5) \quad F(t) > 0, \quad \forall t \in \mathbb{R} \setminus \{0\}.$$

Notice that $F(t) = at^2 + bt$, $\forall t \in \mathbb{R}$, where $a = (\lambda\eta | \lambda\eta) = \|\eta\|^2$, and $b = 2\text{Re}(\xi - \zeta | \lambda\eta) = 2|(\xi - \zeta | \eta)|$. Of course, the property

$$at^2 + bt > 0, \quad \forall t \in \mathbb{R} \setminus \{0\}$$

forces $b = 0$, so we indeed get $(\xi - \zeta | \eta) = 0$.

“ $\Leftarrow$.” Assume $(\xi - \zeta) \perp \mathcal{X}$. For any $\eta \in \mathcal{X}$, we have $(\xi - \zeta) \perp (\zeta - \eta)$, so using the Pythagorean Theorem, we get

$$\|\xi - \eta\|^2 = \|\xi - \zeta\|^2 + \|\zeta - \eta\|^2,$$

which forces

$$\|\xi - \eta\| \geq \|\xi - \zeta\|, \quad \forall \eta \in \mathcal{X}.$$

This proves that

$$\|\xi - \zeta\| = \text{dist}(\xi, \mathcal{X}),$$

i.e. $\zeta = P_{\mathcal{X}}\xi$.

(ii). This property is pretty clear. If $\xi \in \mathcal{X}$, then $0 = \xi - \xi$ is orthogonal to $\mathcal{X}$, so by (i) we get $\xi = P_{\mathcal{X}}\xi$.

(iii). We prove the linearity of $P_{\mathcal{X}}$. Start with vectors $\xi_1, \xi_2 \in \mathcal{H}$ and a scalar $\lambda \in \mathbb{K}$. Take $\zeta_1 = P_{\mathcal{X}}\xi_1$ and $\zeta_2 = P_{\mathcal{X}}\xi_2$. Consider the vector $\zeta = \lambda\zeta_1 + \zeta_2$. For any $\eta \in \mathcal{X}$, we have

$$\begin{aligned} (\lambda\xi_1 + \xi_2 - \zeta \mid \eta) &= ((\lambda\xi_1 - \lambda\zeta_1) + (\xi_2 - \zeta_2) \mid \eta) = \\ &= (\lambda\xi_1 - \lambda\zeta_1 \mid \eta) + (\xi_2 - \zeta_2 \mid \eta) = \overline{\lambda}(\xi_1 - \zeta_1 \mid \eta) + (\xi_2 - \zeta_2 \mid \eta) = 0. \end{aligned}$$

By (i) we have $(\xi_1 - \zeta_1) \perp \mathcal{X}$ and $(\xi_2 - \zeta_2) \perp \mathcal{X}$, so the above computation proves that

$$(\lambda\xi_1 + \xi_2 - \zeta) \perp \mathcal{X},$$

so using (i) we get

$$P_{\mathcal{X}}(\lambda\xi_1 + \xi_2) = \zeta = \lambda\zeta_1 + \zeta_2 = \lambda P_{\mathcal{X}}\xi_1 + P_{\mathcal{X}}\xi_2,$$

so $P_{\mathcal{X}}$ is indeed linear.

To prove the continuity, we start with an arbitrary vector $\xi \in \mathcal{H}$ and we use the fact that $(\xi - P_{\mathcal{X}}\xi) \perp P_{\mathcal{X}}\xi$. By the Pythagorean Theorem we then have

$$\|\xi\|^2 = \|(\xi - P_{\mathcal{X}}\xi) + P_{\mathcal{X}}\xi\|^2 = \|\xi - P_{\mathcal{X}}\xi\|^2 + \|P_{\mathcal{X}}\xi\|^2 \geq \|P_{\mathcal{X}}\xi\|^2.$$

In other words, we have

$$\|P_{\mathcal{X}}\xi\| \leq \|\xi\|, \quad \forall \xi \in \mathcal{H},$$

so $P_{\mathcal{X}}$ is indeed continuous, and we have $\|P_{\mathcal{X}}\| \leq 1$. Using (ii) we immediately get that, when $\mathcal{X} \neq \{0\}$, we have $\|P_{\mathcal{X}}\| = 1$.

(iv). The equality $\text{Ran } P_{\mathcal{X}} = \mathcal{X}$ is trivial by the construction of $P_{\mathcal{X}}$ and by (ii). If $\xi \in \text{Ker } P_{\mathcal{X}}$, then by (i), we have $\xi \in \mathcal{X}^\perp$. Conversely, if $\xi \perp \mathcal{X}$, then $\zeta = 0$ satisfies the condition in (i), i.e. $P_{\mathcal{X}}\xi = 0$. $\square$

COROLLARY 6.4. *If $\mathcal{H}$ is a Hilbert space, and $\mathcal{X} \subset \mathcal{H}$ is a closed linear subspace, then*

$$\mathcal{X} + \mathcal{X}^\perp = \mathcal{H} \text{ and } \mathcal{X} \cap \mathcal{X}^\perp = \{0\}.$$

In other words the map

$$(6) \quad \mathcal{X} \times \mathcal{X}^\perp \ni (\eta, \zeta) \longmapsto \eta + \zeta \in \mathcal{H}$$

is a linear isomorphism.

PROOF. If $\xi \in \mathcal{H}$ then $P_{\mathcal{X}}\xi \in \mathcal{X}$, and $\xi - P_{\mathcal{X}}\xi \in \mathcal{X}^\perp$, and then the equality

$$\xi = P_{\mathcal{X}}\xi + (\xi - P_{\mathcal{X}}\xi)$$

proves that $\xi \in \mathcal{X} + \mathcal{X}^\perp$. The equality $\mathcal{X} \cap \mathcal{X}^\perp = \{0\}$ is trivial, since for $\zeta \in \mathcal{X} \cap \mathcal{X}^\perp$, we must have $\zeta \perp \zeta$, which forces $\zeta = 0$. $\square$

Exercise 3. Let $\mathcal{H}$ be a Hilbert space.

(i) Prove that, for any closed subspace $\mathcal{X} \subset \mathcal{H}$, one has the equality

$$P_{\mathcal{X}^\perp} = I - P_{\mathcal{X}}.$$

- (ii) Prove that two closed subspaces $\mathcal{X}, \mathcal{Y} \subset \mathcal{H}$, the following are equivalent:
- $\mathcal{X} \perp \mathcal{Y}$;
 - $P_{\mathcal{X}}P_{\mathcal{Y}} = 0$;
 - $P_{\mathcal{Y}}P_{\mathcal{X}} = 0$.
- (iii) Prove that two closed subspaces $\mathcal{X}, \mathcal{Y} \subset \mathcal{H}$, the following are equivalent:
- $\mathcal{X} \subset \mathcal{Y}$;
 - $P_{\mathcal{X}}P_{\mathcal{Y}} = P_{\mathcal{X}}$;
 - $P_{\mathcal{Y}}P_{\mathcal{X}} = P_{\mathcal{X}}$.
- (iv) Let $\mathcal{X}, \mathcal{Y} \subset \mathcal{H}$ are closed subspaces, such that $\mathcal{X} \perp \mathcal{Y}$, then
- $\mathcal{X} + \mathcal{Y}$ is a closed linear subspace of $\mathcal{H}$;
 - $P_{\mathcal{X}+\mathcal{Y}} = P_{\mathcal{X}} + P_{\mathcal{Y}}$.

COROLLARY 6.5. *Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{X} \subset \mathcal{H}$ be a linear (not necessarily closed) subspace. Then one has the equality*

$$\overline{\mathcal{X}} = (\mathcal{X}^\perp)^\perp.$$

PROOF. Denote the closed subspace $(\mathcal{X}^\perp)^\perp$ by $\mathcal{Z}$. Since $\mathcal{X}^\perp = \overline{\mathcal{X}}^\perp$, by the previous exercise we have

$$P_{\mathcal{Z}} = I - P_{\mathcal{X}^\perp} = I - P_{\overline{\mathcal{X}}^\perp} = I - (I - P_{\overline{\mathcal{X}}}) = P_{\overline{\mathcal{X}}},$$

which forces

$$\mathcal{Z} = \text{Ran } P_{\mathcal{Z}} = \text{Ran } P_{\overline{\mathcal{X}}} = \overline{\mathcal{X}}.$$

□

THEOREM 6.1 (Riesz' Representation Theorem). *Let $\mathcal{H}$ be a Hilbert space over $\mathbb{K}$, and let $\phi : \mathcal{H} \rightarrow \mathbb{K}$ be a linear continuous map. Then there exists a unique vector $\xi \in \mathcal{H}$, such that*

$$\phi(\eta) = (\xi | \eta), \quad \forall \eta \in \mathcal{H}.$$

Moreover one has $\|\xi\| = \|\phi\|$.

PROOF. First we show the existence. If $\phi = 0$, we simply take $\xi = 0$. Assume $\phi \neq 0$. Define the subspace $\mathcal{X} = \text{Ker } \phi$. Notice that $\mathcal{X}$ is *closed*. Using the linear isomorphism (6) we see that the composition

$$\mathcal{X}^\perp \hookrightarrow \mathcal{H} \xrightarrow{\text{quotient map}} \mathcal{H}/\mathcal{X}$$

is a linear isomorphism. Since

$$\mathcal{H}/\mathcal{X} = \mathcal{H}/\text{Ker } \phi \simeq \text{Ran } \phi = \mathbb{K},$$

it follows that $\dim(\mathcal{X}^\perp) = 1$. In other words, there exists $\xi_0 \in \mathcal{X}^\perp$, $\xi_0 \neq 0$, such that

$$\mathcal{X}^\perp = \mathbb{K}\xi_0.$$

Start now with some arbitrary vector $\eta \in \mathcal{H}$. On the one hand, using the equality $\mathbb{K}\xi_0 + \mathcal{X} = \mathcal{H}$, there exists $\lambda \in \mathbb{K}$ and $\zeta \in \mathcal{X}$, such that

$$\eta = \lambda\xi_0 + \zeta,$$

and since $\zeta \in \mathcal{X} = \text{Ker } \phi$, we get

$$\phi(\eta) = \phi(\lambda\xi_0) = \lambda\phi(\xi_0).$$

On the other hand, we have

$$(\xi_0 | \eta) = (\xi_0 | \lambda\xi_0) + (\xi_0 | \zeta) = \lambda\|\xi_0\|^2,$$

so if we define $\xi = \overline{\phi(\xi_0)} \|\xi_0\|^{-2}$ we will have

$$(\xi | \eta) = (\overline{\phi(\xi_0)} \|\xi_0\|^{-2} \xi_0 | \eta) = \phi(\xi_0) \|\xi_0\|^{-2} (\xi_0 | \eta) = \lambda \phi(\xi_0) = \phi(\eta).$$

To prove uniqueness, assume $\xi' \in \mathcal{H}$ is another vector with

$$\phi(\eta) = (\xi' | \eta), \quad \forall \eta \in \mathcal{H}.$$

In particular, we have

$$\|\xi - \xi'\|^2 = (\xi - \xi' | \xi - \xi') = (\xi | \xi - \xi') - (\xi' | \xi - \xi') = \phi(\xi - \xi') - \phi(\xi - \xi') = 0,$$

which forces $\xi = \xi'$.

Finally, to prove the norm equality, we first observe that when $\xi = 0$, the equality is trivial. If $\xi \neq 0$, then on the one hand, using C-B-S inequality we have

$$|\phi(\eta)| = |(\xi | \eta)| \leq \|\xi\| \cdot \|\eta\|, \quad \forall \eta \in \mathcal{H},$$

so we immediately get $\|\phi\| \leq \|\xi\|$. If we take the vector $\zeta = \|\xi\|^{-1} \xi$, then $\|\zeta\| = 1$, and

$$\phi(\zeta) = (\xi | \|\xi\|^{-1} \xi) = \|\xi\|,$$

so we also have $\|\phi\| \geq \|\xi\|$. $\square$

In the remainder of this section we discuss a Hilbert space notion of linear independence. This should be thought as a “rigid” linear independence.

DEFINITION. Let $\mathcal{X}$ be a $\mathbb{K}$ -vector space, equipped with an inner product. A set $\mathcal{F} \subset \mathcal{X}$ is said to be *orthogonal*, if $0 \notin \mathcal{F}$, and

$$\xi \perp \eta, \quad \forall \xi, \eta \in \mathcal{F}, \text{ with } \xi \neq \eta.$$

A set $\mathcal{F} \subset \mathcal{X}$ is said to be *orthonormal*, if it is orthogonal, but it also satisfies:

$$\|\xi\| = 1, \quad \forall \xi \in \mathcal{F}.$$

Remark that, if one starts with an orthogonal set $\mathcal{F} \subset \mathcal{X}$, then the set

$$\mathcal{F}^{(1)} = \{\|\xi\|^{-1} \xi : \xi \in \mathcal{F}\}$$

is orthonormal.

PROPOSITION 6.5. *Let $\mathcal{X}$ be a $\mathbb{K}$ -vector space equipped with an inner product. Any orthogonal set $\mathcal{F} \subset \mathcal{X}$ is linearly independent.*

PROOF. Indeed, if one starts with a vanishing linear combination

$$\lambda_1 \xi_1 + \cdots + \lambda_n \xi_n = 0,$$

with $\lambda_1, \dots, \lambda_n \in \mathbb{K}$, $\xi_1, \dots, \xi_n \in \mathcal{X}$, such that $\xi_k \neq \xi_\ell$, for all $k, \ell \in \{1, \dots, n\}$ with $k \neq \ell$, then for each $k \in \{1, \dots, n\}$ we clearly have

$$\lambda_k \|\xi_k\|^2 = (\xi_k | \lambda_1 \xi_1 + \cdots + \lambda_n \xi_n) = 0,$$

and since $\xi_k \neq 0$, we get $\lambda_k = 0$. $\square$

LEMMA 6.2. *Let $\mathcal{X}$ be a $\mathbb{K}$ -vector space equipped with an inner product, and let $\mathcal{F} \subset \mathcal{X}$ be an orthogonal set. Then there exists a maximal (with respect to inclusion) orthogonal set $\mathcal{G} \subset \mathcal{X}$ with $\mathcal{F} \subset \mathcal{G}$.*

PROOF. Consider the sets

$$\mathfrak{A} = \{\mathcal{G} : \mathcal{G} \text{ orthogonal subset of } \mathcal{X}\},$$

$$\mathfrak{A}_{\mathcal{F}} = \{\mathcal{G} \in \mathfrak{A} : \mathcal{G} \supset \mathcal{F}\},$$

ordered with the inclusion. We are going to apply Zorn's Lemma to $\mathfrak{A}_{\mathcal{F}}$. Let $\mathfrak{T} \subset \mathfrak{A}_{\mathcal{F}}$ be a subcollection, which is totally ordered, i.e. for any $\mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{T}$ one has $\mathcal{G}_1 \subset \mathcal{G}_2$ or $\mathcal{G}_1 \supset \mathcal{G}_2$. Define the set

$$\mathcal{M} = \bigcup_{\mathcal{G} \in \mathfrak{T}} \mathcal{G}.$$

Since $\mathcal{G} \subset \mathcal{X} \setminus \{0\}$, for all $\mathcal{G} \in \mathfrak{T}$, it is clear that $\mathcal{M} \subset \mathcal{X} \setminus \{0\}$. If $\xi_1, \xi_2 \in \mathcal{M}$ are vectors with $\xi_1 \neq \xi_2$, then we can find $\mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{T}$ with $\xi_1 \in \mathcal{G}_1$ and $\xi_2 \in \mathcal{G}_2$. Using the fact that $\mathfrak{T}$ is totally ordered, it follows that there is $k \in \{1, 2\}$ such that $\xi_1, \xi_2 \in \mathcal{G}_k$, so we indeed get $\xi_1 \perp \xi_2$. It is now clear that $\mathcal{M} \in \mathfrak{A}_{\mathcal{F}}$, and $\mathcal{M} \supset \mathcal{G}$, for all $\mathcal{G} \in \mathfrak{T}$. In other words, we have shown that *every totally ordered subset of $\mathfrak{A}_{\mathcal{F}}$ has an upper bound, in $\mathfrak{A}_{\mathcal{F}}$* . By Zorn's Lemma, $\mathfrak{A}_{\mathcal{F}}$ has a maximal element. Finally, it is clear that any maximal element for $\mathfrak{A}_{\mathcal{F}}$ is also a maximal element in $\mathfrak{A}$. $\square$

REMARK 6.2. Using the notations from the proof above, given an *orthonormal* set $\mathcal{M} \subset \mathcal{X}$, the following are equivalent:

- (i) $\mathcal{M}$ is maximal in $\mathfrak{A}$;
- (ii) $\mathcal{M}$ is maximal in

$$\mathfrak{A}^{(1)} = \{\mathcal{G} : \mathcal{G} \text{ orthonormal subset of } \mathcal{X}\}.$$

The implication (i) $\Rightarrow$ (ii) is trivial. Conversely, if $\mathcal{M}$ is maximal in $\mathfrak{A}^{(1)}$, we use the Lemma to find a maximal $\mathcal{N} \in \mathfrak{A}$ with $\mathcal{N} \supset \mathcal{M}$. But then $\mathcal{N}^{(1)}$ is orthonormal, and $\mathcal{N}^{(1)} \supset \mathcal{M}$, which by the maximality of $\mathcal{M}$ in $\mathfrak{A}^{(1)}$ will force $\mathcal{N}^{(1)} = \mathcal{M}$. Since $\mathcal{N}$ is linearly independent, the relations

$$\mathcal{N}^{(1)} = \mathcal{M} \subset \mathcal{N},$$

will force $\mathcal{N} = \mathcal{N}^{(1)} = \mathcal{M}$.

COMMENT. In linear algebra we know that a linearly independent set is maximal, if and only if it spans the whole space. In the case of orthogonal sets, this statement has a version described by the following result.

THEOREM 6.2. *Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{F}$ be an orthogonal set in $\mathcal{H}$. The following are equivalent:*

- (i) $\mathcal{F}$ is maximal among all orthogonal subsets of $\mathcal{H}$;
- (ii) $\text{Span } \mathcal{F}$ is dense in $\mathcal{H}$ in the norm topology.

PROOF. (i) $\Rightarrow$ (ii). Assume $\mathcal{F}$ is maximal. We are going to show that $\text{Span } \mathcal{F}$ is dense in $\mathcal{H}$, by contradiction. Denote the closure $\overline{\text{Span } \mathcal{F}}$ simply by $\mathcal{X}$, and assume $\mathcal{X} \subsetneq \mathcal{H}$. Since

$$\mathcal{X} = (\mathcal{X}^\perp)^\perp,$$

we see that, the strict inclusion $\mathcal{X} \subsetneq \mathcal{H}$ forces $\mathcal{X}^\perp \neq \{0\}$. But now if we take a non-zero vector $\xi \in \mathcal{X}^\perp$, we immediately see that the set $\mathcal{F} \cup \{\xi\}$ is still orthogonal, thus contradicting the maximality of $\mathcal{F}$.

(ii) $\Rightarrow$ (i). Assume $\text{Span } \mathcal{F}$ is dense in $\mathcal{H}$, and let us prove that $\mathcal{F}$ is maximal. We do this by contradiction. If $\mathcal{F}$ is not maximal, then there exists $\xi \in \mathcal{H} \setminus \mathcal{F}$, such that $\mathcal{F} \cup \{\xi\}$ is still orthogonal. This would force $\xi \perp \mathcal{F}$, so we will also have

$$\xi \perp \overline{\text{Span } \mathcal{F}}.$$

But since $\text{Span } \mathcal{F}$ is dense in $\mathcal{H}$, this will give $\xi \perp \mathcal{H}$. In particular we have $\xi \perp \xi$, which would force $\xi = 0$, thus contradicting the fact that $\mathcal{F} \cup \{\xi\}$ is orthogonal. (Recall that all elements of an orthogonal set are non-zero.) $\square$

DEFINITION. Let $\mathcal{H}$ be a Hilbert space. An orthonormal set $\mathcal{B} \subset \mathcal{H}$, which is maximal among all orthogonal (or orthonormal) subsets of $\mathcal{H}$, is called an *orthonormal basis* for $\mathcal{H}$.

By Lemma ??, we know that given any orthonormal set $\mathcal{F} \subset \mathcal{H}$, there exists an orthonormal basis $\mathcal{B} \supset \mathcal{F}$.

By the above result, an orthonormal set $\mathcal{B} \subset \mathcal{H}$ is an orthonormal basis for $\mathcal{H}$, if and only if $\text{Span } \mathcal{B}$ is dense in $\mathcal{H}$.

EXAMPLE 6.2. Let I be a non-empty set. Consider the Hilbert space $\ell_{\mathbb{K}}^2(I)$. Consider (see section II.2) the set

$$\mathcal{B} = \{\delta^i : i \in I\}.$$

Then

$$\text{Span } \mathcal{B} = \text{fin}_{\mathbb{K}}(I),$$

which is dense in $\ell_{\mathbb{K}}^2(I)$. The above result then says that $\mathcal{B}$ is an orthonormal basis for $\ell_{\mathbb{K}}^2(I)$.

The following exercise will be useful in the discussion of another interesting example.

Exercise 4. Equip the space $C([0, 1])$ with the inner product

$$(f | g) = \int_0^1 \overline{f(t)}g(t) dt, \quad f, g \in C([0, 1]).$$

The norm defined by this inner product is

$$\|f\|_2 = \left(\int_0^1 |f(t)|^2 dt \right)^{\frac{1}{2}}, \quad f \in C([0, 1]).$$

Define the maps $e_n : [0, 1] \ni t \mapsto \exp(2n\pi it) \in \mathbb{T}$, $n \in \mathbb{Z}$. (Here $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$.) Prove that the set

$$\mathcal{B} = \{e_n : n \in \mathbb{Z}\}$$

is orthonormal in $C([0, 1])$, and $\text{Span } \mathcal{B}$ is dense in $C([0, 1])$ in the topology defined by the norm $\|\cdot\|_2$.

HINTS: Define the space

$$\mathcal{P} = \{f \in C([0, 1]) : f(0) = f(1)\}.$$

Prove that $\mathcal{P}$ is dense in $C([0, 1])$ in the topology defined by the norm $\|\cdot\|_2$.

Prove that the map

$$\Phi : C(\mathbb{T}) \ni F \mapsto F \circ e \in \mathcal{P}$$

is a linear isomorphism, which is isometric with respect to the *uniform* norms.

In order to prove that $\text{Span } \mathcal{B}$ is dense in $C([0, 1])$ with respect to $\|\cdot\|_2$, it suffices to show that $\text{Span } \mathcal{B}$ is dense in $\mathcal{P}$ in the *uniform* norm. Equivalently, it suffices to show that

$$\Phi^{-1}(\text{Span } \mathcal{B})$$

is dense in $C(\mathbb{T})$, with respect to the uniform norm. To get this density use Stone-Weierstrass Theorem, plus the fact that the functions $\zeta_n = \Phi^{-1}(e_n) \in C(\mathbb{T})$ are defined by

$$\zeta_n(z) = z^n, \quad \forall z \in \mathbb{T}, n \in \mathbb{Z}.$$

EXAMPLE 6.3. We define $L^2([0, 1])$ to be the completion of $C([0, 1])$ with respect to the norm $\|\cdot\|_2$. Regard $C([0, 1])$ as a dense linear subspace in $L^2([0, 1])$, so we also regard

$$\mathcal{B} = \{e_n : n \in \mathbb{Z}\}$$

as a subset in $L^2([0, 1])$. Then $\text{Span } \mathcal{B}$ is dense in $L^2([0, 1])$, so $\mathcal{B}$ is an *orthonormal basis* for $L^2([0, 1])$.

LEMMA 6.3. Let $\mathcal{B}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$, and let $\mathcal{F} \subsetneq \mathcal{B}$ be an arbitrary non-empty subset.

(i) $\mathcal{F}$ is an orthonormal basis for the Hilbert space $\overline{\text{Span } \mathcal{F}}$.

(ii) $(\overline{\text{Span } \mathcal{F}})^\perp = \overline{\text{Span}(\mathcal{B} \setminus \mathcal{F})}$.

PROOF. (i). This is clear, since $\mathcal{F}$ is orthonormal and has dense span.

(ii). Denote for simplicity $\overline{\text{Span } \mathcal{F}} = \mathcal{X}$ and $\overline{\text{Span}(\mathcal{B} \setminus \mathcal{F})} = \mathcal{Y}$. Since

$$\xi \perp \eta, \quad \forall \xi \in \mathcal{F}, \eta \in \mathcal{B} \setminus \mathcal{F},$$

it is pretty obvious that $\mathcal{X} \perp \mathcal{Y}$. Since $\mathcal{X} + \mathcal{Y}$ clearly contains $\text{Span } \mathcal{B}$, it follows that $\mathcal{X} + \mathcal{Y}$ is dense in $\mathcal{H}$. We know however that $\mathcal{X} + \mathcal{Y}$ is closed, so we have in fact the equality

$$\mathcal{X} + \mathcal{Y} = \mathcal{H}.$$

This will then give

$$I = P_{\mathcal{H}} = P_{\mathcal{X}} + P_{\mathcal{Y}},$$

so we get

$$P_{\mathcal{Y}} = I - P_{\mathcal{X}} = P_{\mathcal{X}^\perp},$$

so

$$\mathcal{X}^\perp = \text{Ran } P_{\mathcal{X}^\perp} = \text{Ran } P_{\mathcal{Y}} = \mathcal{Y}.$$

□

THEOREM 6.3. Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{B}$ be an orthonormal basis for $\mathcal{H}$, labelled⁶ as $\mathcal{B} = \{\xi_j : j \in I\}$. For every vector $\eta \in \mathcal{H}$, let $\alpha^\eta : I \rightarrow \mathbb{K}$ be the map defined by

$$\alpha^\eta(j) = (\xi_j | \eta), \quad \forall j \in I.$$

(i) For every $\eta \in \mathcal{H}$, the map α^η belongs to $\ell_{\mathbb{K}}^2(I)$.

(ii) The map

$$T : \mathcal{H} \ni \eta \longmapsto \alpha^\eta \in \ell_{\mathbb{K}}^2(I)$$

is an isometric linear isomorphism.

PROOF. (i). Fix for the moment $\eta \in \mathcal{H}$. We must show that

$$\sup \left\{ \sum_{j \in F} |\alpha^\eta(j)|^2 : F \subset I, \text{ finite} \right\} < \infty.$$

For any non-empty finite subset $F \subset I$, we define the subspace

$$\mathcal{H}_F = \text{Span}\{\xi_j : j \in F\},$$

⁶ This notation implicitly assumes that $\xi_j \neq \xi_k$, for all $j, k \in I$ with $j \neq k$.

and define the vector

$$\eta^F = \sum_{j \in F} \text{big}(\xi_j | \eta) \cdot \xi_j.$$

Claim: For every finite set $F \subset I$, one has the equality

$$\eta^F = P_{\mathcal{H}_F} \eta.$$

It suffices to prove that

$$(\eta - \eta^F) \perp \mathcal{H}_F.$$

But this is obvious, since if we start with some $k \in F$, then using the fact that $(\xi_k | \xi_j) = 0$, for all $j \in F \setminus \{k\}$, together with the equality $\|\xi_k\| = 1$, we get

$$(\xi_k | \eta - \eta^F) = (\xi_k | \eta) - \sum_{j \in F} (\xi_j | \eta) \cdot (\xi_k | \xi_j) = (\xi_k | \eta) - (\xi_k | \eta) \cdot (\xi_k | \xi_k) = 0.$$

Having proven the Claim, let us observe that, since the terms in the sum that defines η^F are all orthogonal, we get

$$\|\eta^F\|^2 = \sum_{j \in F} \|(\xi_j | \eta) \cdot \xi_j\|^2 = \sum_{j \in F} |(\xi_j | \eta)|^2 \cdot \|\xi_j\|^2 = \sum_{j \in F} |\alpha^\eta(j)|^2.$$

Combining this computation with the Claim, we now have

$$\sum_{j \in F} |\alpha^\eta(j)|^2 = \|\eta^F\|^2 = \|P_{\mathcal{H}_F} \eta\|^2 \leq \|\eta\|^2,$$

which proves that

$$\sup \left\{ \sum_{j \in F} |\alpha^\eta(j)|^2 : F \subset I, \text{ finite} \right\} < \|\eta\|^2.$$

(ii). The linearity of T is obvious. The above inequality actually proves that

$$\|T\eta\| \leq \|\eta\|, \quad \forall \eta \in \mathbb{H}.$$

We now prove that in fact T is isometric. Since T is linear and continuous, it suffices to prove that $T|_{\text{Span } \mathcal{B}}$ is isometric. Start with some vector $\eta \in \text{Span } \mathcal{B}$, which means that there exists some finite set $F \subset I$, and scalars $(\lambda_k)_{k \in F} \subset \mathbb{K}$, such that $\eta = \sum_{k \in F} \lambda_k \xi_k$. Remark that

$$(\xi_j | \eta) = \sum_{k \in F} \lambda_k (\xi_j | \xi_k) = \begin{cases} \lambda_j & \text{if } j \in F \\ 0 & \text{if } j \notin F \end{cases}$$

so the element $\alpha^\eta = T\eta \in \ell_{\mathbb{K}}^2(I)$ is defined by

$$\alpha^\eta(k) = \begin{cases} \lambda_k & \text{if } k \in F \\ 0 & \text{if } k \notin F \end{cases}$$

This gives

$$\|\eta\|^2 = \sum_{j,k \in F} \lambda_j \bar{\lambda}_k (\xi_j | \xi_k) = \sum_{k \in F} |\lambda_k|^2 = \sum_{k \in F} |\alpha^\eta(k)|^2 = \|\alpha^\eta\|^2,$$

so we indeed get

$$\|\eta\| = \|T\eta\|, \quad \forall \eta \in \text{Span } \mathcal{B}.$$

Let us prove that T is surjective. Notice that, the above computation, applied to singleton sets $F = \{k\}$, $k \in I$, proves that

$$T\xi_k = \delta^k, \quad \forall k \in I.$$

In particular, we have

$$\begin{aligned}\text{Ran } T &\supset T(\text{Span } \mathcal{B}) = \text{Span } T(\mathcal{B}) = \\ &= \text{Span}\{T\xi_k : k \in I\} = \text{Span}\{\delta^k : k \in I\} = \text{fin}_{\mathbb{K}}(I),\end{aligned}$$

which proves that $\text{Ran } T$ is dense in $\ell_{\mathbb{K}}^2(I)$. We know however that T is isometric, so $\text{Ran } T \subset \ell_{\mathbb{K}}^2(I)$ is closed. This forces $\text{Ran } T = \ell_{\mathbb{K}}^2(I)$. $\square$

COROLLARY 6.6 (Parseval Identity). *Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{B} = \{\xi_j : j \in I\}$ be an orthonormal basis for $\mathcal{H}$. One has:*

$$(\zeta | \eta) = \sum_{j \in I} (\zeta | \xi_j) \cdot (\xi_j | \eta), \quad \forall \zeta, \eta \in \mathcal{H}.$$

PROOF. If we define $\alpha(j) = (\xi_j | \zeta)$ and $(\xi_j | \eta)$, $\forall j \in I$, then by construction we have $\alpha = T\zeta$ and $\beta = T\eta$. Using the fact that T is isometric, the right hand side of the above equality is the equal to

$$\sum_{j \in I} \overline{\alpha(j)}\beta(j) = (\alpha | \beta) = (T\zeta | T\eta) = (\zeta | \eta).$$

$\square$

NOTATION. Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{B} = \{\xi_j : j \in I\}$ be an orthonormal basis for $\mathcal{H}$, and let $T : \mathcal{H} \rightarrow \ell_{\mathbb{K}}^2(I)$ be the isometric linear isomorphism defined in the previous theorem. Given an element $\alpha \in \ell_{\mathbb{K}}^2(I)$, we denote the vector $T^{-1}\alpha \in \mathcal{H}$ by

$$\sum_{j \in I} \alpha(j)\xi_j.$$

The summation notation is justified by the following fact.

PROPOSITION 6.6. *With the above notations, for every $\varepsilon > 0$, there exists some finite subset $F_\varepsilon \subset I$, such that*

$$\left\| \sum_{j \in I} \alpha(j)\xi_j - \sum_{k \in F} \alpha(k)\xi_k \right\|^2 < \varepsilon, \text{ for all finite sets } F \subset I \text{ with } F \supset F_\varepsilon.$$

PROOF. Define the vector $\eta = \sum_{j \in I} \alpha(j)\xi_j$. By construction we have $T\eta = \alpha$. Likewise, if we define, for each finite set $F \subset I$, the element $\alpha_F \in \ell_{\mathbb{K}}^2(I)$ by

$$\alpha_F(k) = \begin{cases} \alpha(k) & \text{if } k \in F \\ 0 & \text{if } k \in I \setminus F \end{cases}$$

then $T^{-1}\alpha_F = \sum_{k \in F} \alpha(k)\xi_k$. Using the fact that T is an isometry, we have

$$\|\eta - T^{-1}\alpha_F\| = \|T\eta - \alpha_F\| = \|\alpha - \alpha_F\|,$$

and the desired property follows from the well-known properties of $\ell_{\mathbb{K}}^2(I)$. $\square$

Exercise 5. Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{F} = \{\xi_j : j \in J\}$ be an orthonormal set. Define the closed linear subspace $\mathcal{H}_{\mathcal{F}} = \overline{\text{Span } \mathcal{F}}$. Prove that the orthogonal projection $P_{\mathcal{H}_{\mathcal{F}}}$ is defined by

$$P_{\mathcal{H}_{\mathcal{F}}}\eta = \sum_{j \in J} (\xi_j | \eta)\xi_j, \quad \forall \eta \in \mathcal{H}.$$

HINTS: Extend $\mathcal{F}$ to an orthonormal basis $\mathcal{B}$. Let $\mathcal{B}$ be labelled as $\{\xi_i : i \in I\}$ for some set $I \supset J$. First prove that for any $\eta \in \mathcal{H}$, the map $\beta^\eta = T\eta|_J$ belongs to $\ell_{\mathbb{K}}^2(J)$. In particular, the sum

$$\eta_{\mathcal{F}} = \sum_{j \in J} (\xi_j | \eta) \xi_j$$

is “legitimate” and defines an element in $\mathcal{H}_{\mathcal{F}}$ (use the fact that $\mathcal{F}$ is an orthonormal basis for $\mathcal{H}_{\mathcal{F}}$). Finally, prove that $(\eta - \eta_{\mathcal{F}}) \perp \mathcal{F}$, using Parseval Identity.

EXAMPLE 6.4. Let us analyze the space $L^2([0, 1])$. Use the orthonormal basis $\{e_n : n \in \mathbb{Z}\}$ defined by

$$e_n(t) = \exp(2n\pi it), \quad \forall t \in [0, 1], n \in \mathbb{Z}.$$

For any $f \in C([0, 1])$ we define

$$\hat{f}(n) = \int_0^1 \exp(-2n\pi it) f(t) dt = (e_n | f).$$

We then know that

$$f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n.$$

One can think the right hand side as a series, but the reader should be aware of the fact that this series is *convergent only in the norm* $\|\cdot\|_2$. One can define for example for any $N \geq 1$, a partial sum $f_N : [0, 1] \rightarrow \mathbb{C}$ by

$$f_N(t) = \sum_{n=-N}^N \hat{f}(n) \exp(2n\pi it), \quad t \in [0, 1].$$

We will have

$$\lim_{N \rightarrow \infty} \|f - f_N\|_2 = 0,$$

but in general there are (many) values of $t \in [0, 1]$ for which the limit $\lim_{N \rightarrow \infty} f_N(t)$ does not exist. One can consider a formal infinite series

$$(7) \quad \sum_{n=-\infty}^{\infty} \hat{f}(n) \exp(2n\pi it).$$

Although this series is not convergent (pointwise) for all $t \in [0, 1]$, it plays an important role in analysis. The series (7) is called the *complex Fourier series of f* .

Note that Parseval's Identity gives

$$\int_0^1 \overline{f(t)} g(t) dt = \sum_{n=-\infty}^{\infty} \overline{\hat{f}(n)} \hat{g}(n).$$

One can construct another orthonormal basis for $L^2([0, 1])$, by taking real and imaginary parts of e_n . More explicitly, we define the sequences of functions $(g_n)_{n=0}^{\infty}$ and $(h_n)_{n=1}^{\infty}$ by

$$\begin{aligned} g_0(t) &= 1, \quad \forall t \in [0, 1]; \\ g_n(t) &= \sqrt{2} \cos(2n\pi t), \quad \forall t \in [0, 1], n \geq 1; \\ h_n(t) &= \sqrt{2} \sin(2n\pi t), \quad \forall t \in [0, 1], n \geq 1. \end{aligned}$$

Then $\mathcal{B}' = \{g_n : n \geq 0\} \cup \{h_n : n \geq 1\}$ is again an orthonormal basis for $L^2([0, 1])$. (It is clear that $\mathcal{B}'$ is orthonormal, and $\text{Span } \mathcal{B}' \ni e_n, \forall n \in \mathbb{Z}$, so $\text{Span } \mathcal{B}'$ is dense in $L^2([0, 1])$.) For $f \in C([0, 1])$ one can then define its *real Fourier series*

$$\hat{f}(0) + \sum_{n=1}^{\infty} [a_n \cos(2n\pi t) + b_n \sin(2n\pi t)],$$

where

$$a_n = \sqrt{2} \int_0^1 f(t) \cos(2n\pi t) dt \text{ and } b_n = \sqrt{2} \int_0^1 f(t) \sin(2n\pi t) dt, \quad \forall n \geq 1.$$

Note that

$$a_n = \frac{\sqrt{2}}{2} [\hat{f}(-n) + \hat{f}(n)] \text{ and } b_n = \frac{\sqrt{2}}{2i} [\hat{f}(-n) - \hat{f}(n)], \quad \forall n \geq 1.$$

The next result discusses the appropriate notion of dimension for Hilbert spaces.

THEOREM 6.4. *Let $\mathcal{H}$ be a Hilbert space. Then any two orthonormal bases of $\mathcal{H}$ have the same cardinality.*

PROOF. Fix two orthonormal bases $\mathcal{B}$ and $\mathcal{B}'$. There are two possible cases.

CASE I: *One of the sets $\mathcal{B}$ or $\mathcal{B}'$ is finite.*

In this case $\mathcal{H}$ is finite dimensional, since the linear span of a finite set is automatically closed. Since both $\mathcal{B}$ and $\mathcal{B}'$ are linearly independent, it follows that both $\mathcal{B}$ and $\mathcal{B}'$ are finite, hence their linear spans are both closed. It follows that

$$\text{Span } \mathcal{B} = \text{Span } \mathcal{B}' = \mathcal{H},$$

so $\mathcal{B}$ and $\mathcal{B}'$ are in fact linear bases for $\mathcal{H}$, and then we get

$$\text{Card } \mathcal{B} = \text{Card } \mathcal{B}' = \dim \mathcal{H}.$$

CASE II: *Both $\mathcal{B}$ and $\mathcal{B}'$ are infinite.*

The key step we need in this case is the following.

Claim 1: There exists a dense subset $\mathcal{Z} \subset \mathcal{H}$, with

$$\text{Card } \mathcal{Z} = \text{Card } \mathcal{B}'.$$

To prove this fact, we define the set

$$\mathcal{X} = \text{Span}_{\mathbb{Q}} \mathcal{B}'.$$

It is clear that

$$\text{Card } \mathcal{X} = \text{Card } \mathcal{B}'.$$

Notice that $\mathcal{X}$ is dense in $\text{Span}_{\mathbb{R}} \mathcal{B}'$. If we work over $\mathbb{K} = \mathbb{R}$, then we are done. If we work over $\mathbb{K} = \mathbb{C}$, we define

$$\mathcal{Z} = \mathcal{X} + i\mathcal{X},$$

and we will still have

$$\text{Card } \mathcal{Z} = \text{Card } \mathcal{X} = \text{Card } \mathcal{B}'.$$

Now we are done, since clearly $\mathcal{Z}$ is dense in $\text{Span}_{\mathbb{C}} \mathcal{B}'$.

Choose $\mathcal{Z}$ as in Claim 1. For every $\xi \in \mathcal{B}$ we choose a vector $\zeta_{\xi} \in \mathcal{Z}$, such that

$$\|\xi - \zeta_{\xi}\| \leq \frac{\sqrt{2}-1}{2}.$$

Claim 2: The map $\mathcal{B} \ni \xi \mapsto \zeta_{\xi} \in \mathcal{Z}$ is injective.

Start with two vectors $\xi_1, \xi_2 \in \mathcal{B}$, such that $\xi_1 \neq \xi_2$. In particular, $\xi_1 \perp \xi_2$, so we also have $\xi_1 \perp (-\xi_2)$, and using the Pythagorean Theorem we get

$$\|\xi_1 - \xi_2\|^2 = \|\xi_2\|^2 + \|-\xi_2\|^2 = 2,$$

which gives

$$\|\xi_1 - \xi_2\| = \sqrt{2}.$$

Using the triangle inequality, we now have

$$\sqrt{2} = \|\xi_1 - \xi_2\| \leq \|\xi_1 - \zeta_{\xi_1}\| + \|\xi_2 - \zeta_{\xi_2}\| + \|\zeta_{\xi_1} - \zeta_{\xi_2}\| \leq \sqrt{2} - 1 + \|\zeta_{\xi_1} - \zeta_{\xi_2}\|.$$

This gives

$$\|\zeta_{\xi_1} - \zeta_{\xi_2}\| \geq 1,$$

which forces $\zeta_{\xi_1} \neq \zeta_{\xi_2}$.

Using Claim 2, we have constructed an injective map $\mathcal{B} \rightarrow \mathcal{Z}$. In particular, using Claim 1 and the cardinal arithmetic rules, we get

$$\text{Card } \mathcal{B} \leq \text{Card } \mathcal{Z} = \text{Card } \mathcal{B}'.$$

By symmetry we also have

$$\text{Card } \mathcal{B}' \leq \text{Card } \mathcal{B},$$

and then using the Cantor-Bernstein Theorem, we finally get

$$\text{Card } \mathcal{B} = \text{Card } \mathcal{B}'.$$

□

COROLLARY 6.7 (of the proof). *A Hilbert space is separable, in the norm topology, if and only if it has an orthonormal basis which is at most countable.*

PROOF. Use Claims 1 and 2 from the proof of the Theorem. □

DEFINITION. Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{B}$ be an orthonormal basis for $\mathcal{H}$. By the above theorem, the cardinal number $\text{Card } \mathcal{B}$ does not depend on the choice of $\mathcal{B}$. This number is called the *hilbertian* (or *orthogonal*) *dimension* of $\mathcal{H}$, and is denoted by $\text{H-dim } \mathcal{H}$.

COROLLARY 6.8. *For two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, the following are equivalent:*

- (i) $\text{H-dim } \mathcal{H} = \text{H-dim } \mathcal{H}'$;
- (ii) *There exists an isometric linear isomorphism $U : \mathcal{H} \rightarrow \mathcal{H}'$.*

PROOF. (i) $\Rightarrow$ (ii). Choose a set I with $\text{H-dim } \mathcal{H} = \text{H-dim } \mathcal{H}' = \text{Card } I$. Apply Theorem ?? to produce isometric linear isomorphisms $T : \mathcal{H} \rightarrow \ell_{\mathbb{K}}^2(I)$ and $T' : \mathcal{H}' \rightarrow \ell_{\mathbb{K}}^2(I)$. Then define $U = T'^{-1} \circ T$.

(ii) $\Rightarrow$ (i). Assume one has an isometric linear isomorphism $U : \mathcal{H} \rightarrow \mathcal{H}'$. Choose an orthonormal basis $\mathcal{B}$ for $\mathcal{H}$. Then $U(\mathcal{B})$ is clearly an orthonormal basis for $\mathcal{H}'$, and since $U : \mathcal{B} \rightarrow U(\mathcal{B})$ is bijective, we get

$$\text{H-dim } \mathcal{H} = \text{Card } \mathcal{B} = \text{Card } U(\mathcal{B}) = \text{H-dim } \mathcal{H}'.$$

□

Chapter III

Measure Theory

LECTURE 18

1. Set arithmetic: (σ -)rings, (σ -)algebras, and monotone classes

In this section we discuss various types of set collections used in Measure Theory.

NOTATION. Given a (non-empty) set X , we denote by $\mathcal{P}(X)$ the collection of all subsets of X .

DEFINITION. Let X be a non-empty set. For $A \in \mathcal{P}(X)$, we define the function $\kappa_A : X \rightarrow \{0, 1\}$ by

$$\kappa_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in X \setminus A \end{cases}$$

The function κ_A is called the *characteristic function* of A .

The basic properties of characteristic functions are summarized in the following.

Exercise 1. Let X be a non-empty set. Prove:

- (i) $\kappa_\emptyset = 0$ and $\kappa_X = 1$.
- (ii) For $A, B \in \mathcal{P}(X)$ one has

$$\begin{aligned} A \subset B &\Leftrightarrow \kappa_A \leq \kappa_B; \\ A = B &\Leftrightarrow \kappa_A = \kappa_B. \end{aligned}$$

- (iii) $\kappa_{A \cap B} = \kappa_A \cdot \kappa_B, \forall A, B \in \mathcal{P}(X)$.
- (iv) $\kappa_{A \setminus B} = \kappa_A \cdot (1 - \kappa_B), \forall A, B \in \mathcal{P}(X)$.
- (v) $\kappa_{A \cup B} = \kappa_A + \kappa_B - \kappa_A \cdot \kappa_B, \forall A, B \in \mathcal{P}(X)$.
- (vi) $\kappa_{A_1 \cup \dots \cup A_n} = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} \kappa_{A_{i_1}} \cdots \kappa_{A_{i_k}}, \forall A_1, \dots, A_n \in \mathcal{P}(X)$.
- (vii) $\kappa_{A \Delta B} = |\kappa_A - \kappa_B| = \kappa_A + \kappa_B - 2\kappa_A \cdot \kappa_B, \forall A, B \in \mathcal{P}(X)$. Here Δ stands for the symmetric set difference, defined by $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Property (vi) is called the *Inclusion-Exclusion Formula*.

HINT: (vi). Show that the right hand side is equal to $1 - (1 - \kappa_{A_1}) \cdots (1 - \kappa_{A_n})$.

REMARK 1.1. The Inclusion-Exclusion formula has an interesting application in Combinatorics. If the ambient set X is finite, then the number of elements of any subset $A \subset X$ is given by

$$|A| = \sum_{x \in X} \kappa_A(x).$$

Using the Inclusion-Exclusion formula, we then get

$$|A_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} |A_{i_1} \cap \dots \cap A_{i_k}|.$$

This is known as the *Inclusion-Exclusion Principle*.

DEFINITION. Let X be a non-empty set, and let $\mathbb{K}$ be one of the fields⁷ $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. An function $\phi : X \rightarrow \mathbb{K}$ is said to be *elementary*, if its range $\phi(X)$ is finite. Remark that this gives

$$\phi = \sum_{\lambda \in \phi(X)} \lambda \cdot \kappa_{\phi^{-1}(\{\lambda\})} = \sum_{\lambda \in \phi(X) \setminus \{0\}} \lambda \cdot \kappa_{\phi^{-1}(\{\lambda\})}.$$

We define

$$\text{Elem}_{\mathbb{K}}(X) = \{\phi : X \rightarrow \mathbb{K} : \phi \text{ elementary}\}.$$

Given a collection $\mathcal{M} \subset \mathcal{P}(X)$, a function $\phi : X \rightarrow \mathbb{K}$ is said to be $\mathcal{M}$ -*elementary*, if ϕ is elementary, and moreover,

$$\phi^{-1}(\{\lambda\}) \in \mathcal{M}, \quad \forall \lambda \in \mathbb{K} \setminus \{0\}.$$

We define

$$\mathcal{M}\text{-Elem}_{\mathbb{K}}(X) = \{\phi : X \rightarrow \mathbb{K} : \phi \text{ } \mathcal{M}\text{-elementary}\}.$$

Exercise 2. With the above notations, prove that $\text{Elem}_{\mathbb{K}}(X)$ is a unital $\mathbb{K}$ -algebra.

PROPOSITION 1.1. *Given a non-empty set X , the collection $\mathcal{P}(X)$ is a unital ring, with the operations*

$$A + B = A \triangle B \text{ and } A \cdot B = A \cap B, \quad A, B \in \mathcal{P}(X).$$

PROOF. First of all, it is clear that $\triangle$ is commutative.

To prove the associativity of $\triangle$, we simply observe that

$$\begin{aligned} \kappa_{(A \triangle B) \triangle C} &= \kappa_{A \triangle B} + \kappa_C - 2\kappa_{A \triangle B} \kappa_C = \\ &= \kappa_A + \kappa_B - 2\kappa_A \kappa_B + \kappa_C - (\kappa_A + \kappa_B - 2\kappa_A \kappa_B) \cdot \kappa_C = \\ &= \kappa_A + \kappa_B + \kappa_C - 2(\kappa_A \kappa_B + \kappa_A \kappa_C + \kappa_B \kappa_C) + 2\kappa_A \kappa_B \kappa_C. \end{aligned}$$

Since the final result is symmetric in A, B, C , we see that we get

$$\kappa_{A \triangle (B \triangle C)} = \kappa_{(A \triangle B) \triangle C},$$

so we indeed get

$$(A \triangle B) \triangle C = A \triangle (B \triangle C).$$

The neutral element for $\triangle$ is the empty set $\emptyset$. Since we obviously have $A \triangle A = \emptyset$, it follows that $(\mathcal{P}(X), \triangle)$ is indeed an abelian group.

The operation $\cap$ is clearly commutative, associative, and has the total set X as the unit.

To check distributivity, we again use characteristic functions:

$$\begin{aligned} \kappa_{(A \cap C) \triangle (B \cap C)} &= \kappa_{A \cap C} + \kappa_{B \cap C} - 2\kappa_{A \cap C} \kappa_{B \cap C} = \\ &= \kappa_A \kappa_C + \kappa_B \kappa_C - 2\kappa_A \kappa_B \kappa_C = (\kappa_A + \kappa_B - 2\kappa_A \kappa_B) \kappa_C = \\ &= \kappa_{A \triangle B} \kappa_C = \kappa_{(A \triangle B) \cap C}, \end{aligned}$$

so we indeed have the equality

$$(A \cap C) \triangle (B \cap C) = (A \triangle B) \cap C.$$

□

⁷ $\mathbb{K}$ can be *any* field.

DEFINITIONS. Let X be a non-empty set. A *ring on X* is a non-empty sub-ring $\mathcal{R} \subset \mathcal{P}(X)$. We do not require the unit X to belong to $\mathcal{R}$, but we do require $\emptyset \in \mathcal{R}$. An *algebra on X* is a ring $\mathcal{A}$ which contains the unit X .

Rings and algebras of sets are characterized as follows.

PROPOSITION 1.2. *Let X be a non-empty set.*

A. *For a non-empty collection $\mathcal{R} \subset \mathcal{P}(X)$, the following are equivalent:*

- (i) *$\mathcal{R}$ is a ring on X ;*
- (ii) *For any $A, B \in \mathcal{R}$, we have $A \setminus B \in \mathcal{R}$ and $A \cup B \in \mathcal{R}$.*

B. *For a non-empty collection $\mathcal{A} \subset \mathcal{P}(X)$, the following are equivalent:*

- (i) *$\mathcal{A}$ is an algebra on X ;*
- (ii) *For any $A \in \mathcal{A}$, we have $X \setminus A \in \mathcal{A}$, and for any $A, B \in \mathcal{A}$, we have $A \cup B \in \mathcal{A}$.*

PROOF. A. (i) $\Rightarrow$ (ii). Assume $\mathcal{R}$ is a ring on X , and let $A, B \in \mathcal{R}$. Then $A \cap B$ belongs to $\mathcal{R}$, so

$$A \setminus B = A \Delta (A \cap B)$$

also belongs to $\mathcal{R}$. It follows that

$$A \cup B = (A \Delta B) \Delta (A \cap B)$$

again belongs to $\mathcal{R}$.

(ii) $\Rightarrow$ (i). Assume $\mathcal{R}$ satisfies property (ii). Start with $A, B \in \mathcal{R}$. Then $A \setminus B$ belongs to $\mathcal{R}$, and

$$A \cap B = A \setminus (A \setminus B)$$

again belongs to $\mathcal{R}$. Since $A \cup B$ also belongs to $\mathcal{R}$, it follows that the set

$$A \Delta B = (A \cup B) \setminus (A \cap B)$$

again belongs to $\mathcal{R}$.

B. (i) $\Rightarrow$ (ii). This is clear from the implication A.(i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). Assume $\mathcal{A}$ satisfies property (ii). Start with two sets $A, B \in \mathcal{A}$. Then the complements $X \setminus A$ and $X \setminus B$ both belong to $\mathcal{A}$, hence their union

$$(X \setminus A) \cup (X \setminus B) = X \setminus (A \cap B)$$

belongs to $\mathcal{A}$, and the complement of this union

$$X \setminus [X \setminus (A \cap B)] = A \cap B$$

will also belong to $\mathcal{A}$.

If $A, B \in \mathcal{A}$, then since $X \setminus B$ belongs to $\mathcal{A}$, by the above considerations, it follows that the intersection

$$A \cap (X \setminus B) = A \setminus B$$

also belongs to $\mathcal{A}$. Likewise, the difference $B \setminus A$ also belongs to $\mathcal{A}$, hence the union

$$(A \setminus B) \cup (B \setminus A) = A \Delta B$$

also belongs to $\mathcal{A}$. By part A, it follows that $\mathcal{A}$ is a ring.

Finally, since $\mathcal{A}$ is non-empty, if we choose some $A \in \mathcal{A}$, then $A \Delta A = \emptyset$ belongs to $\mathcal{A}$, so its complement $X \setminus \emptyset = X$ also belongs to $\mathcal{A}$. $\square$

It will be useful to introduce the following terminology.

DEFINITION. A system of sets $(A_i)_{i \in I}$ is said to be *pair-wise disjoint*, if $A_i \cap A_j = \emptyset$, for all $i, j \in I$ with $i \neq j$.

LEMMA 1.1. *Let X be a non-empty set, let $\mathbb{K}$ be one of fields $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, and let $\mathcal{R}$ be a ring on X . For a function $\phi : X \rightarrow \mathbb{K}$, the following are equivalent:*

- (i) *ϕ is $\mathcal{R}$ -elementary;*
- (ii) *there exist an integer $n \geq 1$ and sets $A_1, \dots, A_n \in \mathcal{R}$, and numbers $\lambda_1, \dots, \lambda_n \in \mathbb{K}$, such that*

$$\phi = \lambda_1 \chi_{A_1} + \dots + \lambda_n \chi_{A_n}.$$

- (ii) *there exist an integer $m \geq 1$, and a finite pair-wise disjoint system $(B_j)_{j=1}^m \subset \mathcal{R}$, and numbers $\mu_1, \dots, \mu_m \in \mathbb{K}$, such that*

$$\phi = \mu_1 \chi_{B_1} + \dots + \mu_m \chi_{B_m}.$$

PROOF. (i) $\Rightarrow$ (ii). Assume ϕ is $\mathcal{R}$ -elementary. If $\phi = 0$, there is nothing to prove, because we have $\phi = \chi_{\emptyset}$. If ϕ is not identically zero, then we can obviously write

$$\phi = \sum_{\lambda \in \phi(X) \setminus \{0\}} \lambda \chi_{\phi^{-1}(\{\lambda\})},$$

with all sets $\phi^{-1}(\{\lambda\})$ in $\mathcal{R}$.

(ii) $\Rightarrow$ (iii). Define

$$\mathcal{E} = \{\psi : X \rightarrow \mathbb{K} : \psi \text{ satisfies property (iii)}\}.$$

Assume ϕ satisfies (ii), i.e.

$$\phi = \lambda_1 \chi_{A_1} + \dots + \lambda_n \chi_{A_n},$$

with $A_1, \dots, A_n \in \mathcal{R}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{K}$. We are going to prove that $\phi \in \mathcal{E}$, by induction on n . The case $n = 1$ is trivial (either $\phi = 0$, so $\phi = \chi_{\emptyset} \in \mathcal{E}$, or $\phi = \lambda \chi_A$ for some $A \in \mathcal{R}$ and $\lambda \neq 0$, in which case we also have $\phi \in \mathcal{E}$).

Assume

$$\alpha_1 \chi_{D_1} + \dots + \alpha_k \chi_{D_k} \in \mathcal{E},$$

for all $D_1, \dots, D_k \in \mathcal{R}$, $\alpha_1, \dots, \alpha_k \in \mathbb{K}$. Start with a function

$$\phi = \lambda_1 \chi_{A_1} + \dots + \lambda_k \chi_{A_k} + \lambda_{k+1} \chi_{A_{k+1}},$$

with $A_1, \dots, A_{k+1} \in \mathcal{R}$ and $\lambda_1, \dots, \lambda_{k+1} \in \mathbb{K}$, and based on the above inductive hypothesis, let us show that $\phi \in \mathcal{E}$. Using the inductive hypothesis, the function

$$\psi = \lambda_2 \chi_{A_2} + \dots + \lambda_k \chi_{A_k} + \lambda_{k+1} \chi_{A_{k+1}}$$

belongs to $\mathcal{E}$, so there exist scalars $\eta_1, \dots, \eta_p \in \mathbb{K}$, an integer $p \geq 1$, and a pair-wise disjoint system $(C_j)_{j=1}^p \subset \mathcal{R}$, such that

$$\psi = \eta_1 \chi_{C_1} + \dots + \eta_p \chi_{C_p}.$$

With this notation, we have

$$\phi = \lambda_1 \chi_{A_1} + \eta_1 \chi_{C_1} + \dots + \eta_p \chi_{C_p}.$$

Put then

$$B_{2j} = A_1 \cap C_j \text{ and } B_{2j-1} = C_j \setminus A_1, \text{ for all } j \in \{1, \dots, p\};$$

$$B_{2p+1} = A_1 \setminus (C_1 \cup \dots \cup C_p).$$

It is clear that $(B_k)_{k=1}^{2p+1} \subset \mathcal{R}$ is pair-wise disjoint. Notice now that the equalities

$$C_j = B_{2j-1} \cup B_{2j}, \quad \forall j \in \{1, \dots, p\},$$

$$A_1 = B_1 \cup B_3 \cup \dots \cup B_{2p+1},$$

combined with the fact that the B 's are pairwise disjoint, give

$$\begin{aligned}\kappa_{C_j} &= \kappa_{B_{2j-1}} + \kappa_{B_{2j}}, \quad \forall j \in \{1, \dots, p\}, \\ \kappa_{A_1} &= \kappa_{B_1} + \kappa_{B_3} + \dots + \kappa_{B_{2p+1}},\end{aligned}$$

which give

$$\phi = \sum_{j=1}^p \eta_j \kappa_{B_{2j}} + \sum_{j=1}^p (\eta_j + \lambda_1) \kappa_{B_{2j-1}} + \lambda_1 \kappa_{B_{2p+1}},$$

which proves that ϕ indeed belongs to $\mathcal{E}$.

(iii) $\Rightarrow$ (i). Assume there exists a finite pair-wise disjoint system $(B_j)_{j=1}^m \subset \mathcal{R}$, and numbers $\mu_1, \dots, \mu_m \in \mathbb{K}$, such that

$$\phi = \mu_1 \kappa_{B_1} + \dots + \mu_m \kappa_{B_m},$$

and let us prove that ϕ is $\mathcal{R}$ -elementary.

If all the μ 's are zero, there is nothing to prove, since $\phi = 0$.

Assume the μ 's are not all equal to zero. Since the μ 's that are equal to zero do not have any contribution, we can in fact assume that all the μ 's are non-zero. Notice that

$$\phi(X) \setminus \{0\} = \{\mu_j : 1 \leq j \leq m\}.$$

In particular ϕ is elementary.

If we start with an arbitrary $\lambda \in \mathbb{K} \setminus \{0\}$, then either $\lambda \notin \phi(X)$, or $\lambda \in \phi(X) \setminus \{0\}$. In the first case we clearly have $\phi^{-1}(\{\lambda\}) = \emptyset \in \mathcal{R}$. In the second case, we have the equality

$$\phi^{-1}(\{\lambda\}) = \bigcup_{j \in M_\lambda} B_j,$$

where

$$M_\lambda = \{j : 1 \leq j \leq m \text{ and } \mu_j = \lambda\}.$$

Since all B 's belong to $\mathcal{R}$, it follows that $\phi^{-1}(\{\lambda\})$ again belongs to $\mathcal{R}$. Having shown that ϕ is elementary, and $\phi^{-1}(\{\lambda\}) \in \mathcal{R}$, for all $\lambda \in \mathbb{K} \setminus \{0\}$, it follows that ϕ is indeed $\mathcal{R}$ -elementary. $\square$

PROPOSITION 1.3. *Let X be a non-empty set, and let $\mathbb{K}$ be one of the fields $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$.*

A. For a non-empty collection $\mathcal{R} \subset \mathcal{P}(X)$, the following are equivalent:

- (i) $\mathcal{R}$ is a ring on X ;
- (ii) $\mathcal{R}\text{-Elem}_{\mathbb{K}}(X)$ is a $\mathbb{K}$ -subalgebra of $\text{Elem}_{\mathbb{K}}(X)$.

B. For a non-empty collection $\mathcal{A} \subset \mathcal{P}(X)$, the following are equivalent:

- (i) $\mathcal{A}$ is an algebra on X ;
- (ii) $\mathcal{A}\text{-Elem}_{\mathbb{K}}(X)$ is a $\mathbb{K}$ -subalgebra of $\text{Elem}_{\mathbb{K}}(X)$, which contains the constant function 1.

PROOF. A. (i) $\Rightarrow$ (ii). Assume $\mathcal{R}$ is a ring on X . Using Lemma 1.1 we see that we have the equality:

$$\mathcal{R}\text{-Elem}_{\mathbb{K}}(X) = \text{Span}\{\kappa_A : A \in \mathcal{R}\}.$$

In particular, this shows that $\mathcal{R}\text{-Elem}_{\mathbb{K}}(X)$ is a $\mathbb{K}$ -linear subspace of $\text{Elem}_{\mathbb{K}}(X)$. Moreover, in order to prove that $\mathcal{R}\text{-Elem}_{\mathbb{K}}(X)$ is a $\mathbb{K}$ -subalgebra, it suffices to prove the implication

$$A, B \in \mathcal{R} \implies \kappa_A \cdot \kappa_B \in \mathcal{R}\text{-Elem}_{\mathbb{K}}(X).$$

But this implication is trivial, since $\varkappa_A \cdot \varkappa_B = \varkappa_{A \cap B}$, and $A \cap B$ belongs to $\mathcal{R}$.

(ii) $\Rightarrow$ (i). Assume $\mathcal{R}\text{-Elem}_{\mathbb{K}}(X)$ is a $\mathbb{K}$ -subalgebra of $\text{Elem}_{\mathbb{K}}(X)$. First of all, since $\varkappa_{\emptyset} = 0 \in \mathcal{R}\text{-Elem}_{\mathbb{K}}(X)$, it follows that $\emptyset \in \mathcal{R}$.

Start now with two sets $A, B \in \mathcal{R}$. Then $\varkappa_A$ and $\varkappa_B$ belong to $\mathcal{R}\text{-Elem}_{\mathbb{K}}(X)$.

Since $\mathcal{R}\text{-Elem}_{\mathbb{K}}(X)$ is an algebra, the function

$$\varkappa_{A \cap B} = \varkappa_A \cdot \varkappa_B$$

belongs to $\mathcal{R}\text{-Elem}_{\mathbb{K}}(X)$, so we immediately see that $A \cap B \in \mathcal{R}$.

Likewise, the function

$$\varkappa_{A \Delta B} = \varkappa_A + \varkappa_B - 2\varkappa_A \varkappa_B$$

belongs to $\mathcal{R}\text{-Elem}_{\mathbb{K}}(X)$, so we also get $A \Delta B \in \mathcal{R}$.

B. This equivalence is clear from part A, plus the identity $\varkappa_X = 1$. $\square$

Algebras of elementary functions give in fact a complete description for rings or algebras of sets, as indicated in the result below.

PROPOSITION 1.4. *Let X be a non-empty set, and let $\mathbb{K}$ be one of the fields $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$.*

A. *The map*

$$\mathcal{R} \longmapsto \mathcal{R}\text{-Elem}_{\mathbb{K}}(X)$$

is a bijective correspondence from the collection of all rings on X , and the collection of all $\mathbb{K}$ -subalgebras of $\text{Elem}_{\mathbb{K}}(X)$.

B. *The map*

$$\mathcal{A} \longmapsto \mathcal{A}\text{-Elem}_{\mathbb{K}}(X)$$

is a bijective correspondence from the collection of all algebras on X , and the collection of all $\mathbb{K}$ -subalgebras of $\text{Elem}_{\mathbb{K}}(X)$ that contain 1.

PROOF. A. We start by proving surjectivity. Let $\mathcal{E} \subset \text{Elem}_{\mathbb{K}}(X)$ be an arbitrary $\mathbb{K}$ -subalgebra. Define the collection

$$\mathcal{R} = \{A \subset X : \varkappa_A \in \mathcal{E}\}.$$

If $A, B \in \mathcal{R}$, then the equalities

$$\varkappa_{A \cap B} = \varkappa_A \varkappa_B \text{ and } \varkappa_{A \Delta B} = \varkappa_A + \varkappa_B - 2\varkappa_A \varkappa_B,$$

combined with the fact that $\mathcal{E}$ is a subalgebra, prove that $\varkappa_{A \cap B}$ and $\varkappa_{A \Delta B}$ both belong to $\mathcal{E}$, hence $A \cap B$ and $A \Delta B$ both belong to $\mathcal{R}$. This shows that $\mathcal{R}$ is a ring.

It is pretty clear (see Lemma 1.1) that $\mathcal{R}\text{-Elem}_{\mathbb{K}}(X) \subset \mathcal{E}$. To prove the other inclusion, start with some arbitrary function $\phi \in \mathcal{E}$, and let us prove that $\phi \in \mathcal{R}\text{-Elem}_{\mathbb{K}}(X)$. If $\phi = 0$, there is nothing to prove. Assume ϕ is not identically zero. We write $\phi(X) \setminus \{0\}$ as $\{\lambda_1, \dots, \lambda_n\}$, with $\lambda_i \neq \lambda_j$ for all $i, j \in \{1, \dots, n\}$ with $i \neq j$. For each $i \in \{1, \dots, n\}$, we set $A_i = \phi^{-1}(\{\lambda_i\})$, so that

$$\phi = \sum_{i=1}^n \lambda_i \cdot \varkappa_{A_i}.$$

Since all λ 's are different, the matrix

$$T = \begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^n & \lambda_2^n & \dots & \lambda_n^n \end{bmatrix}$$

is invertible. Take $[\alpha_{ij}]_{i,j=1}^n$ to be the inverse of T . The obvious equalities

$$\phi^k = \sum_{j=1}^n \lambda_j^k \varkappa_{A_j}, \quad \forall k = 1, \dots, n$$

can be written in matrix form as

$$\begin{bmatrix} \phi \\ \phi^2 \\ \vdots \\ \phi^n \end{bmatrix} = T \cdot \begin{bmatrix} \varkappa_{A_1} \\ \varkappa_{A_2} \\ \vdots \\ \varkappa_{A_n} \end{bmatrix},$$

so multiplying by T^{-1} yields

$$\varkappa_{A_j} = \sum_{k=1}^n \alpha_{jk} \phi^k, \quad \forall j = 1, \dots, n,$$

which proves that $\varkappa_{A_1}, \dots, \varkappa_{A_n} \in \mathcal{E}$, so $A_1, \dots, A_n \in \mathcal{R}$. This then shows that $\phi \in \mathcal{R}\text{-Elem}_{\mathbb{K}}(X)$.

We now prove injectivity. Suppose first that $\mathcal{R}$ and $\mathcal{S}$ are rings such that $\mathcal{R}\text{-Elem}_{\mathbb{K}}(X) = \mathcal{S}\text{-Elem}_{\mathbb{K}}(X)$, and let us prove that $\mathcal{R} = \mathcal{S}$. For every $A \in \mathcal{R}$, the function $\varkappa_A \in \mathcal{R}\text{-Elem}_{\mathbb{K}}(X)$ is also $\mathcal{S}$ -elementary, which means that $A \in \mathcal{S}$. This proves the inclusion $\mathcal{R} \subset \mathcal{S}$. By symmetry we also have the inclusion $\mathcal{S} \subset \mathcal{R}$, so indeed $\mathcal{R} = \mathcal{S}$.

B. This part is obvious from A. □

DEFINITIONS. Let X be a (non-empty) set. A collection $\mathcal{U} \subset \mathcal{P}(X)$ is called a σ -ring, if it is a ring, and it has the property:

(σ) Whenever $(A_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{U}$, it follows that $\bigcup_{n=1}^{\infty} A_n$ also belongs to $\mathcal{U}$.

A collection $\mathcal{S} \subset \mathcal{P}(X)$ is called a σ -algebra, if it is an algebra, and it has property (σ).

Clearly, every σ -algebra is a σ -ring.

REMARKS 1.2. A. For σ -rings and σ -algebras, one of the properties in the definition of rings and algebras is redundant. More explicitly:

- (i) A collection $\mathcal{U} \subset \mathcal{P}(X)$ is a σ -ring, if and only if it has the property (σ) and the property: $A, B \in \mathcal{U} \implies A \setminus B \in \mathcal{U}$.
- (ii) A collection $\mathcal{S} \subset \mathcal{P}(X)$ is a σ -algebra, if and only if it has the property (σ) and the property: $A \in \mathcal{S} \implies X \setminus A \in \mathcal{S}$.

B. If $\mathcal{U}$ is a σ -ring, then it also has the property

(δ) $(A_n)_{n=1}^{\infty} \subset \mathcal{U} \implies \bigcap_{n=1}^{\infty} A_n \in \mathcal{U}$.

Since σ -algebras are σ -rings, they will also have property (δ).

DEFINITIONS. Let X be a non-empty set. A sequence $(A_n)_{n \geq 1}$ of subsets of X is said to be *monotone*, if it satisfies one of the following conditions:

- ($\uparrow$) $A_n \subset A_{n+1}, \forall n \geq 1$,
- ($\downarrow$) $A_n \supset A_{n+1}, \forall n \geq 1$.

In the case ($\uparrow$) the sequence is said to be *increasing*, and we define

$$\lim_{n \rightarrow \infty} A_n = \bigcup_{n=1}^{\infty} A_n.$$

In the case ($\downarrow$) the sequence is said to be *decreasing*, and we define

$$\lim_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} A_n.$$

A collection $\mathcal{M} \subset \mathcal{P}(X)$ is said to be a *monotone class on X* , if it satisfies the condition:

- (M) *whenever $(A_n)_{n \geq 1}$ is a monotone sequence in $\mathcal{M}$, it follows that its limit $\lim_{n \rightarrow \infty} A_n$ also belongs to $\mathcal{M}$.*

PROPOSITION 1.5. *Let $\mathcal{R}$ be a ring on X . Then the following are equivalent:*

- (i) $\mathcal{R}$ is a σ -ring;
- (ii) $\mathcal{R}$ is a monotone class.

PROOF. (i) $\Rightarrow$ (ii). This is immediate from the definition and Remark 1.2.B.

(ii) $\Rightarrow$ (i). Assume $\mathcal{R}$ is a monotone class, and let us prove that it is a σ -ring. By Remark 1.2.A, we only need to prove that $\mathcal{R}$ has property (σ). Start with an arbitrary sequence $(A_n)_{n \geq 1}$ in $\mathcal{R}$, and let us prove that $\bigcup_{n=1}^{\infty} A_n$ again belongs to $\mathcal{R}$. For every integer $n \geq 1$, we define $B_n = \bigcup_{k=1}^n A_k$. Since $\mathcal{R}$ is a ring, it follows that $B_n \in \mathcal{R}$, $\forall n \geq 1$. Moreover, the sequence $(B_n)_{n \geq 1}$ is increasing, so by assumption, the set $\bigcup_{n=1}^{\infty} A_n = \lim_{n \rightarrow \infty} B_n$ indeed belongs to $\mathcal{R}$. $\square$

LECTURE 19

2. Constructing (σ -)rings and (σ -)algebras

In this section we outline three methods of constructing (σ -)rings and (σ -)algebras. It turns out that one can devise some general procedures, which work for all the types of set collections considered, so it will be natural to begin with some very general considerations.

DEFINITION. Suppose one has a *type* Θ of set collections. In other words, for any set X , one defines what it means for a collection $\mathcal{C} \subset \mathcal{P}(X)$ to be of *type* Θ . The type Θ is said to be *consistent*, if for every set X , one has the following conditions:

- the collection $\mathcal{P}(X)$, of all subsets of X , is of type Θ ;
- if $\mathcal{C}_i, i \in I$ are collections of type Θ , then the intersection $\bigcap_{i \in I} \mathcal{C}_i$ is again of type Θ .

EXAMPLES 2.1. The following types are consistent:

- The type $\mathbf{R}$ of *rings*;
- The type $\mathbf{A}$ of *algebras*;
- The type $\mathbf{S}$ of σ -*rings*;
- The type Σ of σ -*algebras*;
- The type $\mathbf{M}$ of *monotone classes*.

The reason for the consistency is simply the fact that each of these types is defined by means of set operations.

DEFINITION. Let Θ be a consistent type, let X be a set, and let $\mathcal{E} \subset \mathcal{P}(X)$ be an arbitrary collection of sets. Define

$$\mathcal{F}_\Theta(\mathcal{E}, X) = \{\mathcal{C} \subset \mathcal{P}(X) : \mathcal{E} \subset \mathcal{C}, \text{ and } \mathcal{C} \text{ is of type } \Theta \text{ on } X\}.$$

Notice that the family $\mathcal{F}_\Theta(\mathcal{E}, X)$ is non-empty, since it contains at least the collection $\mathcal{P}(X)$. The collection

$$\Theta_X(\mathcal{E}) = \bigcap_{\mathcal{C} \in \mathcal{F}_\Theta(\mathcal{E}, X)} \mathcal{C}$$

is of type Θ on X , and is called the *type Θ class generated by $\mathcal{E}$* . When there is no danger of confusion, the ambient set X will be omitted.

COMMENT. In the above setting, the class $\Theta(\mathcal{E})$ is the *smallest collection of type Θ on X , which contains $\mathcal{E}$* . In other words, if $\mathcal{C}$ is a collection of type Θ on X , with $\mathcal{E} \subset \mathcal{C}$, then $\mathcal{C} \supset \Theta(\mathcal{E})$. This follows immediately from the fact that $\mathcal{C}$ belongs to $\mathcal{F}_\Theta(\mathcal{E}, X)$.

EXAMPLES 2.2. Let X be a (non-empty) set, and let $\mathcal{E}$ be an arbitrary collection of subsets of X . According to the previous list of consistent types $\mathbf{R}$, $\mathbf{A}$, $\mathbf{S}$, Σ , and $\mathbf{M}$, one can construct the following collections.

- (i) $\mathbf{R}(\mathcal{E})$, the ring generated by $\mathcal{E}$; this is the smallest ring that contains $\mathcal{E}$.

- (ii) $\mathbf{A}(\mathcal{E})$, the algebra generated by $\mathcal{E}$; this is the smallest algebra that contains $\mathcal{E}$.
- (iii) $\mathbf{S}(\mathcal{E})$, the σ -ring generated by $\mathcal{E}$; this is the smallest σ -ring that contains $\mathcal{E}$.
- (iv) $\mathbf{\Sigma}(\mathcal{E})$, the σ -algebra generated by $\mathcal{E}$; this is the smallest σ -algebra that contains $\mathcal{E}$.
- (v) $\mathbf{M}(\mathcal{E})$, the monotone class generated by $\mathcal{E}$; this is the smallest monotone class that contains $\mathcal{E}$.

COMMENT. Assume Θ is a consistent type. Suppose $\mathcal{E}$ is an arbitrary collection of subsets of some fixed non-empty set X . There are instances when we would like to decide whether a class $\mathcal{C} \supset \mathcal{E}$ coincides with $\Theta(\mathcal{E})$. The following is a useful test:

- (i) check that $\mathcal{C}$ is of type Θ ;
- (ii) check the inclusion $\mathcal{C} \subset \Theta(\mathcal{E})$.

By (i) we must have $\mathcal{C} \supset \Theta(\mathcal{E})$, so by (ii) we will indeed have equality.

A simple illustration of the above technique allows one to describe the ring and the algebra generated by a collection of sets.

PROPOSITION 2.1. *Let X be a non-empty set, and let $\mathcal{E}$ be an arbitrary collection of subsets of X .*

A. For a set $A \subset X$, the following are equivalent:

- (i) $A \in \mathbf{R}(\mathcal{E})$;
- (ii) *There exist sets $A_1, A_2, \dots, A_n$ such that $A = A_1 \triangle A_2 \triangle \dots \triangle A_n$, and each A_k , $k = 1, \dots, n$ is a finite intersection of sets in $\mathcal{E}$.*

B. The algebra generated by $\mathcal{E}$ is

$$\mathbf{A}(\mathcal{E}) = \mathbf{R}(\mathcal{E}) \cup \{X \setminus A : A \in \mathbf{R}(\mathcal{E})\} = \mathbf{R}(\mathcal{E} \cup \{X\}).$$

PROOF. A. Define $\mathcal{R}$ to be the class of all subsets $A \subset X$, which satisfy property (ii), so that what we have to prove is the equality

$$\mathcal{R} = \mathbf{R}(\mathcal{E}).$$

It is clear that $\mathcal{E} \subset \mathcal{R}$. Since every finite intersection of sets in $\mathcal{E}$ belongs to $\mathbf{R}(\mathcal{E})$, and the latter is a ring, it follows that $\mathcal{R} \subset \mathbf{R}(\mathcal{E})$. So in order to prove the desired equality, all we have to do is to prove that $\mathcal{R}$ is a ring. But this is pretty clear, if we think $\triangle$ as the sum operation, and $\cap$ as the product operation. More explicitly, let us take $\Pi(\mathcal{E})$ to be the collection of all finite intersections of sets in $\mathcal{E}$, so that

$$(1) \quad A \cap B \in \Pi(\mathcal{E}), \quad \forall A, B \in \Pi(\mathcal{E}).$$

Now if we start with two sets $A, B \in \mathcal{R}$, written as $A = A_1 \triangle \dots \triangle A_m$ and $B = B_1 \triangle \dots \triangle B_n$, with $A_1, \dots, A_m, B_1, \dots, B_n \in \Pi(\mathcal{E})$, then the equality

$$A \cap B = [(A_1 \cap B_1) \triangle \dots \triangle (A_m \cap B_1)] \triangle [(A_1 \cap B_2) \triangle \dots \triangle (A_m \cap B_2)] \triangle \dots \\ \dots \triangle [(A_1 \cap B_n) \triangle \dots \triangle (A_m \cap B_n)],$$

combined with (1) proves that $A \cap B \in \mathcal{R}$, while the equality

$$A \triangle B = A_1 \triangle \dots \triangle A_m \triangle B_1 \triangle \dots \triangle B_n$$

proves that $A \triangle B$ also belongs to $\mathcal{R}$.

B. Define

$$\mathcal{A} = \mathbf{R}(\mathcal{E}) \cup \{X \setminus A : A \in \mathbf{R}(\mathcal{E})\}.$$

Since we clearly have $\mathcal{E} \subset \mathcal{A} \subset \mathbf{A}(\mathcal{E})$, all we need to prove is the fact that $\mathcal{A}$ is an algebra. It is clear that, whenever $A \in \mathcal{A}$, it follows that $X \setminus A \in \mathcal{A}$. Therefore (see Section III.1), we only need to show that

$$A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}.$$

There are four cases to examine: (i) $A, B \in \mathbf{R}(\mathcal{E})$; (ii) $A \in \mathbf{R}(\mathcal{E})$ and $X \setminus B \in \mathbf{R}(\mathcal{E})$; (iii) $X \setminus A \in \mathbf{R}(\mathcal{E})$ and $B \in \mathbf{R}(\mathcal{E})$; (iv) $X \setminus A \in \mathbf{R}(\mathcal{E})$ and $X \setminus B \in \mathbf{R}(\mathcal{E})$.

Case (i) is clear, since it will force $A \cup B \in \mathbf{R}(\mathcal{E})$.

In case (ii), we use

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B) = (X \setminus B) \setminus A,$$

which proves that $X \setminus (A \cup B) \in \mathbf{R}(\mathcal{E})$.

Case (iii) is proven exactly as case (ii).

In case (iv) we use

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B),$$

which proves that $X \setminus (A \cup B) \in \mathbf{R}(\mathcal{E})$.

The equality $\mathbf{A}(\mathcal{E}) = \mathbf{R}(\mathcal{E} \cup \{X\})$ is trivial. $\square$

COMMENT. Unfortunately, for σ -rings and σ -algebras, no easy constructive description is available. There is an analogue of Proposition 2.1 uses transfinite induction. In order to formulate such a statement, we introduce the following notations. For every collection $\mathcal{C}$ of subsets of X , we define

$$\mathcal{C}^* = \left\{ \bigcup_{n=1}^{\infty} (A_n \setminus B_n) : A_n, B_n \in \mathcal{C} \cup \{\emptyset\}, \forall n \geq 1 \right\}.$$

Notice that

$$(2) \quad \mathcal{C} \cup \{\emptyset\} \subset \mathcal{C}^* \subset \mathbf{S}(\mathcal{C}).$$

THEOREM 2.1. *Let X be a non-empty set, and let $\mathcal{E}$ be an arbitrary collection of subsets of X . For every ordinal number η define the set*

$$P_\eta = \{\alpha : \alpha \text{ ordinal number with } \alpha < \eta\}.$$

Let Ω denote the smallest uncountable ordinal number, and define the classes $\mathcal{E}_\alpha$, $\alpha \in P_\Omega$ recursively by $\mathcal{E}_0 = \mathcal{E}$, and

$$\mathcal{E}_\alpha = \left(\bigcup_{\beta \in P_\alpha} \mathcal{E}_\beta \right)^*, \quad \forall \alpha \in P_\Omega \setminus \{0\}.$$

Then the σ -ring generated by $\mathcal{E}$ is

$$\mathbf{S}(\mathcal{E}) = \bigcup_{\alpha \in P_\Omega} \mathcal{E}_\alpha.$$

PROOF. Denote the union $\bigcup_{\alpha \in P_\Omega} \mathcal{E}_\alpha$ simply by $\mathcal{U}$. It is obvious that $\mathcal{E} \subset \mathcal{U}$.

Let us prove that $\mathcal{U} \subset \mathbf{S}(\mathcal{E})$. We do this by showing that $\mathcal{E}_\alpha \subset \mathbf{S}(\mathcal{E})$, $\forall \alpha \in P_\Omega$. We use transfinite induction. The case $\alpha = 0$ is clear. Assume $\alpha \in P_\Omega$ has the property that $\mathcal{E}_\beta \subset \mathbf{S}(\mathcal{E})$, for all $\beta \in P_\alpha$, and let us show that we also have the inclusion $\mathcal{E}_\alpha \subset \mathbf{S}(\mathcal{E})$. On the one hand, if we take the class

$$\mathcal{C} = \bigcup_{\beta \in P_\alpha} \mathcal{E}_\beta,$$

then $\mathcal{E}_\alpha = \mathcal{C}^*$. On the other hand, by the inductive hypothesis, we have $\mathcal{C} \subset \mathbf{S}(\mathcal{E})$, which clearly forces $\mathbf{S}(\mathcal{C}) \subset \mathbf{S}(\mathcal{E})$. Then the desired inclusion follows from (2).

In order to finish the proof, we only need to prove that $\mathcal{U}$ is a σ -ring. It suffices to prove the equality $\mathcal{U}^* = \mathcal{U}$, which in turn is equivalent to the inclusion $\mathcal{U}^* \subset \mathcal{U}$. Start with some $U \in \mathcal{U}^*$, written as

$$U = \bigcup_{n=1}^{\infty} (A_n \setminus B_n),$$

for two sequences $(A_n)_{n=1}^{\infty}$ and $(B_n)_{n=1}^{\infty}$ in $\mathcal{U}$. For each $n \geq 1$ choose $\alpha_n, \beta_n \in P_\Omega$, such that $A_n \in \mathcal{E}_{\alpha_n}$ and $B_n \in \mathcal{E}_{\beta_n}$. Form then the countable set

$$Z = \{\alpha_n : n \in \mathbb{N}\} \cup \{\beta_n : n \in \mathbb{N}\} \subset P_\Omega.$$

Then we clearly have

$$U \in \left(\bigcup_{\nu \in Z} \mathcal{E}_\nu \right)^*.$$

Since Z is countable, there is a strict upper bound for Z in P_Ω , i.e. there exists $\gamma \in P_\Omega$, such that $\alpha_n < \gamma$ and $\beta_n < \gamma$, $\forall n \geq 1$. In other words we have $Z \subset P_\gamma$, so

$$U \in \left(\bigcup_{\nu \in P_\gamma} \mathcal{E}_\nu \right)^* = \mathcal{E}_\gamma,$$

so U indeed belongs to $\mathcal{U}$. □

COROLLARY 2.1. *Given a non-empty set X , and an arbitrary collection $\mathcal{E}$ of subsets of X , with $\text{card } \mathcal{E} \geq 2$, one has the inequality*

$$\text{card } \mathbf{S}(\mathcal{E}) \leq (\text{card } \mathcal{E})^{\aleph_0}.$$

PROOF. Using the notations from the proof of the above theorem, we will first prove, by transfinite induction, that

$$(3) \quad \text{card } \mathcal{E}_\alpha \leq (\text{card } \mathcal{E})^{\aleph_0}, \quad \forall \alpha \in P_\Omega.$$

The case $\alpha = 0$ is clear. Assume now we have $\alpha \in P_\Omega \setminus \{0\}$, such that

$$\text{card } \mathcal{E}_\beta \leq (\text{card } \mathcal{E})^{\aleph_0}, \quad \forall \beta \in P_\alpha,$$

and let us prove that we also have the inequality $\text{card } \mathcal{E}_\alpha \leq (\text{card } \mathcal{E})^{\aleph_0}$. If we take $\mathcal{C} = \bigcup_{\beta \in P_\alpha} \mathcal{E}_\beta$, we know that $\mathcal{C}$ is a countable union of sets, each having cardinality $\leq (\text{card } \mathcal{E})^{\aleph_0}$, so we immediately get

$$\text{card } \mathcal{C} \leq \aleph_0 \cdot (\text{card } \mathcal{E})^{\aleph_0} = (\text{card } \mathcal{E})^{\aleph_0}.$$

Then the collection

$$D(\mathcal{C}) = \{A \setminus B : A, B \in \mathcal{C}\}$$

has cardinality at most $(\text{card } \mathcal{C})^2$, so we also have

$$\text{card } D(\mathcal{C}) \leq (\text{card } \mathcal{E})^{\aleph_0}.$$

Finally, the collection $\mathcal{E}_\alpha = \mathcal{C}^*$ has cardinality at most $(\text{card } D(\mathcal{C}))^{\aleph_0}$, so we get

$$\text{card } \mathcal{E}_\alpha \leq [(\text{card } \mathcal{E})^{\aleph_0}]^{\aleph_0} = (\text{card } \mathcal{E})^{\aleph_0}.$$

Having proven (3), we now have

$$\text{card } \mathbf{S}(\mathcal{E}) = \text{card} \left(\bigcup_{\alpha \in P_\Omega} \mathcal{E}_\alpha \right) \leq (\text{card } P_\Omega) \cdot (\text{card } \mathcal{E})^{\aleph_0} = \aleph_1 \cdot (\text{card } \mathcal{E})^{\aleph_0}.$$

Since $\aleph_1 \leq \mathfrak{c} = 2^{\aleph_0} \leq (\text{card } \mathcal{E})^{\aleph_0}$, the above estimate gives

$$\text{card } \mathbf{S}(\mathcal{E}) \leq [(\text{card } \mathcal{E})^{\aleph_0}]^2 = (\text{card } \mathcal{E})^{\aleph_0}. \quad \square$$

COMMENT. Suppose Θ is a consistent type. There is a very useful technique for proving results on classes of the form $\Theta(\mathcal{E})$. More explicitly, suppose $\mathcal{E}$ is an arbitrary collection of subsets of X , and (P) is a certain property which refers to subsets of X . Suppose now we want to prove a statement like:

(*) *Every set $A \in \Theta(\mathcal{E})$ has property (P).*

In order to prove such a statement, one defines

$$\mathcal{U} = \{A \in \Theta(\mathcal{E}) : A \text{ has property (P)}\},$$

and it suffices to prove that:

- (i) $\mathcal{U}$ is of type Θ ;
- (ii) $\mathcal{U} \supset \mathcal{E}$, i.e. every set $A \in \mathcal{E}$ has property (P).

Indeed, if we prove the above two facts, that would force $\mathcal{U} \supset \Theta(\mathcal{E})$, and since by construction we have $\mathcal{U} \supset \Theta(\mathcal{E})$, we will in fact get $\mathcal{U} = \Theta(\mathcal{E})$, thus proving (*).

As a first illustration of the above technique, we prove the following.

PROPOSITION 2.2. *Let X be a non-empty set, and let $\mathcal{R}$ be a ring on X . Then the σ -ring generated by $\mathcal{R}$ is the same as the monotone class generated by $\mathcal{R}$, that is, one has the equality*

$$\mathbf{S}(\mathcal{R}) = \mathbf{M}(\mathcal{R}).$$

PROOF. Since $\mathbf{S}(\mathcal{R})$ is a monotone class, and contains $\mathcal{R}$, we have the inclusion $\mathbf{S}(\mathcal{R}) \supset \mathbf{M}(\mathcal{R})$.

To prove the other inclusion, using the fact that $\mathbf{M}(\mathcal{R})$ contains $\mathcal{R}$, it suffices to show that $\mathbf{M}(\mathcal{R})$ is a σ -ring. Since $\mathbf{M}(\mathcal{R})$ is already a monotone class, we only need to show that it is a ring. In other words, we need to show that whenever $A, B \in \mathbf{M}(\mathcal{R})$, it follows that both $A \setminus B$ and $A \cup B$ belong to $\mathbf{M}(\mathcal{R})$. Define then, for every $A \in \mathbf{M}(\mathcal{R})$ the set

$$\mathcal{M}_A = \{B \in \mathbf{M}(\mathcal{R}) : A \cap B, A \setminus B, B \setminus A \in \mathbf{M}(\mathcal{R})\},$$

so that what we need to prove is:

(*) $\mathcal{M}_A = \mathbf{M}(\mathcal{R})$, $\forall A \in \mathbf{M}(\mathcal{R})$.

Before we proceed with the proof of (*), let us first remark that, for $A, B \in \mathbf{M}(\mathcal{R})$, one has

$$(4) \quad B \in \mathcal{M}_A \iff A \in \mathcal{M}_B.$$

Secondly, we have the following

Claim 1: For every $A \in \mathbf{M}(\mathcal{R})$, the collection $\mathcal{M}_A$ is a monotone class.

To prove this, we start with a monotone sequence $(B_n)_{n=1}^\infty$ in $\mathcal{M}_A$, and we prove that the limit $B = \lim_{n \rightarrow \infty} B_n$ again belongs to $\mathcal{M}_A$. First of all, clearly B belongs to $\mathbf{M}(\mathcal{R})$. Second, since the sequences $(A \cap B_n)_{n=1}^\infty$, $(A \setminus B_n)_{n=1}^\infty$, and $(B_n \setminus A)_{n=1}^\infty$ are all monotone sequences in $\mathbf{M}(\mathcal{R})$, and since $\mathbf{M}(\mathcal{R})$ is a monotone class, it follows

that the limits $A \cap B = \lim_{n \rightarrow \infty} (A \cap B_n)$, $A \setminus B = \lim_{n \rightarrow \infty} (A \setminus B_n)$, and $B \setminus A = \lim_{n \rightarrow \infty} (B_n \setminus A)$ all belong to $\mathbf{M}(\mathcal{R})$, so B indeed belongs to $\mathcal{M}_A$.

Having proven Claim 1, we now prove (*) in a particular case:

Claim 2: $\mathcal{M}_A = \mathbf{M}(\mathcal{R})$, $\forall A \in \mathcal{R}$.

Fix $A \in \mathcal{R}$. We know that $\mathcal{M}_A \subset \mathbf{M}(\mathcal{R})$ is a monotone class, so it suffices to prove that $\mathcal{M}_A \supset \mathcal{R}$. But this is obvious, since $\mathcal{R}$ is a ring.

We now proceed with the proof of (*) in the general case. If we define

$$\mathcal{U} = \{A \in \mathbf{M}(\mathcal{R}) : \mathcal{M}_A = \mathbf{M}(\mathcal{R})\},$$

all we need to prove is the equality $\mathcal{U} = \mathbf{M}(\mathcal{R})$. By Claim 2, we know that $\mathcal{U} \supset \mathcal{R}$, so it suffices to prove that $\mathcal{U}$ is a monotone class. Start then with a monotone sequence $(A_n)_{n=1}^\infty$, and let us show that the limit $A = \lim_{n \rightarrow \infty} A_n$ again belongs to $\mathcal{U}$. First of all, A belongs to $\mathbf{M}(\mathcal{R})$. What we then have to prove is that $\mathcal{M}_A = \mathbf{M}(\mathcal{R})$. Start with some arbitrary $B \in \mathbf{M}(\mathcal{R})$. We know that $B \in \mathcal{M}_{A_n}$, $\forall n \geq 1$. Using (4) we have $A_n \in \mathcal{M}_B$, $\forall n \geq 1$, and using the fact that $\mathcal{M}_B$ is a monotone class (see Claim 1), it follows that $A = \lim_{n \rightarrow \infty} A_n$ belongs to $\mathcal{M}_A$. Using (4) again, this gives $B \in \mathcal{M}_A$. This way we have proven that any $B \in \mathbf{M}(\mathcal{R})$ also belongs to $\mathcal{M}_A$, so we indeed have the equality $\mathcal{M}_A = \mathbf{M}(\mathcal{R})$. $\square$

COROLLARY 2.2. *Let X be a non-empty set, and let $\mathcal{E}$ be an arbitrary family of subsets of X . Then the σ -ring, and the σ -algebra generated by $\mathcal{E}$ respectively, are given as the monotone classes generated by the ring, and by the algebra generated by $\mathcal{E}$ respectively. That is, one has the equalities:*

- (i) $\mathbf{S}(\mathcal{E}) = \mathbf{M}(\mathbf{R}(\mathcal{E}))$;
- (ii) $\Sigma(\mathcal{E}) = \mathbf{M}(\mathbf{A}(\mathcal{E}))$.

PROOF. (i). By the above result, since $\mathbf{R}(\mathcal{E})$ is a ring, we have

$$(5) \quad \mathbf{M}(\mathbf{R}(\mathcal{E})) = \mathbf{S}(\mathbf{R}(\mathcal{E})).$$

Since $\mathbf{S}(\mathbf{R}(\mathcal{E}))$ is a σ -ring, and contains $\mathcal{E}$, it follows that

$$\mathbf{S}(\mathbf{R}(\mathcal{E})) \supset \mathbf{S}(\mathcal{E}).$$

Conversely, since $\mathbf{S}(\mathcal{E})$ is a ring, and contains $\mathcal{E}$, we get the inclusion

$$\mathbf{S}(\mathcal{E}) \supset \mathbf{R}(\mathcal{E}),$$

and since $\mathbf{S}(\mathcal{E})$ is a σ -ring, we will now get

$$\mathbf{S}(\mathcal{E}) \supset \mathbf{S}(\mathbf{R}(\mathcal{E})),$$

so we get

$$\mathbf{S}(\mathbf{R}(\mathcal{E})) = \mathbf{S}(\mathcal{E}).$$

Using (5), the desired equality follows.

(ii). This follows from Proposition 2.1 and part (i) applied to $\mathcal{E} \cup \{X\}$, combined with the obvious equality $\Sigma(\mathcal{E}) = \mathbf{S}(\mathcal{E} \cup \{X\})$. $\square$

The σ -ring and the σ -algebra, generated by an arbitrary collection of sets, are related by means of the following result.

PROPOSITION 2.3. *Let X be a non-empty set, and let $\mathcal{E}$ be an arbitrary collection of subsets of X . Define the collection*

$$\mathcal{P}_\sigma^\mathcal{E}(X) = \left\{ A \subset X : \text{there exists } (E_n)_{n=1}^\infty \subset \mathcal{E}, \text{ with } A \subset \bigcup_{n=1}^\infty E_n \right\}.$$

- (i) $\mathcal{P}_\sigma^\mathcal{E}(X)$ is a σ -ring on X ;
- (ii) the σ -ring $\mathbf{S}(\mathcal{E})$ and the σ -algebra $\Sigma(\mathcal{E})$, generated by $\mathcal{E}$, satisfy the equality

$$\mathbf{S}(\mathcal{E}) = \Sigma(\mathcal{E}) \cap \mathcal{P}_\sigma^\mathcal{E}(X).$$

PROOF. Part (i) is trivial.

To prove part (ii), we first observe that the intersection $\Sigma(\mathcal{E}) \cap \mathcal{P}_\sigma^\mathcal{E}(X)$ is a σ -ring, which obviously contains $\mathcal{E}$, so we immediately get the inclusion

$$\mathbf{S}(\mathcal{E}) \subset \Sigma(\mathcal{E}) \cap \mathcal{P}_\sigma^\mathcal{E}(X).$$

The key ingredient in proving the inclusion “ $\supset$ ” is contained in the following.

Claim: Given a set $E \in \mathcal{E}$, the collection

$$\mathcal{A}_E(X) = \{ A \subset X : A \cap E \in \mathbf{S}(\mathcal{E}) \}$$

is a σ -algebra on X .

To prove this we need to check:

- (a) if A belongs to $\mathcal{A}_E(X)$, then $X \setminus A$ also belongs to $\mathcal{A}_E(X)$;
- (b) whenever $(A_n)_{n=1}^\infty$ is a sequence of sets in $\mathcal{A}_E(X)$, it follows that the union $\bigcup_{n=1}^\infty A_n$ also belongs to $\mathcal{A}_E(X)$.

To check (a) we simply remark that, since both E and $A \cap E$ belong to $\mathbf{S}(\mathcal{E})$, it follows immediately that $(X \setminus A) \cap E = E \setminus (A \cap E)$, also belongs to $\mathbf{S}(\mathcal{E})$, which means that $X \setminus A$ indeed belongs to $\mathcal{A}_E(X)$.

Property (b) is clear. Since the fact that $A_n \cap E$ belongs to $\mathbf{S}(\mathcal{E})$, for all n , immediately gives the fact that $(\bigcup_{n=1}^\infty A_n) \cap E = \bigcup_{n=1}^\infty (A_n \cap E)$ belongs to $\mathbf{S}(\mathcal{E})$, which means precisely that $\bigcup_{n=1}^\infty A_n$ belongs to $\mathcal{A}_E$.

Having proven the Claim, we now proceed with the proof of the inclusion $\mathbf{S}(\mathcal{E}) \supset \Sigma(\mathcal{E}) \cap \mathcal{P}_\sigma^\mathcal{E}(X)$. Start with some set $A \in \Sigma(\mathcal{E}) \cap \mathcal{P}_\sigma^\mathcal{E}(X)$, and we will show that A belongs to $\mathbf{S}(\mathcal{E})$. First of all, there exists a sequence $(E_n)_{n=1}^\infty \subset \mathcal{E}$, such that

$$(6) \quad A \subset \bigcup_{n=1}^\infty E_n.$$

Using the Claim, we know that for each $n \in \mathbb{N}$, the collection $\mathcal{A}_{E_n}$ is a σ -algebra. This σ -algebra clearly contains $\mathcal{E}$, so we have

$$\Sigma(\mathcal{E}) \subset \mathcal{A}_{E_n}, \quad \forall n \in \mathbb{N}.$$

In particular, we get the fact that $A \in \mathcal{A}_{E_n}$, which means that $A \cap E_n$ belongs to $\mathbf{S}(\mathcal{E})$, for all $n \in \mathbb{N}$. But then the inclusion (6) forces the equality

$$A = \bigcup_{n=1}^\infty (A \cap E_n),$$

which then gives the fact that A indeed belongs to $\mathbf{S}(\mathcal{E})$. □

The above result motivates the following.

DEFINITION. A collection $\mathcal{E}$ of subsets of X is said to be σ -total in X , if $X \in \mathcal{P}_\sigma^\mathcal{E}(X)$, i.e. there exists some sequence $(E_n)_{n=1}^\infty \subset \mathcal{E}$ with $\bigcup_{n=1}^\infty E_n = X$. By the above result, this is equivalent to the fact that X belongs to the σ -ring $\mathbf{S}(\mathcal{E})$ generated by $\mathcal{E}$, which in turn is equivalent to the equality $\Sigma(\mathcal{E}) = \mathbf{S}(\mathcal{E})$.

We discuss now two more methods of constructing (σ) -rings, (σ) -algebras, or monotone classes.

NOTATIONS. Let $f : X \rightarrow Y$ be a function, and let $\mathcal{E} \subset \mathcal{P}(X)$ and $\mathcal{G} \subset \mathcal{P}(Y)$ be two arbitrary collections of sets. We define

$$\begin{aligned} f_*\mathcal{E} &= \{A \in \mathcal{P}(Y) : f^{-1}(A) \in \mathcal{E}\} \subset \mathcal{P}(Y); \\ f^*\mathcal{G} &= \{f^{-1}(G) : G \in \mathcal{G}\} \subset \mathcal{P}(X). \end{aligned}$$

DEFINITIONS. Let Θ be a type of set collections. We say that Θ is *natural*, if for any map $f : X \rightarrow Y$, one has the implications

- (i) $\mathcal{C}$ of type Θ on $X \implies f_*\mathcal{C}$ of type Θ on Y ;
- (ii) $\mathcal{D}$ of type Θ on $Y \implies f^*\mathcal{D}$ of type Θ on X .

EXAMPLES 2.3. The types $\mathbf{R}$, $\mathbf{A}$, $\mathbf{S}$, Σ , and $\mathbf{M}$ are natural.

The term “natural” is justified by the following.

Exercise 1. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps.

- (i) Prove that, for any collection $\mathcal{C} \subset \mathcal{P}(X)$, one has the equality $g_*(f_*\mathcal{C}) = (g \circ f)_*\mathcal{C}$.
- (ii) Prove that, for any collection $\mathcal{D} \subset \mathcal{P}(Y)$, one has the equality $f^*(g^*\mathcal{D}) = (g \circ f)^*\mathcal{D}$.

THEOREM 2.2 (Generating Theorem). *Suppose Θ is a consistent class type, which is natural. Let X and Y be non-empty sets, and let $f : X \rightarrow Y$ be a map. For any collection $\mathcal{G} \subset \mathcal{P}(Y)$, one has the equality*

$$f^*\Theta(\mathcal{G}) = \Theta(f^*\mathcal{G}).$$

PROOF. On the one hand, by naturality, we know that $f^*\Theta(\mathcal{G})$ is of type Θ . On the other hand, it is pretty clear that, since $\Theta(\mathcal{G}) \supset \mathcal{G}$, we also have the inclusion $f^*\Theta(\mathcal{G}) \supset f^*\mathcal{G}$. Since Θ is consistent, it then follows that we have the inclusion

$$f^*\Theta(\mathcal{G}) \supset \Theta(f^*\mathcal{G}).$$

To prove the other inclusion, we consider the class

$$\mathcal{C} = f_*[\Theta(f^*\mathcal{G})] \subset \mathcal{P}(Y).$$

By naturality, it follows that $\mathcal{C}$ is of type Θ on Y . For any $G \in \mathcal{G}$, the obvious relation

$$f^{-1}(G) \in f^*\mathcal{G} \subset \Theta(f^*\mathcal{G})$$

proves that $G \in \mathcal{C}$. This means that we have the inclusion $\mathcal{C} \supset \mathcal{G}$, and since $\mathcal{C}$ is of class Θ , it follows that we have the inclusion

$$\Theta(\mathcal{G}) \subset \mathcal{C}.$$

This means that, for every $A \in \Theta(\mathcal{G})$, we have $f^{-1}(A) \in \Theta(f^*\mathcal{G})$, which means precisely that we have the desired inclusion

$$f^*\Theta(\mathcal{G}) \subset \Theta(f^*\mathcal{G}). \quad \square$$

EXAMPLE 2.4. Let Θ be a consistent class type, which is both covariant and contravariant. Let Y be some set, and let $\mathcal{C}$ be a collection of type Θ on Y . Given a subset $X \subset Y$, we consider the inclusion map $\iota : X \hookrightarrow Y$. The collection $\iota^*\mathcal{C}$ is then of type Θ on X . It will be denoted by $\mathcal{C}|_X$. Since $\iota^{-1}A = A \cap X$, $\forall A \in \mathcal{P}(Y)$, we have

$$\mathcal{C}|_X = \{A \cap X : A \in \mathcal{C}\}.$$

If $\mathcal{E} \subset \mathcal{P}(Y)$ is a collection with $\mathcal{C} = \Theta(\mathcal{E})$, then by the Generating Theorem we have the equality

$$(7) \quad \Theta(\mathcal{E})|_X = \Theta(\{E \cap X : E \in \mathcal{E}\}).$$

COMMENT. The exercise below shows that a “forward” version of the Generating Theorem does not hold in general. In other words, an equality of the type $f_*\Theta(\mathcal{G}) = \Theta(f_*\mathcal{G})$ may fail. The reason is the fact that the collection $f_*\mathcal{G}$ may be relatively “small.”

Exercise 2. Consider the sets $X = \{1, 2, 3\}$, $Y = \{1, 2\}$, the function $f : X \rightarrow Y$, defined by $f(1) = f(2) = 1$, $f(3) = 2$, and the collection $\mathcal{C} = \{\{1\}, \{2\}, \emptyset\}$. Describe the collection $f_*\mathcal{C}$, the algebra $\mathbf{A}(\mathcal{C})$ generated by $\mathcal{C}$ (on X), and the algebra $\mathbf{A}(f_*\mathcal{C})$ generated by $f_*\mathcal{C}$ (on Y). Prove that one has a strict inclusion $\mathbf{A}(f_*\mathcal{C}) \subsetneq f_*\mathbf{A}(\mathcal{C})$.

Exercise 3. Let Θ be a consistent natural type, let $f : X \rightarrow Y$ be a *surjective* map, and let $\mathcal{G}$ be a collection of subsets of X . Assume one has the inclusion

$$(8) \quad \mathcal{G} \subset f^*\Theta(f_*\mathcal{G}).$$

Prove that one has the equality

$$f_*\Theta(\mathcal{G}) = \Theta(f_*\mathcal{G}).$$

(One instance when (8) holds is for example when $f^{-1}(f(G)) = G$, $\forall G \in \mathcal{G}$.)

Exercise 4*. Let Θ be one of the types $\mathbf{A}$, $\mathbf{R}$, $\mathbf{S}$, $\mathbf{\Sigma}$, or $\mathbf{M}$. Let $f : X \rightarrow Y$ be an *injective* map, and let $\mathcal{G} \subset \mathcal{P}(X)$ be some arbitrary collection. Prove the equality

$$f_*\Theta(\mathcal{G}) = \Theta(f_*\mathcal{G}).$$

Natural consistent types are useful, because it is possible to construct product structures.

DEFINITION. Let Θ be a consistent type which is natural. Let $(X_i)_{i \in I}$ be a collection of non-empty sets. Assume that, for each $i \in I$, a collection $\mathcal{E}_i \subset \mathcal{P}(X_i)$ of type Θ is given. Consider the product cartesian product $X = \prod_{i \in I} X_i$, together with the projection maps $\pi_i : X \rightarrow X_i$, $i \in I$. The collection

$$\Theta\text{-}\bigtimes_{i \in I} \mathcal{E}_i = \Theta\left(\bigcup_{i \in I} \pi_i^* \mathcal{E}_i\right)$$

is a collection of type Θ on X , which is called the Θ -product. When there is no danger of confusion, we use the notation $\bigtimes$.

REMARK 2.1. Use the notations from the above definition. Assume that, for each $i \in I$, a collection $\mathcal{G}_i \subset \mathcal{P}(X_i)$ is given. Then one has the equality

$$\bigtimes_{i \in I} \Theta(\mathcal{G}_i) = \Theta\left(\bigcup_{i \in I} \pi_i^* \mathcal{G}_i\right).$$

Indeed, if we define $\mathcal{E}_i = \Theta(\mathcal{G}_i)$, the inclusion $\supset$ follows from the obvious inclusions

$$\bigcap_{i \in I} \mathcal{E}_i \supset \pi_i^* \mathcal{E}_i \supset \pi_i^* \mathcal{G}_i.$$

The inclusion $\subset$, follows from the inclusions

$$\pi_i^* \mathcal{G}_i \subset \Theta\left(\bigcup_{i \in I} \pi_i^* \mathcal{G}_i\right),$$

which combined with the fact that the right hand side is of type Θ , and the Generating Theorem, give the inclusions

$$\pi_i^* \mathcal{E}_i = \pi_i^* \Theta(\mathcal{G}_i) = \Theta(\pi_i^* \mathcal{G}_i) \subset \Theta\left(\bigcup_{i \in I} \pi_i^* \mathcal{G}_i\right).$$

Natural consistent types also allow one to define disjoint union structures.

DEFINITIONS. Let $(X_i)_{i \in I}$ be a collection of non-empty sets. Assume that, for each $i \in I$, a collection $\mathcal{C}_i \subset \mathcal{P}(X_i)$ is given. On the disjoint union $X = \bigsqcup_{i \in I} X_i$ one defines the collection

$$\bigvee_{i \in I} \mathcal{C}_i = \{C \subset X : C \cap X_i \in \mathcal{C}_i, \forall i \in I\}.$$

Assume now Θ is a natural consistent, and $\mathcal{C}_i$ is of type Θ on X_i , for each $i \in I$. If we consider the inclusion maps $\epsilon_i : X_i \rightarrow X$, $i \in I$, then one clearly has the equality

$$\bigvee_{i \in I} \mathcal{C}_i = \bigcap_{i \in I} \epsilon_{i*} \mathcal{C}_i,$$

which means that $\bigvee_{i \in I} \mathcal{C}_i$ is a collection of type Θ on X .

Exercise 5. Let I be countable, and let $(X_i)_{i \in I}$ be a collection of non-empty sets. Assume that, for each $i \in I$, a collection $\mathcal{C}_i \subset \mathcal{P}(X_i)$ is given, such that $\emptyset \in \mathcal{C}_i$. Prove the equalities

$$\bigvee_{i \in I} \mathbf{S}(\mathcal{C}_i) = \mathbf{S}\left(\bigvee_{i \in I} \mathcal{C}_i\right) \text{ and } \bigvee_{i \in I} \Sigma(\mathcal{C}_i) = \Sigma\left(\bigvee_{i \in I} \mathcal{C}_i\right).$$

We conclude with a discussion on certain constructions related to topology.

DEFINITIONS. Let X be a topological Hausdorff space. We consider the collection $\mathcal{T}$ of all open sets in X . The σ -algebra $\Sigma(\mathcal{T})$ on X , generated by $\mathcal{T}$ is denoted by $Bor(X)$. The sets in $Bor(X)$ are called *Borel* sets.

Remark that singleton sets are Borel, since they are closed. Moreover

- every countable set $B \subset X$ is Borel.

One also defines the σ -algebra $Bor_c(X) = \Sigma(\mathcal{C}_X)$ generated by the class $\mathcal{C}_X$ of all compact subsets of X .

Another class of sets will also be of interest. Its construction uses the following terminology.

A subset $A \subset X$ is said to be σ -compact, if there exists a sequence $(K_n)_{n=1}^\infty$ of compact subsets of X , such that $A = \bigcup_{n=1}^\infty K_n$. A set $B \subset X$ is said to be *relatively σ -compact*, if there exists a σ -compact set A with $B \subset A$. We set

$$\mathcal{P}_{\sigma c}(X) = \{B \in \mathcal{P}(X) : B \text{ relatively } \sigma\text{-compact}\},$$

and we define

$$Bor_{\sigma c}(X) = Bor(X) \cap \mathcal{P}_{\sigma c}(X).$$

PROPOSITION 2.4. *Let X be a topological Hausdorff space.*

- (i) $\mathcal{P}_{\sigma c}(X)$ is a σ -ring on X ;
- (ii) the σ -ring $Bor_{\sigma c}(X)$ coincides with the σ -ring $\mathbf{S}(\mathcal{C}_X)$ generated by the collection $\mathcal{C}_X$ of all compact subsets of X .

PROOF. Using the notations from Proposition 2.3, we have $\mathcal{P}_{\sigma c}(X) = \mathcal{P}_{\sigma}^{\mathcal{C}_X}(X)$, so part (i) is a consequence of Proposition 2.3.(i). By Proposition 2.3.(ii) we also know that

$$\mathbf{S}(\mathcal{C}_X) = \Sigma(\mathcal{C}_X) \cap \mathcal{P}_{\sigma c}(X) = Bor_c(X) \cap \mathcal{P}_{\sigma c}(X),$$

and since $Bor_c(X) \subset Bor(X)$, we have the inclusion

$$\mathbf{S}(\mathcal{C}_X) \subset Bor_{\sigma c}(X).$$

To prove the other inclusion, all we need to show is the inclusion

$$Bor_{\sigma c}(X) \subset Bor_c(X).$$

Start with some arbitrary set $B \in Bor_{\sigma c}(X)$, and let us prove that $B \in Bor_c(X)$. Since B is relatively σ -compact, there exists a sequence $(K_n)_{n=1}^{\infty}$ of compact sets, such that $B \subset \bigcup_{n=1}^{\infty} K_n$. Define, for each integer $n \geq 1$, the set $B_n = B \cap K_n$. Since $B = \bigcup_{n=1}^{\infty} B_n$, It suffices to show that

$$(9) \quad B_n \in Bor_c(X), \quad \forall n \in \mathbb{N}.$$

Fix n , and let us analyze the inclusion $\iota_n : K_n \hookrightarrow X$. Denote by $\mathcal{T}$ the collection of all open sets in X , and denote by $\mathcal{T}_{K_n}$ the collection of all sets $D \subset K_n$, which are open in the induced topology, that is,

$$\mathcal{T}_{K_n} = \{D \cap K_n : D \in \mathcal{T}\}.$$

By the Generating Theorem (Example 2.4), we know that

$$Bor(X)|_{K_n} = \Sigma(\mathcal{T})|_{K_n} = \Sigma_{K_n}(\{D \cap K_n : D \in \mathcal{T}\}) = \Sigma_{K_n}(\mathcal{T}_{K_n}) = Bor(K_n).$$

(Here the notation Σ_{K_n} indicates that the σ -algebra is taken on K_n .) In particular, we get

$$(10) \quad B_n = B \cap K_n \in Bor(X)|_{K_n} = Bor(K_n), \quad \forall n \in \mathbb{N}.$$

Since K_n is compact, the σ -ring $\mathbf{S}(\mathcal{C}_{K_n})$, generated by all compact subsets of K_n , is a σ -algebra on K_n (simply because it contains K_n .) Notice that every set $D \in \mathcal{T}_{K_n}$ is of the form $K_n \setminus F$, with $F \subset K_n$ compact (in X), therefore D belongs to $\mathbf{S}(\mathcal{C}_{K_n})$. Since $\mathbf{S}(\mathcal{C}_X)$ is a σ -algebra, which contains $\mathcal{T}_{K_n}$, we have

$$Bor(K_n) = \Sigma_{K_n}(\mathcal{T}_{K_n}) \subset \mathbf{S}(\mathcal{C}_{K_n}) \subset \mathbf{S}(\mathcal{C}_X) \subset Bor_c(X), \quad \forall n \in \mathbb{N}.$$

Now (9) immediately follows from the above inclusions, combined with (10). $\square$

REMARK 2.2. For a topological Hausdorff space, we always have the inclusions

$$Bor_{\sigma c}(X) \subset Bor_c(X) \subset Bor(X).$$

The following are equivalent

- (i) $Bor_{\sigma c}(X) = Bor(X)$;
- (ii) X is σ -compact.

The following result explains when a minimal set of generators can be chosen for the Borel sets.

PROPOSITION 2.5. *Let X be a topological space which is second countable, i.e. there is a countable base for the topology. If $\mathcal{S}$ is any sub-base for the topology (countable or not), then*

$$\text{Bor}(X) = \Sigma(\mathcal{S}).$$

PROOF. Denote by $\mathcal{T}$ the collection of all open sets in X . Denote by $\mathcal{V}$ the collection of all subsets of X , which can be written as finite intersections of sets in $\mathcal{S}$. It is obvious that $\mathcal{S} \subset \mathcal{V} \subset \Sigma(\mathcal{S})$, so we have the equality $\Sigma(\mathcal{S}) = \Sigma(\mathcal{V})$. This means that it suffices to prove the equality

$$(11) \quad \Sigma(\mathcal{T}) = \Sigma(\mathcal{V}).$$

Notice that $\mathcal{V}$ is a base for the topology, which means that every open subset $D \subsetneq X$ can be written as a union of sets in $\mathcal{V}$. What we want to prove is

Claim: Every open set $D \subsetneq X$ is a countable union of sets in $\mathcal{V}$.

To prove this fact, we fix an open set $D \subsetneq X$, as well as a countable base $\mathcal{B} = \{B_n\}_{n=1}^\infty$ for the topology. For every $x \in D$ we define the set

$$M_x = \{n \in \mathbb{N} : \text{there exists } V \in \mathcal{V}, \text{ such that } x \in B_n \subset V \subset D\}.$$

It is pretty clear that $M_x \neq \emptyset$, $\forall x \in D$. (First use the fact that $\mathcal{V}$ is a base, to find $V \in \mathcal{V}$ such that $x \in V \subset D$, and then use the fact that $\mathcal{B}$ is a base to find n such that $x \in B_n \subset V$.) If we put $M = \bigcup_{x \in D} M_x$, then it is pretty obvious that $\bigcup_{n \in M} B_n = D$. For every $n \in M$ we choose some $V_n \in \mathcal{V}$ with $B_n \subset V_n \subset D$ (use the fact that n must belong to some M_x). It is then clear that $D = \bigcup_{n \in M} V_n$, and the claim follows.

As a consequence of the Claim, we see that any open set $D \subsetneq X$ automatically belongs to $\Sigma(\mathcal{V})$, and then we have the inclusion $\mathcal{T} \subset \Sigma(\mathcal{V}) \subset \Sigma(\mathcal{T})$. This clearly forces the equality (11). $\square$

COROLLARY 2.3. *Let I be a set which is at most countable, and let $(X_i)_{i \in I}$ be a collection of second countable topological spaces. Then one has the equality*

$$(12) \quad \text{Bor}\left(\prod_{i \in I} X_i\right) = \Sigma\text{-}\bigtimes_{i \in I} \text{Bor}(X_i),$$

where the product space $\prod_{i \in I} X_i$ is equipped with the product topology.

PROOF. By the definition of the product σ -algebra, we know that

$$\Sigma\text{-}\bigtimes_{i \in I} \text{Bor}(X_i) = \Sigma\left(\bigcup_{j \in I} \pi_j^* \text{Bor}(X_j)\right),$$

where $\pi_j : \prod_{i \in I} X_i \rightarrow X_j$, $j \in I$, denote the projection maps. Choose, for each $j \in I$, a countable sub-base $\mathcal{S}_j$ for X_j , so that we have the equalities

$$\text{Bor}(X_j) = \Sigma(\mathcal{S}_j), \quad \forall j \in I.$$

By Remark 2.1 we have the equality

$$\Sigma\text{-}\bigtimes_{i \in I} \text{Bor}(X_i) = \Sigma\left(\bigcup_{j \in I} \pi_j^* \mathcal{S}_j\right),$$

where $\pi_j : \prod_{i \in I} X_i \rightarrow X_j$, $j \in I$, denote the projection maps. Since the collection $\bigcup_{i \in I} \pi_i^* \mathcal{S}_i$ is a countable sub-base for the product topology, the above equality, combined with Proposition 2.5 immediately gives (12). $\square$

Exercise 6. A. Prove that, if X is second countable, and $\mathcal{S}$ is a sub-base for its topology (countable or not), with $\bigcup_{S \in \mathcal{S}} S = X$, then we have in fact the equality

$$\text{Bor}(X) = \mathbf{S}(\mathcal{S}).$$

B. Prove that, if X is Hausdorff, second countable, with $\text{card } X \geq 2$, then for *any* sub-base $\mathcal{S}$ (countable or not), we have the equality

$$\text{Bor}(X) = \mathbf{S}(\mathcal{S}).$$

HINTS: Follow the proof above. Remark that every open set $D \subset X$, which is a countable union of sets in $\mathcal{V}$, belongs in fact to the σ -ring $\mathbf{S}(\mathcal{V}) = \mathbf{S}(\mathcal{S})$. So in either case, we only have to show that X is a countable union of sets in $\mathcal{V}$.

In case A, we trace the proof of the Claim, and we notice that the only property that we used was the fact that, for every $x \in D$, there exists $V \in \mathcal{V}$ with $x \in V \subset D$, i.e. D is a (possibly uncountable) union of sets in $\mathcal{V}$. Since X itself satisfies this property, it follows that X is also a countable union of sets in $\mathcal{V}$.

In case B, we use the Hausdorff property to write $X = D_1 \cup D_2$, with $D_1, D_2 \subsetneq X$ open.

COROLLARY 2.4. *If X is a topological Hausdorff space, which is second countable, and X is infinite (as a set), then $\text{card } \text{Bor}(X) = \mathfrak{c}$.*

PROOF. First of all, since X is infinite, one can choose an infinite countable subset $A \subset X$. Then A , and all its subsets are Borel, i.e. we have the inclusion $\mathcal{P}(A) \subset \text{Bor}(X)$, thus proving the inequality

$$\text{card } \text{Bor}(X) \geq \text{card } \mathcal{P}(A) = 2^{\aleph_0} = \mathfrak{c}.$$

Secondly, one can choose a base $\mathcal{V}$ for the topology, which is countable. We now have $\text{Bor}(X) = \mathbf{S}(\mathcal{V} \cup \{X\})$, so by Corollary 2.1. we get

$$\text{card } \text{Bor}(X) \leq \text{card}(\mathcal{V} \cup \{X\})^{\aleph_0} \leq \aleph_0^{\aleph_0} = \mathfrak{c},$$

and the desired equality follows. $\square$

EXAMPLES 2.5. A. Consider the extended real line $[-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$, thought as a compact space, homeomorphic to the interval $[-\pi/2, \pi/2]$, via the map $f : [-\pi/2, \pi/2] \rightarrow [-\infty, \infty]$, defined by

$$f(t) = \begin{cases} -\infty & \text{if } t = -\pi/2 \\ \tan t & \text{if } -\pi/2 < t < \pi/2 \\ \infty & \text{if } t = \pi/2 \end{cases}$$

Notice that, when restricted to $\mathbb{R} = (-\infty, \infty)$, this topology agrees with the usual topology. In particular, this gives the equality $\text{Bor}([-\infty, \infty])|_{\mathbb{R}} = \text{Bor}(\mathbb{R})$.

Let $A \subset \mathbb{R}$ be a dense subset. Consider the collections

$$\begin{aligned} \mathcal{E}_1 &= \{(a, \infty] : a \in A\}; & \mathcal{E}_2 &= \{[a, \infty] : a \in A\}; \\ \mathcal{E}_3 &= \{[-\infty, a) : a \in A\}; & \mathcal{E}_4 &= \{[-\infty, a] : a \in A\}. \end{aligned}$$

With these notations we have the equalities

$$\text{Bor}([-\infty, \infty]) = \Sigma(\mathcal{E}_1) = \Sigma(\mathcal{E}_2) = \Sigma(\mathcal{E}_3) = \Sigma(\mathcal{E}_4).$$

First of all, we notice that each set in $\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_4$ is either open or closed, which means that

$$\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_4 \subset \text{Bor}([-\infty, \infty]),$$

thus giving the inclusions

$$\Sigma(\mathcal{E}_k) \subset \text{Bor}([-\infty, \infty]), \quad \forall k \in \{1, 2, 3, 4\}.$$

Second, we observe that $\mathcal{E}_1 \cup \mathcal{E}_3$ is a sub-base for the topology, and since $[-\infty, \infty]$ is obviously second countable, we will have the equality

$$\text{Bor}([-\infty, \infty]) = \Sigma(\mathcal{E}_1 \cup \mathcal{E}_3).$$

So, in order to finish the proof we only need to show the inclusions

$$(13) \quad \mathcal{E}_1 \cup \mathcal{E}_3 \subset \Sigma(\mathcal{E}_k), \quad \forall k \in \{1, 2, 3, 4\}.$$

Since every set in $\mathcal{E}_2$ has its complement in $\mathcal{E}_3$, and viceversa, we have the inclusions

$$\mathcal{E}_2 \subset \Sigma(\mathcal{E}_3) \text{ and } \mathcal{E}_3 \subset \Sigma(\mathcal{E}_2),$$

which prove the equality

$$(14) \quad \Sigma(\mathcal{E}_2) = \Sigma(\mathcal{E}_3).$$

Likewise, we have the equality

$$(15) \quad \Sigma(\mathcal{E}_1) = \Sigma(\mathcal{E}_4).$$

This means that we only have to prove (13) for $k = 2$ and $k = 4$. The case $k = 2$ amounts to proving that $\mathcal{E}_1 \subset \Sigma(\mathcal{E}_2)$. Fix some $a \in A$. For every integer $n \geq 1$ we choose $a_n \in (a, a + \frac{1}{n}) \cap A$. Then the equality

$$(a, \infty] = \bigcup_{n=1}^{\infty} [a_n, \infty]$$

clearly shows that $(a, \infty] \in \Sigma(\mathcal{E}_2)$.

The case $k = 4$ amounts to proving that $\mathcal{E}_3 \subset \Sigma(\mathcal{E}_4)$. Fix some $a \in A$. For every integer $n \geq 1$ we choose $a_n \in (a - \frac{1}{n}, a) \cap A$. Then the equality

$$[-\infty, a) = \bigcup_{n=1}^{\infty} [-\infty, a_n]$$

clearly shows that $[-\infty, a) \in \Sigma(\mathcal{E}_4)$.

B. If we work on $\mathbb{R}$, and we consider the collections

$$\mathcal{E}_k^0 = \{E \cap \mathbb{R} : E \in \mathcal{E}_k\}, \quad k = 1, 2, 3, 4,$$

then by the Generating Theorem (Example 2.4) we have the equalities

$$\text{Bor}(\mathbb{R}) = \Sigma(\mathcal{E}_k^0) = \mathbf{S}(\mathcal{E}_k^0), \quad k = 1, 2, 3, 4.$$

(The fact that the σ -algebra $\Sigma(\mathcal{E}_k^0)$ and the σ -ring $\mathbf{S}(\mathcal{E}_k^0)$ coincide is a consequence of the fact that $\mathcal{E}_k^0$ is σ -total in $\mathbb{R}$.)

C. Let X be a separable metric space. Let $A \subset X$ be a dense set, and let $R \subset (0, \infty)$ be a subset with $\inf R = 0$. Then the collection

$$\mathcal{S}_{A,R} = \{\mathcal{B}_r(a) : r \in R, a \in A\}$$

is clearly a base for the metric topology. Since X is separable, one can choose both A and R to be countable, which proves that X is automatically second countable. Then for *any* choice of A and R , we will have the equality

$$(16) \quad \text{Bor}(X) = \Sigma(\{\mathcal{B}_r(a) : r \in R, a \in A\}) = \mathbf{S}(\{\mathcal{B}_r(a) : r \in R, a \in A\}).$$

(The equality between the generated σ -algebra and σ -ring follows from Exercise 1.A.) As particular cases when the equality (16) holds, one has the metric spaces which are σ -compact.

Exercise 7.* Let I be an *uncountable* set, and let $(X_i)_{i \in I}$ be a collection of topological spaces. Assume that for each $i \in I$, there exists at least one non-empty closed subset $F_i \subsetneq X_i$. (This is the case for example when X_i is Hausdorff, and $\text{card } X_i \geq 2$.) Prove that one has a strict inclusion

$$\text{Bor}\left(\prod_{i \in I} X_i\right) \supsetneq \Sigma\text{-}\bigtimes_{i \in I} \text{Bor}(X_i).$$

HINT: For every subset $J \subset I$, define the projection map $\pi_J : \prod_{i \in I} X_i \rightarrow \prod_{i \in J} X_i$. Consider the collection

$$\mathcal{A} = \left\{ A \subset \prod_{i \in I} X_i : \text{there exists } J \subset I \text{ countable, such that } A = \pi_J^{-1}(\pi_J(A)) \right\}.$$

Prove that $\mathcal{A} \cup \{\emptyset\}$ is a σ -algebra, which contains $\bigcup_{i \in I} \pi_i^* \text{Bor}(X_i)$. Prove that one has a strict inclusion $\text{Bor}\left(\prod_{i \in I} X_i\right) \supsetneq \mathcal{A} \cup \{\emptyset\}$, by constructing a non-empty closed set $F \subset \prod_{i \in I} X_i$, which does not belong to $\mathcal{A}$.

LECTURE 20

3. Measurable spaces and measurable maps

In this section we discuss a certain type of maps related to σ -algebras.

DEFINITIONS. A *measurable space* is a pair $(X, \mathcal{A})$ consisting of a (non-empty) set X and a σ -algebra $\mathcal{A}$ on X .

Given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, a *measurable map* $T : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ is simply a map $T : X \rightarrow Y$, with the property

$$(1) \quad T^{-1}(B) \in \mathcal{A}, \quad \forall B \in \mathcal{B}.$$

REMARK 3.1. In terms of the constructions outlined in Section 2, measurability for maps can be characterized as follows. *Given measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, and a map $T : X \rightarrow Y$, the following are equivalent:*

- (i) $T : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ is measurable;
- (ii) $T^*\mathcal{B} \subset \mathcal{A}$;
- (iii) $T_*\mathcal{A} \supset \mathcal{B}$.

Recall

$$\begin{aligned} T^*\mathcal{B} &= \{T^{-1}(B) : B \in \mathcal{B}\}; \\ T_*\mathcal{A} &= \{B \subset Y : T^{-1}(B) \in \mathcal{A}\}. \end{aligned}$$

With these equalities, everything is immediate.

The following summarizes some useful properties of measurable maps.

PROPOSITION 3.1. *Let $(X, \mathcal{A})$ be a measurable space.*

- (i) *If $\mathcal{A}'$ is any σ -algebra, with $\mathcal{A}' \subset \mathcal{A}$, then the identity map $\text{Id}_X : (X, \mathcal{A}) \rightarrow (X, \mathcal{A}')$ is measurable.*
- (ii) *For any subset $M \subset X$, the inclusion map $\iota : (M, \mathcal{A}|_M) \hookrightarrow (X, \mathcal{A})$ is measurable.*
- (iii) *If $(Y, \mathcal{B})$ and $(Z, \mathcal{C})$ are measurable spaces, and if $(X, \mathcal{A}) \xrightarrow{T} (Y, \mathcal{B}) \xrightarrow{S} (Z, \mathcal{C})$ are measurable maps, then the composition $S \circ T : (X, \mathcal{A}) \rightarrow (Z, \mathcal{C})$ is again a measurable map.*

PROOF. (i). This is trivial, since $(\text{Id}_X)^*\mathcal{A}' = \mathcal{A}' \subset \mathcal{A}$.

(ii). This is again trivial, since $\iota^*\mathcal{A} = \mathcal{A}|_M$.

(iii). Start with some set $C \in \mathcal{C}$, and let us prove that $(S \circ T)^{-1}(C) \in \mathcal{A}$. We know that $(S \circ T)^{-1} = T^{-1}(S^{-1}(C))$. Since S is measurable, we have $S^{-1}(C) \in \mathcal{B}$, and since T is measurable, we have $T^{-1}(S^{-1}(C)) \in \mathcal{A}$. $\square$

Often, one would like to check the measurability condition (1) on a small collection of B 's. Such a criterion is the following.

LEMMA 3.1. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. Assume $\mathcal{B} = \Sigma(\mathcal{E})$, for some collection of sets $\mathcal{E} \subset \mathcal{P}(Y)$. For a map $T : X \rightarrow Y$, the following are equivalent:*

- (i) $T : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ is measurable;
- (ii) $T^{-1}(E) \in \mathcal{A}, \forall E \in \mathcal{E}$.

PROOF. The implication (i) $\Rightarrow$ (ii) is trivial.

To prove the implication (ii) $\Rightarrow$ (i), assume (ii) holds. We first observe that condition (ii) reads $f^*\mathcal{E} \subset \mathcal{A}$. Since $\mathcal{A}$ is a σ -algebra, we get the inclusion

$$\Sigma(f^*\mathcal{E}) \subset \mathcal{A}.$$

Using the Generating Theorem 2.2, we have

$$f^*\mathcal{B} = f^*\Sigma(\mathcal{E}) = \Sigma(f^*\mathcal{E}) \subset \mathcal{A},$$

and, by the preceding remark, we are done. $\square$

COROLLARY 3.1. *Let $(X, \mathcal{A})$ be a measurable space, let Y be a topological Hausdorff space which is second countable, and let $\mathcal{S}$ be a sub-base for the topology of Y . For a map $T : X \rightarrow Y$, the following are equivalent:*

- (i) $T : (X, \mathcal{A}) \rightarrow (Y, \text{Bor}(Y))$ is a measurable map;
- (ii) $T^{-1}(S) \in \mathcal{A}, \forall S \in \mathcal{S}$.

PROOF. Immediate from the above Lemma, and Proposition 2.2, which states that $\text{Bor}(Y) = \Sigma(\mathcal{S})$. $\square$

We know (see Section 19) that the type Σ is consistent and natural. In particular, measurability behaves nicely with respect to products and disjoint unions. More explicitly one has the following.

PROPOSITION 3.2. *Let $(X_i, \mathcal{A}_i)_{i \in I}$ be a collection of measurable spaces. Consider the sets $X = \prod_{i \in I} X_i$ and $Y = \bigsqcup_{i \in I} X_i$, and the σ -algebras*

$$\mathcal{A} = \Sigma\text{-}\bigtimes_{i \in I} \mathcal{A}_i \text{ and } \mathcal{B} = \bigvee_{i \in I} \mathcal{A}_i.$$

Let $(Z, \mathcal{G})$ be a measurable space.

- (i) *If we denote by $\pi_i : X \rightarrow X_i, i \in I$, the projection maps, then a map $f : (Z, \mathcal{G}) \rightarrow (X, \mathcal{A})$ is measurable, if and only if, all the maps $\pi_i \circ f : (Z, \mathcal{G}) \rightarrow (X_i, \mathcal{A}_i), i \in I$, are measurable.*
- (ii) *If we denote by $\epsilon_i : X_i \rightarrow Y, i \in I$, the inclusion maps, then a map $g : (Y, \mathcal{B}) \rightarrow (Z, \mathcal{G})$ is measurable, if and only if, all the maps $g \circ \epsilon_i \circ f : (X_i, \mathcal{A}_i) \rightarrow (Z, \mathcal{G}), i \in I$, are measurable.*

PROOF. (i). By the definition of the product σ -algebra, we know that

$$(2) \quad \mathcal{A} = \Sigma\left(\bigcup_{i \in I} \pi_i^* \mathcal{A}_i\right).$$

If we fix some index $i \in I$, then the obvious inclusion $\pi_i^* \mathcal{A}_i \subset \mathcal{A}$ immediately shows that $\pi_i : (X, \mathcal{A}) \rightarrow (X_i, \mathcal{A}_i)$ is measurable. Therefore, if $f : (Z, \mathcal{G}) \rightarrow (X, \mathcal{A})$ is measurable, then by Proposition 3.1 it follows that all compositions $\pi_i \circ f : (Z, \mathcal{G}) \rightarrow (X_i, \mathcal{A}_i), i \in I$, are measurable.

Conversely, assume all the compositions $\pi_i \circ f$ are measurable, and let us show that $f : (Z, \mathcal{G}) \rightarrow (X, \mathcal{A})$ is measurable. By Lemma 3.1 and (2), all we need to prove is the fact that

$$f^*\left(\bigcup_{i \in I} \pi_i^* \mathcal{A}_i\right) \subset \mathcal{G},$$

which is equivalent to

$$f^*(\pi_i^* \mathcal{A}_i) \subset \mathcal{G}, \quad \forall i \in I.$$

But this is obvious, because $f^*(\pi_i^* \mathcal{A}_i) = (\pi_i \circ f)^* \mathcal{A}_i$, and $\pi_i \circ f$ is measurable, for all $i \in I$.

(ii). By the definition of the σ -algebra sum, we know that

$$(3) \quad \mathcal{B} = \bigcap_{i \in I} \epsilon_{i*} \mathcal{A}_i.$$

If we fix some index $i \in I$, then the obvious inclusion $\epsilon_{i*} \mathcal{A}_i \supset \mathcal{B}$ immediately shows that $\epsilon_i : (X_i, \mathcal{A}_i) \rightarrow (Y, \mathcal{B})$ is measurable. Therefore, if $g : (Y, \mathcal{B}) \rightarrow (Z, \mathcal{G})$ is measurable, then by Proposition 3.1 it follows that all compositions $g \circ \epsilon_i : (X_i, \mathcal{A}_i) \rightarrow (Z, \mathcal{G})$, $i \in I$, are measurable.

Conversely, assume all the compositions $g \circ \epsilon_i$ are measurable, and let us show that $g : (Y, \mathcal{B}) \rightarrow (Z, \mathcal{G})$ is measurable. This is equivalent to the inclusion $g_* \mathcal{B} \supset \mathcal{G}$. By (3) we immediately have

$$(4) \quad g_* \mathcal{B} = g_* \left(\bigcap_{i \in I} \epsilon_{i*} \mathcal{A}_i \right) = \bigcap_{i \in I} g_* (\epsilon_{i*} \mathcal{A}_i).$$

We know however that, since $g \circ \epsilon_i$ are all measurable, we have

$$g_* (\epsilon_{i*} \mathcal{A}_i) = (g \circ \epsilon_i)_* \mathcal{A}_i \supset \mathcal{G}, \quad \forall i \in I,$$

so the desired inclusion is an immediate consequence of (4). $\square$

CONVENTIONS. Let $(X, \mathcal{A})$ be a measurable space. An extended real-valued function $f : (X, \mathcal{A}) \rightarrow [-\infty, \infty]$ is said to be a *measurable function*, if it is measurable in the above sense as a map $f : (X, \mathcal{A}) \rightarrow ([-\infty, \infty], \text{Bor}([-\infty, \infty]))$. If f has values in $\mathbb{R}$, this is equivalent to the fact that f is a measurable map $f : (X, \mathcal{A}) \rightarrow (\mathbb{R}, \text{Bor}(\mathbb{R}))$ is measurable. Likewise, a complex valued function $f : (X, \mathcal{A}) \rightarrow \mathbb{C}$ is measurable, if it is measurable as a map $f : (X, \mathcal{A}) \rightarrow (\mathbb{C}, \text{Bor}(\mathbb{C}))$. If $\mathbb{K}$ is one of the fields $\mathbb{R}$ or $\mathbb{C}$, we define the set

$$\mathbf{B}_{\mathbb{K}}(X, \mathcal{A}) = \{f : (X, \mathcal{A}) \rightarrow \mathbb{K} : f \text{ measurable function}\}.$$

REMARK 3.2. Let $(X, \mathcal{A})$ be a measurable space. If $A \subset \mathbb{R}$ is a dense subset, then the results from Section 2, combined with Lemma 2.1, show that the measurability of a function $f : (X, \mathcal{A}) \rightarrow [-\infty, \infty]$ is equivalent to any of the following conditions:

- $f^{-1}((a, \infty]) \in \mathcal{A}, \forall a \in A$;
- $f^{-1}([a, \infty]) \in \mathcal{A}, \forall a \in A$;
- $f^{-1}([-\infty, a)) \in \mathcal{A}, \forall a \in A$;
- $f^{-1}([-\infty, a]) \in \mathcal{A}, \forall a \in A$.

DEFINITION. If X and Y are topological Hausdorff spaces, a map $T : X \rightarrow Y$ is said to be *Borel measurable*, if T is measurable as a map

$$T : (X, \text{Bor}(X)) \rightarrow (Y, \text{Bor}(Y)).$$

In the cases when $Y = \mathbb{R}, \mathbb{C}, [-\infty, \infty]$, a Borel measurable map will be simply called a *Borel measurable function*.

For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, we define

$$\mathbf{B}_{\mathbb{K}}(X) = \{f : X \rightarrow \mathbb{K} : f \text{ Borel measurable function}\}.$$

REMARK 3.3. If X and Y are topological Hausdorff spaces, then any continuous map $T : X \rightarrow Y$ is Borel measurable. This follows from Lemma 3.1, from the fact that

$$\text{Bor}(Y) = \Sigma(\{D \subset Y : D \text{ open}\}),$$

and the fact that $T^{-1}(D)$ is open, hence in $\text{Bor}(X)$, for every open set $D \subset Y$.

Measurable maps behave nicely with respect to “measurable countable operations,” as suggested by the following result.

PROPOSITION 3.3. *Let $(X, \mathcal{A})$ and $(Z, \mathcal{B})$ be measurable spaces, let I be a set which is at most countable, and let $(Y_i)_{i \in I}$ be a family of topological Hausdorff spaces, each of which is second countable. Suppose a measurable map $T_i : (X, \mathcal{A}) \rightarrow (Y_i, \text{Bor}(Y_i))$ is given, for each $i \in I$. Define the map $T : X \rightarrow \prod_{i \in I} Y_i$ by*

$$T(x) = (T_i(x))_{i \in I}, \quad \forall x \in X.$$

Equip the product space $Y = \prod_{i \in I} Y_i$ with the product topology.

For any measurable map $g : (Y, \text{Bor}(Y)) \rightarrow (Z, \mathcal{B})$, the composition $g \circ T : (X, \mathcal{A}) \rightarrow (Z, \mathcal{B})$ is measurable.

PROOF. We know (see Corollary 2.3) that we have the equality

$$\text{Bor}(Y) = \Sigma \text{-} \bigtimes_{i \in I} \text{Bor}(Y_i).$$

By Proposition 3.2, the map $T : (X, \mathcal{A}) \rightarrow (Y, \text{Bor}(Y))$ is measurable, so by Proposition 3.1, the composition $g \circ T : (X, \mathcal{A}) \rightarrow (Z, \mathcal{B})$ is also measurable. $\square$

The above result has many useful applications.

COROLLARY 3.2. *Suppose $(X, \mathcal{A})$ is a measurable space, and $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Then, when equipped with point-wise addition and multiplication, the set $\mathbf{B}_{\mathbb{K}}(X, \mathcal{A})$ is a unital $\mathbb{K}$ -algebra.*

PROOF. Clearly the constant function 1 is measurable.

Also, if $f \in \mathbf{B}_{\mathbb{K}}(X, \mathcal{A})$ and $\lambda \in \mathbb{K}$, then the function λf is again measurable, since it can be written as the composition $M_{\lambda} \circ f$, where $M_{\lambda} : \mathbb{K} \ni \alpha \mapsto \lambda \alpha \in \mathbb{K}$ is obviously continuous.

Finally, let us show that if $f_1, f_2 \in \mathbf{B}_{\mathbb{K}}(X, \mathcal{A})$, then $f_1 + f_2$ and $f_1 \cdot f_2$ again belong to $\mathbf{B}_{\mathbb{K}}(X, \mathcal{A})$. This is however immediate from Proposition 3.3, applied to the index set $I = \{1, 2\}$, the spaces $Y_1 = Y_2 = \mathbb{K}$, and the continuous maps

$$g_1 : \mathbb{K}^2 \ni (\lambda_1, \lambda_2) \mapsto \lambda_1 + \lambda_2 \in \mathbb{K},$$

$$g_2 : \mathbb{K}^2 \ni (\lambda, \lambda_2) \mapsto \lambda_1 \cdot \lambda_2 \in \mathbb{K}. \quad \square$$

COROLLARY 3.3. *If $(X, \mathcal{A})$ is a measurable space, then a complex valued function $f : X \rightarrow \mathbb{C}$ is measurable, if and only if the real valued functions $\text{Re } f, \text{Im } f : X \rightarrow \mathbb{R}$ are measurable.*

PROOF. If f is measurable, the composing f with the continuous maps

$$\rho : \mathbb{C} \ni z \mapsto \operatorname{Re} z \in \mathbb{R} \text{ and } \gamma : \mathbb{C} \ni z \mapsto \operatorname{Im} z \in \mathbb{R},$$

immediately gives the measurability of $\operatorname{Re} f = \rho \circ f$ and $\operatorname{Im} f = \gamma \circ f$.

Conversely, if both $\operatorname{Re} f, \operatorname{Im} f : X \rightarrow \mathbb{R}$ then the measurability of f follows from Proposition 3.3, applied to $Y_1 = Y_2 = \mathbb{R}$, the functions $f_1 = \operatorname{Re} f$ and $f_2 = \operatorname{Im} f$, and to the continuous function

$$g : \mathbb{R}^2 \ni (a, b) \mapsto a + bi \in \mathbb{C}. \quad \square$$

COROLLARY 3.4. *Let $(X, \mathcal{A})$ be a measurable space, let I be a set which is at most countable, and let $f_i : (X, \mathcal{A}) \rightarrow [-\infty, \infty]$, $i \in I$ be collection of measurable functions. Then the functions $g, h : X \rightarrow [-\infty, \infty]$, defined by*

$$g(x) = \inf \{f_i(x) : i \in I\} \text{ and } h(x) = \sup \{f_i(x) : i \in I\}, \quad \forall x \in X,$$

are both measurable.

PROOF. Define the maps $m, M : \prod_{i \in I} [-\infty, \infty] \rightarrow [-\infty, \infty]$ by

$$m(x) = \inf \{x_i : i \in I\} \text{ and } M(x) = \sup \{x_i : i \in I\}, \quad \forall x = (x_i)_{i \in I} \in \prod_{i \in I} [-\infty, \infty].$$

By Proposition 3.3, it suffices to prove the (Borel) measurability of the maps m and M .

To prove the measurability of m , we are going to show that

$$m^{-1}([-\infty, a)) \in \operatorname{Bor}\left(\prod_{i \in I} [-\infty, \infty]\right), \quad \forall a \in \mathbb{R}.$$

But this is quite obvious, since a point $x = (x_i)_{i \in I}$ belongs to $m^{-1}([-\infty, a))$, if and only if there exists some $j \in I$ with $x_j < a$. In other words, if we define the projections $\pi_j : \prod_{i \in I} [-\infty, \infty] \rightarrow [-\infty, \infty]$, then we have

$$m^{-1}([-\infty, a)) = \bigcup_{j \in I} \pi_j^{-1}([-\infty, a)).$$

This shows that in fact $m^{-1}([-\infty, a))$ is open, hence clearly Borel.

To prove the measurability of M , we are going to show that

$$M^{-1}((a, \infty]) \in \operatorname{Bor}\left(\prod_{i \in I} [-\infty, \infty]\right), \quad \forall a \in \mathbb{R}.$$

But this is again clear, since, as before, we have the equality

$$M^{-1}((a, \infty]) = \bigcup_{j \in I} \pi_j^{-1}((a, \infty]),$$

which shows that in fact $M^{-1}((a, \infty])$ is open, hence Borel. $\square$

COROLLARY 3.5. *Let $(X, \mathcal{A})$ be a measurable space, and let $f_n : (X, \mathcal{A}) \rightarrow [-\infty, \infty]$, $n \in \mathbb{N}$ be sequence of measurable functions. Then the functions $g, h : X \rightarrow [-\infty, \infty]$, defined by*

$$g(x) = \liminf_{n \rightarrow \infty} f_n(x) \text{ and } h(x) = \limsup_{n \rightarrow \infty} f_n(x), \quad \forall x \in X,$$

are both measurable.

PROOF. For every $n \in \mathbb{N}$, define the functions $g_n, h_n : X \rightarrow [-\infty, \infty]$ by

$$g_n(x) = \inf \{f_k(x) : k \geq n\} \text{ and } h_n(x) = \sup \{f_k(x) : k \geq n\}, \quad \forall x \in X.$$

By Corollary 3.5, we know that g_n and h_n are measurable for all $n \in \mathbb{N}$. Since

$$g(x) = \sup \{g_n(x) : n \in \mathbb{N}\} \text{ and } h(x) = \inf \{h_n(x) : n \in \mathbb{N}\}, \quad \forall x \in X,$$

the fact that both g and h are measurable follows again from Corollary 3.5. $\square$

COROLLARY 3.6. *Let $(X, \mathcal{A})$ be a measurable space, and let*

$$f_n : (X, \mathcal{A}) \rightarrow [-\infty, \infty], \quad n \in \mathbb{N}$$

be sequence of measurable functions, with the property that, for each $x \in X$, the sequence $(f_n(x))_{n=1}^{\infty} \subset [-\infty, \infty]$ has a limit. Then the function $f : X \rightarrow [-\infty, \infty]$, defined by

$$f(x) = \lim_{n \rightarrow \infty} f_n(x), \quad \forall x \in X,$$

is again measurable.

PROOF. Immediate from the above result. $\square$

Exercise 1. If $f_n : \mathbb{R} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, are continuous functions, and if $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ exists, for every $x \in \mathbb{R}$, then by the above Corollary we know that $f : \mathbb{R} \rightarrow [-\infty, \infty]$ is Borel measurable. Prove that the converse is not true. More explicitly, prove that there is no sequence $(f_n)_{n=1}^{\infty}$ of continuous functions, with

$$\lim_{n \rightarrow \infty} f_n(x) = \chi_{\mathbb{Q}}(x), \quad \forall x \in \mathbb{R}.$$

HINT: Use Baire's Theorem.

Exercise 2. Prove that a function $f : \mathbb{R} \rightarrow \mathbb{R}$, which is continuous everywhere, except for a countable set of points, is Borel measurable. As an application, prove that any monotone function is Borel measurable.

Corollary 3.6 can be generalized, as follows.

THEOREM 3.1. *Let $(X, \mathcal{A})$ be a measurable space, let Y be a separable metric space, and let*

$$T_n : (X, \mathcal{A}) \rightarrow (Y, \text{Bor}(Y)), \quad n \in \mathbb{N}$$

be a sequence of measurable maps. Assume that, for every $x \in X$, the sequence $(T_n(x))_{n=1}^{\infty} \subset Y$ is convergent. Define the map $T : X \rightarrow Y$ by

$$T(x) = \lim_{n \rightarrow \infty} T_n(x), \quad \forall x \in X.$$

Then $T : (X, \mathcal{A}) \rightarrow (Y, \text{Bor}(Y))$ is a measurable map.

PROOF. Denote by d the metric on Y . The collection

$$\mathcal{V} = \{\mathcal{B}_r(y) : y \in Y, r > 0\}$$

is a base for the topology of Y . Since Y is second countable, it suffices then to show that

$$(5) \quad T^{-1}(\mathcal{B}_r(y)) \in \mathcal{A}, \quad \forall y \in Y, r > 0.$$

Claim: For every $y \in Y$ and $r > 0$ one has the equality

$$(6) \quad T^{-1}(\mathcal{B}_r(y)) = \bigcup_{m,n=1}^{\infty} \left[\bigcap_{k=m}^{\infty} T_k^{-1}(\mathcal{B}_{r-\frac{1}{n}}(y)) \right].$$

Denote the set in the right hand side simply by A . Start first with some $x \in A$. There exist some $m, n \in \mathbb{N}$ such that

$$x \in \bigcap_{k=m}^{\infty} T_k^{-1}(\mathcal{B}_{r-\frac{1}{n}}(y)),$$

which means that

$$T_k(x) \in \mathcal{B}_{r-\frac{1}{n}}(y), \quad \forall k \geq m,$$

that is,

$$d(T_k(x), y) < r - \frac{1}{n}, \quad \forall k \geq m.$$

Pasing to the limit ($k \rightarrow \infty$) then yields

$$d(T(x), y) \leq r - \frac{1}{n} < r,$$

which means that $T(x) \in \mathcal{B}_r(y)$, i.e. $x = T^{-1}(\mathcal{B}_r(y))$, thus proving the inclusion $A \subset T^{-1}(\mathcal{B}_r(y))$.

Conversely, if $x \in T^{-1}(\mathcal{B}_r(y))$, we get $T(x) \in \mathcal{B}_r(y)$, i.e. $d(T(x), y) < r$. Choose an integer n such that

$$(7) \quad d(T(x), y) < r - \frac{2}{n}.$$

Since $\lim_{k \rightarrow \infty} T_k(x) = T(x)$, there exists some $m \in \mathbb{N}$ such that

$$d(T_k(x), T(x)) < \frac{2}{n}, \quad \forall k \geq m.$$

Combining this with (7) then gives

$$d(T_k(x), y) \leq d(T(x), y) + d(T_k(x), T(x)) < r - \frac{2}{n} + \frac{2}{n} = r - \frac{1}{n}, \quad \forall k \geq m,$$

which means that

$$x \in \bigcap_{k=m}^{\infty} T_k^{-1}(\mathcal{B}_{r-\frac{1}{n}}(y)),$$

hence x indeed belongs to A .

Having proven (6) we now observe that, since the T_k 's are measurable, it follows that

$$T_k^{-1}(\mathcal{B}_{r-\frac{1}{n}}(y)) \in \mathcal{A}, \quad \forall k, n \in \mathbb{N}, r > 0.$$

Using the fact that $\mathcal{A}$ is closed under countable intersections, it follows that

$$\bigcap_{k=m}^{\infty} T_k^{-1}(\mathcal{B}_{r-\frac{1}{n}}(y)) \in \mathcal{A}, \quad \forall m, n \in \mathbb{N}, r > 0.$$

Finally, using the fact that $\mathcal{A}$ is closed under countable unions, the desired property (5) follows. $\square$

Exercise 3. Let $(X, \mathcal{A})$ be a measurable space, and let $(X_n)_{n=1}^{\infty}$ be a sequence of sets in $\mathcal{A}$, with $X = \bigcup_{n=1}^{\infty} X_n$. Suppose $(Y, \mathcal{B})$ is a measurable space, and $F : X \rightarrow Y$ is a map, such that

$$F|_{X_n} : (X_n, \mathcal{A}|_{X_n}) \rightarrow (Y, \mathcal{B})$$

is measurable, for all $n \in \mathbb{N}$. Prove that $f : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ is measurable.

Exercise 4.* Let $\Omega_1 \subset \mathbb{R}^n$ be an open set, and let $f_1, \dots, f_n : \Omega_1 \rightarrow \mathbb{R}$ be C^1 functions, with the property that the matrix

$$A(p) = \left[\frac{\partial f_j}{\partial x_k}(p) \right]_{j,k=1}^n$$

is invertible, for every point $p \in \Omega_1$. Define the map

$$F : \Omega_1 \ni p \longmapsto (f_1(p), \dots, f_n(p)) \in \mathbb{R}^n.$$

- (i) Prove that the set $\Omega_2 = F(\Omega_1)$ is open in $\mathbb{R}^n$.
- (ii) Although $F : \Omega_1 \rightarrow \Omega_2$ may fail to be injective, prove that there exists a Borel measurable map $\phi : \Omega_2 \rightarrow \Omega_1$, with $F \circ \phi = \text{Id}_{\Omega_2}$.

HINT: Use the Inverse Function Theorem, combined with Exercises 2 and 3. exercise.

Exercise 5.* Let $P(z)$ be a non-constant polynomial with complex coefficients. Prove that there exists a Borel measurable function $f : \mathbb{C} \rightarrow \mathbb{C}$, such that

$$P(f(z)) = z, \quad \forall z \in \mathbb{C}.$$

HINT: Use the preceding exercise, applied to the set $\Omega_1 = \{z \in \mathbb{C} : P'(z) \neq 0\}$.

The preceding exercise can be generalized:

Exercise 6.* Let $\Omega_1 \subset \mathbb{C}$ be a connected open set, and let $f : \Omega_1 \rightarrow \mathbb{C}$ be a non-constant holomorphic function. By the Open Mapping Theorem we know that the set $\Omega_2 = f(\Omega_1)$ is open. Prove that there exists a Borel measurable function $\phi : \Omega_2 \rightarrow \Omega_1$, such that $f \circ \phi = \text{Id}_{\Omega_2}$.

HINT: Use Exercise 4, applied to the set $\Omega_0 = \{z \in \Omega_1 : f'(z) \neq 0\}$. Since f is non-constant, the set $\Omega_1 \setminus \Omega_0$ is countable.

We continue with a discussion on the role of elementary functions.

PROPOSITION 3.4. *Let $(X, \mathcal{A})$ be a measurable space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. For an elementary function $f \in \text{Elem}_{\mathbb{K}}(X)$, the following are equivalent:*

- (i) $f \in \mathcal{A}\text{-Elem}_{\mathbb{K}}(X)$;
- (ii) $f : (X, \mathcal{A}) \rightarrow \mathbb{K}$ is measurable.

PROOF. (i) $\Rightarrow$ (ii). We know that $\mathcal{A}\text{-Elem}_{\mathbb{K}} = \text{Span}_{\mathbb{K}}\{\varkappa_A : A \in \mathcal{A}\}$. Since $\mathbf{B}_{\mathbb{K}}(X, \mathcal{A})$ is a vector space, it suffices to show only that $\varkappa_A : (X, \mathcal{A}) \rightarrow \mathbb{K}$ is measurable, for all $A \in \mathcal{A}$. But this is trivial, since for every Borel set $B \subset \mathbb{R}$ one has either $\varkappa_A^{-1}(B) = \emptyset$, or $\varkappa_A^{-1}(B) = A$, or $\varkappa_A^{-1}(B) = X$.

(ii) $\Rightarrow$ (i). Assume now f is measurable. List the range of f as

$$f(X) = \{\lambda_1, \dots, \lambda_n\},$$

with $\lambda_j \neq \lambda_k$, for all $j, k \in \{1, \dots, n\}$ with $j \neq k$. Since f is measurable, and the singleton sets $\{\lambda_1\}, \dots, \{\lambda_n\}$ are in $\text{Bor}(\mathbb{K})$, it follows that the sets $A_j = f^{-1}(\{\lambda_j\})$, $j = 1, \dots, n$ are all in $\mathcal{A}$. Since we clearly have

$$f = \lambda_1 \varkappa_{A_1} + \dots + \lambda_n \varkappa_{A_n},$$

it follows that f indeed belongs to $\mathcal{A}\text{-Elem}_{\mathbb{K}}(X)$. □

REMARKS 3.4. A. If $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measurable spaces, if $T : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ is a measurable map, and if $f \in \mathcal{B}\text{-Elem}_{\mathbb{K}}(Y)$, then $f \circ T \in \mathcal{A}\text{-Elem}_{\mathbb{K}}(X)$. This follows from the fact that the composition $f \circ T : (X, \mathcal{A}) \rightarrow \mathbb{K}$ is measurable, and elementary.

B. If $(X, \mathcal{A})$ is a measurable space, if $f \in \mathcal{A}\text{-Elem}_{\mathbb{K}}(X)$, and if $g : f(X) \rightarrow \mathbb{K}$ is an arbitrary function, then $g \circ f \in \mathcal{A}\text{-Elem}_{\mathbb{K}}(X)$. This follows from the fact that, if one considers the finite set $Y = f(X)$, and the σ -algebra $\mathcal{P}(Y)$ on it, then

$$(X, \mathcal{A}) \xrightarrow{f} (Y, \mathcal{P}(Y)) \xrightarrow{g} \mathbb{K}$$

are measurable. So $g \circ f$ is also measurable, and obviously elementary.

The following is an interesting converse of Corollary 3.6.

THEOREM 3.2. *Let $(X, \mathcal{A})$ be a measurable space, and let $f : (X, \mathcal{A}) \rightarrow [-\infty, \infty]$ be a measurable function. Then there exists a sequence $(f_n)_{n=1}^{\infty} \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, such that*

- $\inf \{f(y) : y \in X\} \leq f_n(x) \leq \sup \{f(z) : z \in X\}, \forall x \in X, n \geq 1;$
- $\lim_{n \rightarrow \infty} f_n(x) = f(x), \forall x \in X.$

Moreover,

- (i) *if $\inf \{f(x) : x \in X\} > -\infty$, then the sequence $(f_n)_{n=1}^{\infty}$ can be chosen to be non-decreasing, i.e. $f_n \leq f_{n+1}, \forall n \in \mathbb{N};$*
- (ii) *if $\sup \{f(x) : x \in X\} < \infty$, then the sequence $(f_n)_{n=1}^{\infty}$ can be chosen to be non-increasing, i.e. $f_n \geq f_{n+1}, \forall n \in \mathbb{N};$*
- (iii) *if $\inf \{f(x) : x \in X\} > -\infty$ and $\sup \{f(x) : x \in X\} < \infty$, then the sequence $(f_n)_{n=1}^{\infty}$ can be chosen either non-decreasing, or non-increasing, and such that it converges uniformly to f , i.e.*

$$\lim_{n \rightarrow \infty} \left[\sup_{x \in X} |f_n(x) - f(x)| \right] = 0.$$

PROOF. We begin with a special case of (iii). Assume $X = [0, 1]$, $\mathcal{A} = \text{Bor}([0, 1])$, and consider the inclusion $F : [0, 1] \hookrightarrow [-\infty, \infty]$. For each $n \in \mathbb{N}$, define the intervals $I_k^n, J_k^n, 0 \leq k \leq 2^n - 1$ by

$$\begin{aligned} I_k^n &= [k/2^n, (k+1)/2^n), \text{ if } 0 \leq k \leq 2^n - 2; & I_{2^n-1}^n &= [(2^n - 1)/2^n, 1], \\ J_k^n &= (k/2^n, (k+1)/2^n], \text{ if } 1 \leq k \leq 2^n - 1; & J_0^n &= [0, 1/2^n]. \end{aligned}$$

We then define, for each $n \in \mathbb{N}$, the functions $g_n, h_n : [0, 1] \rightarrow \mathbb{R}$ by

$$g_n = 2^{-n} \sum_{k=0}^{2^n-1} k \chi_{I_k^n} \text{ and } h_n = 2^{-n} \sum_{k=0}^{2^n-1} (k+1) \chi_{J_k^n}.$$

Remark that

$$(8) \quad 0 \leq g_n(s) < 1 \text{ and } 0 < h_n(s) \leq 1, \forall s \in [0, 1].$$

Note that, for every $n \in \mathbb{N}$, we have

$$(9) \quad g_n(0) = 0; \quad g_n(1) = (2^n - 1)/2^n;$$

$$(10) \quad h_n(0) = 1/2^n; \quad h_n(1) = 1.$$

Claim 1: The sequence $(g_n)_{n=1}^{\infty}$ is non-decreasing, and the sequence $(h_n)_{n=1}^{\infty}$ is non-increasing.

Using (9) and (10), we only need to examine the restrictions to the open interval $(0, 1)$. Fix some point $s \in (0, 1)$. For every integer $n \geq 1$, define

$$p_n^s = \max \left\{ k \in \mathbb{Z} : 0 \leq \frac{k}{2^n} < s \right\}.$$

We clearly have $p_n^s < 2^n$ and

$$(11) \quad \frac{p_n^s}{2^n} < s \leq \frac{p_n^s + 1}{2^n}.$$

We then have

$$(12) \quad g_n(s) = \begin{cases} p_n^s/2^n & \text{if } s \neq (p_n^s + 1)/2^n \\ (p_n^s + 1)/2^n & \text{if } s = (p_n^s + 1)/2^n \end{cases} \quad \text{and } h_n(s) = \frac{p_n^s + 1}{2^n}$$

We now estimate $g_{n+1}(s)$ and $h_{n+1}(s)$. First of all, using (11), we have

$$\frac{2p_n^s}{2^{n+1}} < x \leq \frac{2p_n^s + 2}{2^{n+1}},$$

which means that either $p_{n+1}^s = 2p_n^s$, or $p_{n+1}^s = 2p_n^s + 1$. This immediately gives

$$h_{n+1}(s) = \frac{p_{n+1}^s + 1}{2^{n+1}} \leq \frac{2p_n^s + 2}{2^{n+1}} = \frac{p_n^s + 1}{2^n} = h_n(s).$$

Note that, if $s = (p_n^s + 1)/2^n$, we will have $p_{n+1}^s = 2p_n^s + 1$ and $s = (p_{n+1}^s + 1)/2^{n+1}$, so we get

$$g_{n+1}(s) = (p_{n+1}^s + 1)/2^{n+1} = (2p_n^s + 2)/2^{n+1} = (p_n^s + 1)/2^n = g_n(s).$$

If $s \neq (p_n^s + 1)/2^n$, then

$$g_n(s) = \frac{p_n^s}{2^n} = \frac{2p_n^s}{2^n} \leq \frac{p_{n+1}^s}{2^{n+1}} \leq g_{n+1}(s).$$

Claim 2: For every $s \in [0, 1]$ one has

$$\lim_{n \rightarrow \infty} \left[\sup_{s \in [0, 1]} |g_n(s) - s| \right] = \lim_{n \rightarrow \infty} \left[\sup_{s \in [0, 1]} |h_n(s) - s| \right] = 0.$$

To prove this fact we are going to estimate the differences $|g_n(s) - s|$ and $|h_n(s) - s|$.

If $s = 0$ or $s = 1$, then the equalities (9) and (10) immediately show that

$$(13) \quad |g_n(s) - s| \leq \frac{1}{2^n} \quad \text{and} \quad |h_n(s) - s| \leq \frac{1}{2^n}, \quad \forall n \in \mathbb{N}.$$

If $s \in (0, 1)$, then the definitions of $g_n(s)$ and $h_n(s)$ clearly show that

$$s, g_n(s), h_n(s) \in [p_n^s/2^n, (p_n^s + 1)/2^n],$$

and then we see that we again have the inequalities (13). Since (13) now holds for all $s \in [0, 1]$, the Claim immediately follows.

We proceed now with the proof of the theorem. Define

$$\alpha = \inf \{ f(x) : x \in X \} \quad \text{and} \quad \beta = \sup \{ f(x) : x \in X \}.$$

If $\alpha = \beta$, there is nothing to prove. Assume $\alpha < \beta$. Depending on the finitude of α and β , we define a homeomorphism $\Phi : [\alpha, \beta] \rightarrow [0, 1]$, as follows.

(a) If $\alpha > -\infty$ and $\beta < \infty$, we define

$$\Phi(s) = \frac{s - \alpha}{\beta - \alpha}, \quad \forall s \in [\alpha, \beta].$$

(b) If $\alpha > -\infty$ and $\beta = \infty$, we define

$$\Phi(s) = \begin{cases} \frac{2}{\pi} \arctan(s - \alpha) & \text{if } s \neq \beta \\ 1 & \text{if } s = \beta \end{cases}$$

(c) If $\alpha = -\infty$ and $\beta < \infty$, we define

$$\Phi(s) = \begin{cases} 1 + \frac{2}{\pi} \arctan(s - \beta) & \text{if } s \neq \alpha \\ 0 & \text{if } s = \alpha \end{cases}$$

(d) If $\alpha = -\infty$ and $\beta = \infty$, we define

$$\Phi(s) = \begin{cases} 0 & \text{if } s = \alpha \\ \frac{1}{2} + \frac{1}{\pi} \arctan(s - \beta) & \text{if } \alpha < s < \beta \\ 1 & \text{if } s = \beta \end{cases}$$

Notice that $\Phi(\alpha) = 0$, $\Phi(\beta) = 1$, and

$$\alpha \leq s < t \leq \beta \Rightarrow \Phi(s) < \Phi(t).$$

After these preparations, we proceed with the proof. We begin with the special cases (i) (ii) and (iii).

If $\alpha > -\infty$, we define the functions $f_n = \Phi^{-1} \circ g_n \circ \Phi \circ f$. Since Φ and Φ^{-1} are increasing, and $(g_n)_{n=1}^\infty$ is non-decreasing, it follows that $(f_n)_{n=1}^\infty$ is non-decreasing. Since $0 \leq g_n(s) < 1$, $\forall s \in [0, 1]$, we see that $\alpha \leq f_n(x) < \beta$, $\forall x \in X$. In particular, we have $-\infty < f_n(x) < \infty$, for all n and x . It is obvious that f_n is elementary, measurable, and since $\lim_{n \rightarrow \infty} g_n(s) = s$, $\forall s \in [0, 1]$ (by Claim 2), we immediately get $\lim_{n \rightarrow \infty} f_n(x) = f(x)$, $\forall x \in X$.

If $\beta < \infty$, we define the functions $f_n = \Phi^{-1} \circ h_n \circ \Phi \circ f$. Since Φ and Φ^{-1} are increasing, and $(h_n)_{n=1}^\infty$ is non-increasing, it follows that $(f_n)_{n=1}^\infty$ is non-increasing. Since $0 < h_n(s) \leq 1$, $\forall s \in [0, 1]$, we see that $\alpha < f_n(x) \leq \beta$, $\forall x \in X$. In particular, we have $-\infty < f_n(x) < \infty$, for all n and x . It is obvious that f_n is elementary, measurable, and since $\lim_{n \rightarrow \infty} h_n(s) = s$, $\forall s \in [0, 1]$ (by Claim 2), we immediately get $\lim_{n \rightarrow \infty} f_n(x) = f(x)$, $\forall x \in X$.

If $\alpha > -\infty$ and $\beta < \infty$, then we can take $f_n = \Phi^{-1} \circ g_n \circ \Phi \circ f$, $\forall n$, or we can take $f_n = \Phi^{-1} \circ h_n \circ \Phi \circ f$, $\forall n$. The inequalities (13), combined with the definition (c) of Φ , show that

$$|f_n(x) - f| \leq \frac{\beta - \alpha}{2^n}, \quad \forall x \in X, \quad n \in \mathbb{N},$$

with any of the above choices for $(f_n)_{n=1}^\infty$.

Having proven the cases (i), (ii) and (iii), we now examine the general situation, when $\alpha = -\infty$ and $\beta = \infty$. Consider the functions $f', f'' : X \rightarrow [-\infty, \infty]$ defined by

$$f'(x) = \max\{f(x), 0\} \text{ and } f''(x) = \min\{f(x), 0\}, \quad \forall x \in X.$$

By Corollary 3.4, both f' and f'' are measurable. Since $\inf_{x \in X} f'(x) \geq 0$, by part (i), there exists a sequence $(f'_n)_{n=1}^\infty \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, such that $\lim_{n \rightarrow \infty} f'_n(x) = f'(x)$, $\forall x \in X$. Since $\sup_{x \in X} f''(x) \leq 0$, by part (ii), there exists a sequence $(f''_n)_{n=1}^\infty \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, such that $\lim_{n \rightarrow \infty} f''_n(x) = f''(x)$, $\forall x \in X$. Define the elementary functions $f_n = f'_n + f''_n$, $n \in \mathbb{N}$. Clearly the f_n 's are all in $\mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$.

We now check that

$$(14) \quad \lim_{n \rightarrow \infty} f_n(x) = f(x), \quad \forall x \in X.$$

There are two cases to examine: (a) $f(x) \geq 0$; (b) $f(x) \leq 0$.

In case (a), we have $f'(x) = f(x)$ and $f''(x) = 0$, so $\lim_{n \rightarrow \infty} f'_n(x) = f(x)$ and $\lim_{n \rightarrow \infty} f''_n(x) = 0$.

In case (b), we have $f'(x) = 0$ and $f''(x) = f(x)$, so $\lim_{n \rightarrow \infty} f'_n(x) = 0$ and $\lim_{n \rightarrow \infty} f''_n(x) = f(x)$.

In either case, the equality (14) follows. $\square$

We conclude this section with a discussion on an interesting measurable space, that appears often in connection with probability theory.

EXAMPLE 3.1. Consider the space $T = \{0, 1\}^{\mathbb{N}}$, i.e.

$$T = \{a = (\alpha_n)_{n=1}^{\infty} : \alpha_n \in \{0, 1\}, \forall n \in \mathbb{N}\}.$$

We call T the *space of infinite coin flippings*, having in mind that an element of T is the same as the outcome of an infinite sequence of coin flips (think 0 as corresponding to tails, and 1 as corresponding to heads). Equip T with the product topology. By Tihonov's Theorem, T is compact. The product topology on T is in fact given by a metric d defined by

$$d(a, b) = \sum_{n=1}^{\infty} \frac{|\alpha_n - \beta_n|}{2^n}, \quad \forall a = (\alpha_n)_{n=1}^{\infty}, b = (\beta_n)_{n=1}^{\infty} \in T.$$

For every number $r \geq 2$ we define a map $\phi_r : T \rightarrow [0, 1]$ by

$$\phi_r(a) = (r-1) \sum_{n=1}^{\infty} \frac{\alpha_n}{r^n}, \quad \forall a = (\alpha_n)_{n=1}^{\infty} \in T.$$

It is pretty clear that

$$|\phi_r(a) - \phi_r(b)| \leq (r-1)d(a, b), \quad \forall a, b \in T,$$

so the maps $\phi_r : T \rightarrow [0, 1]$, $r \geq 2$ are continuous. In particular, the set $K_r = \phi_r(T)$ is a compact subset of $[0, 1]$.

Define

$$T_0 = \{a = (\alpha_n)_{n \in \mathbb{N}} \in T : \text{the set } \{n \in \mathbb{N} : \alpha_n = 0\} \text{ is infinite}\}.$$

The set $T \setminus T_0$ can be described as:

$$T \setminus T_0 = \{(\alpha_n)_{n \in \mathbb{N}} \in T : \text{there exists } N \in \mathbb{N}, \text{ such that } \alpha_n = 1, \forall n \geq N\}.$$

The following are well known (see Appendix B, the proof of Proposition B.2).

Facts: 1. The set $T \setminus T_0$ is countable

2. For any $r \geq 2$, and elements $a = (\alpha_n)_{n=1}^{\infty}, b = (\beta_n)_{n=1}^{\infty} \in T_0$, the following are equivalent:

- there exists $N \in \mathbb{N}$ such that $\alpha_N = 1$, $\beta_N = 0$, and $\alpha_n = \beta_n$, for all $n \in \mathbb{N}$ with $n < N$;
- $\phi_r(a) > \phi_r(b)$.

In particular, the map $\phi_r|_{T_0} : T_0 \rightarrow [0, 1]$ is injective.

The above constructions have a remarkable feature.

THEOREM 3.3. Use the notations above. For a number $r \geq 2$ and subset $A \subset T$, the following are equivalent:

- (i) $A \in \text{Bor}(T)$;
- (ii) $\phi_r(A) \in \text{Bor}(K_r)$.

PROOF. Throughout the proof the number r will be fixed. The map ϕ_r will be denoted by ϕ , and the compact set K_r will be denoted by K .

Since $\phi : T \rightarrow K$ is continuous, it is measurable, i.e. we have the implication

$$(15) \quad B \in \text{Bor}(K) \Rightarrow \phi^{-1}(B) \in \text{Bor}(T).$$

Before we proceed with the actual proof, we need some preparations. Remark that, since $\phi : T \rightarrow K$ is surjective, we have the equality

$$(16) \quad \phi(\phi^{-1}(C)) = C, \quad \forall C \subset K.$$

Claim 1: If a subset $C \subset K$ is at most countable, if and only if the set $\phi^{-1}(C) \subset T$ is at most countable.

Suppose C is at most countable. If we take $A_0 = \phi^{-1}(C) \cap T_0$, and $A_1 = \phi^{-1}(C) \setminus T_0$, then obviously $\phi^{-1}(C) = A_0 \cup A_1$. Since $A_1 \subset T \setminus T_0$, and $T \setminus T_0$ is countable, it follows that A_1 is at most countable, so we only need to prove that A_0 is at most countable. But since $\phi|_{T_0}$ is injective, and $A_0 \subset T_0$, it follows that $\phi|_{A_0} : A_0 \rightarrow C$ is injective, and then the fact that C is at most countable, forces A_0 to be at most countable.

Conversely, if $\phi^{-1}(C)$ is at most countable, then so is $\phi(\phi^{-1}(C))$. By (16) we are done.

For each subset $A \subset T$, we define

$$\langle A \rangle = \phi^{-1}(\phi(A)).$$

Remark that $A \subset \langle A \rangle$, $\forall A \subset T$. Note also that, for any family $(A_i)_{i \in I}$ of subsets of T , one has the equality

$$(17) \quad \langle \bigcup_{i \in I} A_i \rangle = \phi^{-1}\left(\phi\left(\bigcup_{i \in I} A_i\right)\right) = \phi^{-1}\left(\bigcup_{i \in I} \phi(A_i)\right) = \bigcup_{i \in I} \phi^{-1}(\phi(A_i)) = \bigcup_{i \in I} \langle A_i \rangle.$$

As an application of Claim 1, to the set $C = \phi(T \setminus T_0)$, we see that

(*) *the set $\langle T \setminus T_0 \rangle$ is at most countable.*

Claim 2: For any subset $A \subset T_0$, one has the inclusion

$$\langle A \rangle \setminus A \subset \langle T \setminus T_0 \rangle.$$

In particular, the difference $\langle A \rangle \setminus A$ is at most countable.

Start with an arbitrary element $x \in \langle A \rangle \setminus A$. This means that $x \notin A$, but $\phi(x) \in \phi(A)$, which means that there exists some $a \in A$, with $\phi(x) = \phi(a)$. Assume now $x \notin T \setminus T_0$, which means that $x \in T_0$. But then, the fact that $x, a \in T_0$, combined with the injectivity of $\phi|_{T_0}$ will force $x = a$, which is impossible since $a \in A$.

Claim 3: For any set $A \subset T$, the difference $\langle A \rangle \setminus A$ is at most countable.

Take $A_0 = A \cap T_0$ and $A_1 = A \setminus A_0$. Notice that, since $A_1 \subset T \setminus T_0$, we have

$$\langle A_1 \rangle = \phi^{-1}(\phi(A_1)) \subset \phi^{-1}(\phi(T \setminus T_0)) = \langle T \setminus T_0 \rangle,$$

so it follows that $\langle A_1 \rangle$ is at most countable. We obviously have $A = A_0 \cup A_1$, so by (17)

$$\langle A \rangle = \langle A_0 \rangle \cup \langle A_1 \rangle.$$

But now we are done, since

$$\langle A \rangle \setminus A = (\langle A_0 \rangle \cup \langle A_1 \rangle) \setminus (A_0 \cup A_1) \subset (\langle A_0 \rangle \setminus A_0) \cup \langle A_1 \rangle,$$

and both $\langle A_0 \rangle \setminus A_0$ (by Claim 2) and $\langle A_1 \rangle$ are at most countable.

Claim 4: For any subset $A \subset T$, one has the inclusion

$$(18) \quad \phi(T \setminus A) \supset K \setminus \phi(A),$$

and the difference $\phi(T \setminus A) \setminus (K \setminus \phi(A))$ is at most countable.

The inclusion (18) is pretty obvious, from the surjectivity of ϕ . In order to prove that the difference

$$C = \phi(T \setminus A) \setminus (K \setminus \phi(A)) = \phi(T \setminus A) \cap \phi(A)$$

is countable, by Claim 1, it suffices to prove that $\phi^{-1}(C)$ is countable. We have

$$\phi^{-1}(C) = \phi^{-1}(\phi(T \setminus A) \cap \phi(A)) = \phi^{-1}(\phi(T \setminus A)) \cap \phi^{-1}(\phi(A)) = \langle T \setminus A \rangle \cap \langle A \rangle.$$

We can write $\phi^{-1}(C) = A_1 \cup A_2$, where

$$A_1 = (T \setminus A) \cap \langle A \rangle \text{ and } A_2 = [\langle T \setminus A \rangle \setminus (T \setminus A)] \cap \langle A \rangle,$$

so it suffices to prove that both A_1 and A_2 are at most countable. But these facts are immediate from Claim 3, since $A_1 = \langle A \rangle \setminus A$, and $A_2 \subset \langle T \setminus A \rangle \setminus (T \setminus A)$.

We can now proceed with the proof of the theorem. Define

$$\mathcal{A} = \{A \subset T : \phi(A) \in \text{Bor}(K)\},$$

so that what we need to prove is the equality $\mathcal{A} = \text{Bor}(T)$.

First, remark that, if $A \in \mathcal{A}$, then $\phi(A) \in \text{Bor}(K)$, and the fact that ϕ is Borel measurable will force $\langle A \rangle = \phi^{-1}(\phi(A))$ to be a Borel set in T . But since $\langle A \rangle \setminus A$ is countable, hence Borel, it follows that

$$A = \langle A \rangle \setminus (\langle A \rangle \setminus A)$$

is again Borel. Therefore, we have the inclusion $\mathcal{A} \subset \text{Bor}(T)$.

Second, remark that if $F \subset T$ is a compact subset, then the continuity of ϕ gives the fact that $\phi(F)$ is compact, hence Borel. This then forces $F \in \mathcal{A}$. Therefore $\mathcal{A}$ contains the collection $\mathcal{C}_T$ of all compact subsets of T .

Now we have

$$\mathcal{C}_T \subset \mathcal{A} \subset \text{Bor}(T) = \Sigma(\mathcal{C}_T),$$

so all we need to prove is the fact that $\mathcal{A}$ is a σ -algebra, i.e. we have the properties

- (a) $A \in \mathcal{A} \Rightarrow T \setminus A \in \mathcal{A}$;
- (b) for any sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{A}$, the union $\bigcup_{n=1}^{\infty} A_n$ also belongs to $\mathcal{A}$.

To check (a) start with some set $A \in \mathcal{A}$. We know that $\phi(A) \in \text{Bor}(K)$, and we want to show that $\phi(T \setminus A)$ is again Borel. By Claim 4, we know we can write

$$\phi(T \setminus A) = [K \setminus \phi(A)] \cup C,$$

for some set $C \subset K$ which is at most countable. Since C and $K \setminus \phi(A)$ are Borel, this shows that $\phi(T \setminus A)$ is also Borel.

Property (b) is obvious, since $\phi(A_n)$, $n \geq 1$ are all Borel, and

$$\phi\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} \phi(A_n). \quad \square$$

COROLLARY 3.7. *Use the above notations. For a number $r \geq 2$ and a subset $B \subset K_r$, the following are equivalent:*

- (i) $B \in \text{Bor}(K_r)$;
- (ii) $\phi_r^{-1}(B) \in \text{Bor}(T)$.

PROOF. The implication (i) $\Rightarrow$ (ii) is trivial, since ϕ_r is continuous, hence measurable.

Conversely, if the set $A = \phi_r^{-1}(B)$ is Borel, then by the Theorem, $\phi_r(A)$ is Borel. But since ϕ_r is surjective, we have $B = \phi_r(A)$. $\square$

COMMENTS. From the above results, we see that $\phi_r : T \rightarrow K_r$ “almost preserves Borel structures.” More explicitly, if one considers the maps

$$\begin{aligned}\Phi_r : \mathcal{P}(T) \ni A &\longmapsto \phi_r(A) \in \mathcal{P}(K_r), \\ \Psi_r : \mathcal{P}(K_r) \ni B &\longmapsto \phi_r^{-1}(B) \in \mathcal{P}(T),\end{aligned}$$

then

- $(\Phi_r \circ \Psi_r)(B) = B$, for all $B \subset K_r$;
- $(\Psi_r \circ \Phi_r)(A) \supset A$, and $(\Phi_r \circ \Psi_r)(A) \setminus A$ is at most countable, for all $A \subset T$;
- $B \in \text{Bor}(K_r) \Leftrightarrow \Psi_r(B) \in \text{Bor}(T)$;
- $A \in \text{Bor}(T) \Leftrightarrow \Phi_r(A) \in \text{Bor}(K_r)$.

In the particular case $r = 2$, we know that $K_2 = [0, 1]$, so we can think the measurable space $([0, 1], \text{Bor}([0, 1]))$ as “approximately the same” as the measurable space $(T, \text{Bor}(T))$.

The case $r = 3$ will be an interesting one, especially for constructing various counter-examples. The compact set $K_3 \subset [0, 1]$ is called the *ternary Cantor set*.

It turns out that there exists another useful description of the ternary Cantor set K_3 , which yields some interesting properties.

NOTATIONS. We keep the notations above. An element $a = (\alpha_n)_{n=1}^\infty \in T$ will be called *finite*, if there exists some $N \in \mathbb{N}$, such that $\alpha_n = 0$, $\forall n \geq N$. We define

$$T_{fin} = \{a \in T : a \text{ finite}\}.$$

Remark that $T_{fin} \subset T_0$. In particular the map $\phi_3|_{T_{fin}} : T_{fin} \rightarrow K_3$ is injective.

For $a \in T_{fin}$ we define its length as

$$\ell(a) = \min\{N \in \mathbb{N} : \alpha_n = 0, \forall n \geq N\} - 1.$$

With this definition, for every $a = (\alpha_n)_{n=1}^\infty \in T_{fin}$, we have

$$(19) \quad \alpha_{\ell(a)} = 1 \text{ and } \alpha_n = 0, \forall n > \ell(a).$$

We define

$$\Lambda = \{(k, a) \in \mathbb{Z} \times T_{fin} : k \geq \ell(a)\}.$$

Finally, for every pair $\lambda = (k, a) \in \Lambda$, we define the open interval

$$I_\lambda = \left(\phi_3(a) + \frac{1}{3^{k+1}}, \phi_3(a) + \frac{2}{3^{k+1}}\right).$$

Remark that, using (19) we have

$$\phi_3(a) \leq 2 \sum_{n=1}^{\ell(a)} \frac{2}{3^n} = 1 - \frac{1}{3^{\ell(a)}},$$

with the convention that the sum is 0, if $\ell(a) = 0$. We then get

$$\phi_3(a) + \frac{2}{3^{k+1}} \leq 1 - \frac{1}{3^{\ell(a)}} + \frac{2}{3^{k+1}} < 1 - \frac{1}{3^{\ell(a)}} + \frac{1}{3^k} \leq 1,$$

which gives the inclusion $I_\lambda \subset (0, 1)$.

The following result describes an alternative construction of K_3 .

THEOREM 3.4. *Use the notations above.*

- (i) *The set T_{fin} is dense in T ;*
- (ii) *The system $(I_\lambda)_{\lambda \in \Lambda}$ is pair-wise disjoint.*
- (iii) $\bigcup_{\lambda \in \Lambda} = [0, 1] \setminus K_3$.

PROOF. The map ϕ_3 will be simply denoted by ϕ , and the Cantor set K_3 will be denoted simply by K .

(i). Fix some element $a = (\alpha_n)_{n=1}^\infty \in T$. For every integer $k \geq 1$ define the element $a_k = (\alpha_n^k)_{n=1}^\infty \in T$, by

$$\alpha_n^k = \begin{cases} \alpha_n & \text{if } n \leq k \\ 0 & \text{if } n > k \end{cases}$$

It is obvious that $a_k \in T_{fin}$, $\forall k \in \mathbb{N}$. The inequality

$$d(a, a_k) = \sum_{n=k+1}^{\infty} \frac{\alpha_n}{2^n} \leq \sum_{n=k+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^k}, \quad \forall k \in \mathbb{N}$$

then immediately shows that $\lim_{k \rightarrow \infty} a_k = a$.

(ii). Assume $\lambda, \mu \in \Lambda$ are such that $\lambda \neq \mu$, and let us prove that $I_\lambda \cap I_\mu = \emptyset$. Let $\lambda = (j, a)$ and $\mu = (k, b)$, where $a = (\alpha_n)_{n=1}^\infty$ and $b = (\beta_n)_{n=1}^\infty$ are elements in T_{fin} with $\ell(a) \leq j$ and $\ell(b) \leq k$. Since $\lambda \neq \mu$, we have one (or both) of the following cases: (A) $a \neq b$, or (B) $j \neq k$.

In case (A) we take

$$m = \min\{n \in \mathbb{N} : \alpha_n \neq \beta_n\}.$$

Without any loss of generality, we can assume that $\alpha_m = 0$ and $\beta_m = 1$. Note that $k \geq \ell(b) \geq m \geq 1$. We are going to prove that $I_\lambda \cap I_\mu = \emptyset$, by showing that the right end-point of I_λ is not greater than the left end-point of I_μ , that is,

$$(20) \quad \phi(a) + \frac{2}{3^{k+1}} \leq \phi(b) + \frac{1}{3^{k+1}}.$$

Define the number

$$M = \sum_{n=1}^{m-1} \frac{\alpha_n}{3^n} = \sum_{n=1}^{m-1} \frac{\beta_n}{3^n},$$

with the convention that $M = 0$, if $m = 1$. We have:

$$\begin{aligned} \phi(a) &= 2M + 2 \sum_{n=m+1}^{\ell(a)} \frac{\alpha_n}{3^n} \leq 2M + 2 \sum_{n=m+1}^{\ell(a)} \frac{1}{3^n} = 2M + \frac{1}{3^m} - \frac{1}{3^{\ell(a)}}; \\ \phi(b) &= 2M + \frac{2}{3^m} + 2 \sum_{n=m+1}^{\ell(b)} \frac{\beta_n}{3^n} \geq 2M + \frac{2}{3^m}. \end{aligned}$$

The inequality (20) then follows immediately from:

$$\begin{aligned} \phi(a) + \frac{2}{3^{j+1}} &\leq 2M + \frac{1}{3^m} - \frac{1}{3^{\ell(a)}} + \frac{2}{3^{j+1}} << 2M + \frac{1}{3^m} - \frac{1}{3^{\ell(a)}} + \frac{1}{3^j} \leq \\ &\leq 2M + \frac{1}{3^m} < 2M + \frac{2}{3^m} \leq \phi(b) < \phi(b) + \frac{1}{3^{k+1}}. \end{aligned}$$

In case (B), based on the fact that we have proven case (A), we can assume, without any loss of generality, that $a = b$ and $j < k$. In this case we have

$$\phi(b) + \frac{2}{3^{k+1}} = \phi(a) + \frac{2}{3^{k+1}} < \phi(a) + \frac{1}{3^k} \leq \phi(a) + \frac{1}{3^{j+1}},$$

which means that the right end-point of I_μ is not greater than the left end-point of I_λ , so again we get $I_\lambda \cap I_\mu = \emptyset$.

For the proof of (iii) we are going to use the space

$$P = \{0, 1, 2\}^{\aleph_0} = \{(\alpha_n)_{n=1}^\infty : \alpha_n \in \{0, 1, 2\}, \forall n \in \mathbb{N}\}.$$

Exactly as is the case with T , the product space P is compact with respect to the product topology, which is given by the metric

$$d(a, b) = \sum_{n=1}^\infty \frac{|\alpha_n - \beta_n|}{2^n}, \quad \forall a = (\alpha_n)_{n=1}^\infty, b = (\beta_n)_{n=1}^\infty \in P.$$

Then map $\psi : P \rightarrow [0, 1]$, defined by

$$\psi(a) = \sum_{n=1}^\infty \frac{\alpha_n}{3^n}, \quad \forall a = (\alpha_n)_{n=1}^\infty \in P,$$

satisfies

$$|\psi(a) - \psi(b)| \leq d(a, b), \quad \forall a, b \in P,$$

hence it is continuous. Note also that ψ is surjective. We can write $\phi = \psi \circ \rho$, where

$$\rho : \{0, 1\}^{\aleph_0} \ni (\alpha_n)_{n=1}^\infty \mapsto (2\alpha_n)_{n=1}^\infty \in \{0, 1, 2\}^{\aleph_0}.$$

Note also that $\rho : T \rightarrow P$ is continuous, since we clearly have

$$d(\rho(a), \rho(b)) \leq 2d(a, b), \quad \forall a, b \in T.$$

We now proceed with the proof of (iii). Denote the open set $\bigcup_{\lambda \in \Lambda} I_\lambda$ simply by D . Since T_{fin} is dense in T , it follows that $\phi(T_{fin})$ is dense in $K = \phi(T)$. Therefore, in order to prove the inclusion $K \subset [0, 1] \setminus D$, using the surjectivity of ψ , it suffices to prove the inclusion

$$\phi(T_{fin}) \subset [0, 1] \setminus D.$$

Using the map $\psi : P \rightarrow [0, 1]$, the above inclusion is equivalent to

$$(21) \quad P \setminus \rho(T_{fin}) \supset \psi^{-1}(D).$$

In order to prove the inclusion $[0, 1] \setminus D \subset K$, again using the surjectivity of ψ , it suffices to prove the inclusion

$$(22) \quad \psi^{-1}(D) \supset P \setminus \psi^{-1}(K).$$

To prove (21) start with some element $a = (\alpha_n)_{n=1}^\infty \in \psi^{-1}(D)$, which means that there exists some $b \in T_{fin}$, and an integer $k \geq \ell(b)$, such that $\psi(a) \in I_{(k,b)}$, i.e.

$$(23) \quad \frac{2\beta_1}{3} + \cdots + \frac{2\beta_k}{3^k} + \frac{1}{3^{k+1}} < \sum_{n=1}^\infty \frac{\alpha_n}{3^n} < \frac{2\beta_1}{3} + \cdots + \frac{2\beta_k}{3^k} + \frac{2}{3^{k+1}}.$$

We prove that $a \notin \rho(T_{fin})$ by contradiction. Assume $a \in \rho(T_{fin})$, which means that there exists $c = (\gamma_n)_{n=1}^\infty \in T_{fin}$, such that $\alpha_n = 2\gamma_n$, $\forall n \in \mathbb{N}$. Define the element $\tilde{b} = (\tilde{\beta}_n)_{n=1}^\infty \in T_{fin}$ by

$$\tilde{\beta}_n = \begin{cases} \beta_n & \text{if } n \leq k \\ 1 & \text{if } n = k+1 \\ 0 & \text{if } n > k+1 \end{cases}$$

With this definition, the inequalities (23) give

$$(24) \quad \phi(b) < \phi(b) + \frac{1}{3^{k+1}} < \phi(c) < \phi(\tilde{b}).$$

By Fact 2 above, there exist $N, N' \in \mathbb{N}$ such that

- $\gamma_N = 1$, $\beta_N = 0$, and $\gamma_n = \beta_n$, for all $n \in \mathbb{N}$ with $n < N$;
- $\gamma_{N'} = 0$, $\beta_{N'} = 1$, and $\gamma_n = \beta_n$, for all $n \in \mathbb{N}$ with $n < N'$.

We will examine three cases: (A) $N < N'$, (B) $N = N'$, or (C) $N > N''$.

Case (B) is clearly impossible. In case (A), the inequality $N < N'$ forces $\beta_N = 0$, $\gamma_N = 1$ and $\tilde{\beta}_N = \gamma_N$, which means that $\tilde{\beta}_N = 1 \neq \beta_N = 0$. This clearly forces $N = k+1 > \ell(b)$, which in particular gives $\beta_n = \tilde{\beta}_n = 0$, $\forall n > N$, so we clearly have $\gamma_n \geq \tilde{\beta}_n$, $\forall n \in \mathbb{N}$, so we get $\phi(c) \geq \phi(\tilde{b})$, thus contradicting (24). In case (C), we have $\gamma_{N'} = 0$, $\tilde{\beta}_{N'} = 1$, and since $N' < N$, we also have $\beta_{N'} = \gamma_{N'} = 0$. As before this would force $N' = k+1$. We then have

$$\begin{aligned} \phi(c) &= 2 \sum_{n=1}^{\infty} \frac{\gamma_n}{3^n} = 2 \sum_{n=1}^{N'-1} \frac{\gamma_n}{3^n} + \frac{2\gamma_{N'}}{3^{N'}} + 2 \sum_{n=N'+1}^{\infty} \frac{\gamma_n}{3^n} = 2 \sum_{n=1}^k \frac{\beta_n}{3^n} + 0 + 2 \sum_{n=k+2}^{\infty} \frac{\gamma_n}{3^n} = \\ &= \phi(b) + 2 \sum_{n=k+2}^{\infty} \frac{\gamma_n}{3^n} \leq \phi(b) + 2 \sum_{n=k+1}^{\infty} \frac{1}{3^n} = \phi(b) + \frac{1}{3^{k+1}}, \end{aligned}$$

again contradicting (24).

To prove (22), we start with some element $a \in P \setminus \psi^{-1}(K)$, and we show that $\psi(a) \in D$. The fact that $a \notin \psi^{-1}(K)$ forces the fact that $a \notin \rho(T)$. In particular, this gives the fact that $a = (\alpha_n)_{n=1}^\infty \in \{0, 1, 2\}^{\aleph_0}$ and there exists some $n \in \mathbb{N}$ such that $\alpha_n = 1$. Put

$$N = \min\{n \in \mathbb{N} : \alpha_n = 1\}.$$

Define the elements $b = (\beta_n)_{n=1}^\infty \in \{0, 1\}^{\aleph_0}$, by

$$\beta_n = \begin{cases} \alpha_n/2 & \text{if } n < N \\ 0 & \text{if } n \geq N \end{cases}$$

Notice that $b \in T_{fin}$, and $\ell(b) \leq N-1$. Notice also that $2\beta_n = \alpha_n$, for all $n \in \mathbb{N}$ with $n < N-1$. In particular, using the equality $\alpha_N = 1$, this gives

(25)

$$\phi(b) + \frac{1}{3^N} = 2 \sum_{n=1}^{N-1} \frac{\beta_n}{3^n} + \frac{\alpha_N}{3^N} = \sum_{n=1}^N \frac{\alpha_n}{3^n} \leq \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n} = \psi(a);$$

(26)

$$\phi(b) + \frac{2}{3^N} = 2 \sum_{n=1}^{N-1} \frac{\gamma_n}{3^n} + \frac{\alpha_N}{3^N} + \sum_{n=N+1}^{\infty} \frac{2}{3^n} = \sum_{n=1}^N \frac{\alpha_n}{3^n} + \sum_{n=N+1}^{\infty} \frac{2}{3^n} \geq \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n} = \psi(a).$$

Consider the pair $\lambda = (N - 1, \beta) \in \Lambda$. We are going to show that $\psi(a) \in I_\lambda$, i.e. we have the inequalities

$$(27) \quad \phi(b) + \frac{1}{3^N} < \psi(a) < \phi(b) + \frac{2}{3^N}.$$

By (25) and (26) it suffices to prove only that

$$\psi(a) \neq \phi(b) + \frac{1}{3^N} \text{ and } \psi(a) \neq \phi(b) + \frac{2}{3^N}.$$

If $\psi(a) = \phi(b) + \frac{1}{3^N}$, then by the inequalities (25), we are forced to have

$$(28) \quad \alpha_n = 0, \quad \forall n > N..$$

If $\psi(a) = \phi(b) + \frac{2}{3^N}$, then by the inequalities (26), we are forced to have

$$(29) \quad \alpha_n = 2, \quad \forall n > N..$$

If (28) holds, we define $c = (\gamma_n)_{n=1}^\infty \in T$, by

$$\gamma_n = \begin{cases} \alpha_n/2 & \text{if } n < N \\ 0 & \text{if } n = N \\ 1 & \text{if } n > N \end{cases}$$

and we will have

$$\phi(c) = 2 \sum_{n=1}^\infty \frac{\gamma_n}{3^n} = \sum_{n=1}^{N-1} \frac{2\gamma_n}{3^n} + 2 \sum_{n=N+1}^\infty \frac{1}{3^n} = \sum_{n=1}^{N-1} \frac{\alpha_n}{3^n} + \frac{1}{3^N} = \psi(a),$$

thus forcing $\psi(a) \in K$, which is impossible.

If (29) holds, we define $c = (\gamma_n)_{n=1}^\infty \in T$, by

$$\gamma_n = \begin{cases} \alpha_n/2 & \text{if } n \neq N \\ 1 & \text{if } n = N \\ 0 & \text{if } n > N \end{cases}$$

and we will have

$$\phi(c) = 2 \sum_{n=1}^{N-1} \frac{\gamma_n}{3^n} + \frac{2}{3^N} = \sum_{n=1}^{N-1} \frac{2\gamma_n}{3^n} + \frac{1}{3^N} + \sum_{n=N+1}^\infty \frac{2}{3^n} = \sum_{n=1}^\infty \frac{\alpha_n}{3^n} = \psi(a),$$

thus forcing again $\psi(a) \in K$, which is impossible. $\square$

Exercise 7. Using the notations above, prove that the set

$$[0, 1] \setminus K_3 = \bigcup_{\lambda \in \Lambda} I_\lambda$$

is dense in $[0, 1]$.

HINTS: Define the set

$$P_0 = \{(\alpha_n)_{n=1}^\infty \in \{0, 1, 2\}^{\aleph_0} : \text{the set } \{n \in \mathbb{N} : \alpha_n = 1\} \text{ is infinite}\}.$$

Prove that P_0 is dense in P , and prove that $\psi(P) \subset [0, 1] \setminus K$. (Use the arguments employed in the proof of part (iii).)

REMARKS 3.5. If we set $\Lambda_n = \Lambda \cap (\{n\} \times P)$, then we can write the complement of the ternary Cantor set as

$$[0, 1] \setminus K_3 = \bigcup_{n=0}^\infty D_n,$$

where

$$D_n = \bigcup_{\lambda \in \Lambda_n} I_\lambda.$$

Then the system of open sets $(D_n)_{n \geq 0}$ is pair-wise disjoint. Moreover, each D_n is a union of 2^n disjoint intervals of length $1/3^{n+1}$.

Since $\text{card } T_0 = \mathfrak{c}$, and the map $\phi_3|_{T_0} : T_0 \rightarrow K_3$ is injective, we get $\text{card } K_3 \geq \mathfrak{c}$. Since we also have $\text{card } K_3 \leq \text{card } \mathbb{R} = \mathfrak{c}$, we get in fact the equality

$$\text{card } K_3 = \mathfrak{c}.$$

LECTURE 21

4. The concept of measure

DEFINITION. Let X be a non-empty set, and let $\mathcal{E}$ be an arbitrary collection of subsets of X . Assume $\emptyset \in \mathcal{E}$. A *measure on $\mathcal{E}$* is a map $\mu : \mathcal{E} \rightarrow [0, 1]$ with the following properties

- (0) $\mu(\emptyset) = 0$.
- (ADD $_{\sigma}$) Whenever $(E_n)_{n=1}^{\infty} \subset \mathcal{E}$ is a pair-wise disjoint sequence, with $\bigcup_{n=1}^{\infty} E_n \in \mathcal{E}$, it follows that we have the equality

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

Property (ADD $_{\sigma}$) is called *σ -additivity*.

CONVENTION. For a sequence $(\alpha_n)_{n=1}^{\infty} \subset [0, \infty]$ we define

$$\sum_{n=1}^{\infty} \alpha_n = \begin{cases} \sum_{n=1}^{\infty} \alpha_n & \text{if } \alpha_n \in [0, \infty), \forall n \in \mathbb{N} \\ \infty & \text{if there exists } n \in \mathbb{N} \text{ with } \alpha_n = \infty. \end{cases}$$

(Of course, in the first case, it is still possible to have $\sum_{n=1}^{\infty} \alpha_n = \infty$.)

REMARK 4.1. If μ is a measure on $\mathcal{E}$, then μ is *additive*, i.e.

- (ADD) Whenever $(E_n)_{n=1}^N \subset \mathcal{E}$ is a finite pair-wise disjoint system, such that $E_1 \cup \dots \cup E_N \in \mathcal{E}$, it follows that we have the equality

$$\mu(E_1 \cup \dots \cup E_N) = \mu(E_1) + \dots + \mu(E_N).$$

This follows from (ADD $_{\sigma}$) (0), after completing the sequence $E_1, \dots, E_N$ to an infinite sequence by taking $E_n = \emptyset, \forall n > N$.

COMMENT. The most natural setting for measures is the one when $\mathcal{E}$ is a σ -ring. In this case, the stipulation that $\bigcup_{n=1}^{\infty} E_n \in \mathcal{E}$, which appears in the definition, is superfluous.

The purpose of this section is to study measures on more rudimentary collections.

EXAMPLES 4.1. Let X be a non-empty set.

A. If we take $\mathcal{E} = \{\emptyset, X\}$ and we define $\mu(\emptyset) = 0$ and $\mu(X)$ to be any element in $[0, \infty]$, then μ is obviously a measure on $\{\emptyset, X\}$.

B. If we take $\mathcal{E} = \mathcal{P}(X)$ and we define

$$\mu(E) = \begin{cases} 0 & \text{if } E = \emptyset \\ \infty & \text{if } E \neq \emptyset \end{cases}$$

then μ is a measure on $\mathcal{P}(X)$.

C. If we take $\mathcal{E} = \mathcal{P}(X)$ and we define

$$\mu(E) = \begin{cases} \text{card } E & \text{if } E \text{ is finite} \\ \infty & \text{if } E \text{ is infinite} \end{cases}$$

then μ is a measure on $\mathcal{P}(X)$. This is called the *counting measure*.

Exercise 1. Let X_1, X_2 be non-empty spaces, let $\mathcal{E}_k \subset \mathcal{P}(X_k)$ be arbitrary collections with $\emptyset \in \mathcal{E}_k$, $k = 1, 2$. Let μ_1 be a measure on $\mathcal{E}_1$ and μ_2 be a measure on $\mathcal{E}_2$. Consider the collections

$$f_*\mathcal{E}_1 = \{A \subset X_2 : f^{-1}(A) \in \mathcal{E}_1\} \subset \mathcal{P}(X_2);$$

$$f^*\mathcal{E}_2 = \{f^{-1}(A) : A \in \mathcal{E}_2\} \subset \mathcal{P}(X_1).$$

A. Prove that the map $f_*\mu_1 : f_*\mathcal{E}_1 \rightarrow [0, \infty]$, defined by

$$(f_*\mu_1)(A) = \mu_1(f^{-1}(A)), \quad \forall A \in f_*\mathcal{E}_1,$$

is a measure on $f_*\mathcal{E}_1$.

B. If f is *surjective*, prove that the map $f^*\mu_2 : f^*\mathcal{E}_2 \rightarrow [0, \infty]$, defined by

$$(f^*\mu_2)(f^{-1}(B)) = \mu_2(B), \quad \forall B \in \mathcal{E}_2,$$

is a measure on $f^*\mathcal{E}_2$.

We now concentrate on the most rudimentary types of collections $\mathcal{E}$ on which measures can be somehow easily defined. Actually, what we have in mind is a set of easy conditions on a map $\mu : \mathcal{E} \rightarrow [0, \infty]$ which would guarantee that μ is a measure.

DEFINITION. Let X be a non-empty set. A collection $\mathcal{J} \subset \mathcal{P}(X)$ is called a *semiring*, if it satisfies the following properties:

- $\emptyset \in \mathcal{J}$;
- if $A, B \in \mathcal{J}$, then $A \cap B \in \mathcal{J}$;
- if $A, B \in \mathcal{J}$ and $A \subset B$, then there exists an integer $n \geq 1$, and sets $D_0, D_1, \dots, D_n \in \mathcal{J}$, such that $A = D_0 \subset D_1 \subset \dots \subset D_n = B$, and $D_k \setminus D_{k-1} \in \mathcal{J}$, $\forall k \in \{1, \dots, n\}$.

Remark that every ring is a semiring.

Exercise 2. Prove that the semiring type is not consistent. Give an example of two semirings $\mathcal{J}_1, \mathcal{J}_2 \subset \mathcal{P}(X)$, such that $\mathcal{J}_1 \cap \mathcal{J}_2$ is not a semiring.

HINT: Use the set $X = \{1, 2, 3\}$.

Exercise 3. Let $X_1, \dots, X_n$ be non-empty sets, and let $\mathcal{J}_k \subset \mathcal{P}(X_k)$, $k = 1, \dots, n$, be semirings. Prove that

$$\mathcal{J} = \{A_1 \times \dots \times A_n : A_1 \in \mathcal{J}_1, \dots, A_n \in \mathcal{J}_n\} \subset \mathcal{P}(X_1 \times \dots \times X_n)$$

is a semiring.

HINT: First prove the case $n = 2$, and then use induction.

EXAMPLE 4.2. Take $X = \mathbb{R}$. The collection

$$\mathcal{J} = \{\emptyset\} \cup \{[a, b) : a, b \in \mathbb{R}, a < b\} \subset \mathcal{P}(\mathbb{R})$$

is a semiring.

Indeed, the first two axioms are pretty clear. To prove the third axiom, we start with two intervals $A = [a, b)$ and $B = [c, d)$ with $A \subset B$. This means that $a \geq c$ and $b \leq d$. If $a = c$ or $b = d$, we set $D_0 = A$ and $D_1 = B$. If $a > c$ and $b < d$, we set $D_0 = A$, $D_1 = [a, d)$ and $D_2 = B$.

More generally, by Exercise 3, the collection of "half-open boxes"

$$\mathcal{J}_n = \{\emptyset\} \cup \left\{ \prod_{j=1}^n [a_j, b_j) : a_1 < b_1, \dots, a_n < b_n \right\} \subset \mathcal{P}(\mathbb{R}^n)$$

is a semiring.

Exercise 4. Let $\mathcal{J}_n \subset \mathcal{P}(\mathbb{R}^n)$ be the semiring defined above. Prove that the σ -ring $\mathbf{S}(\mathcal{J})$ generated by $\mathcal{J}_n$ coincides with $\text{Bor}(\mathbb{R}^n)$.

The ring generated by a semiring has a particularly nice description (compare to Proposition 2.1):

PROPOSITION 4.1. *Let $\mathcal{J}$ be a semiring on X . For a subset $A \subset X$, the following are equivalent:*

- (i) *A belongs to $\mathbf{R}(\mathcal{J})$, the ring generated by $\mathcal{J}$;*
- (ii) *There exists an integer $n \geq 1$, and a pair-wise disjoint system $(A_j)_{j=1}^n \subset \mathcal{J}$, such that $A = A_1 \cup \dots \cup A_n$.*

PROOF. Denote by $\mathcal{R}$ the collection of all subsets $A \subset X$ that satisfy condition (ii). It is obvious that

$$\mathcal{J} \subset \mathcal{R} \subset \mathbf{R}(\mathcal{J}),$$

so (see Section III.2) we only need to prove that $\mathcal{R}$ is a ring.

Let us first remark that we obviously have the property:

- (i) if $A, B \in \mathcal{R}$, and $A \cap B = \emptyset$, then $A \cup B \in \mathcal{R}$.

Secondly, we remark that we have the implication:

- (II) $A, B \in \mathcal{J} \Rightarrow A \setminus B \in \mathcal{R}$.

Indeed, since $A \cap B \in \mathcal{J}$, by the definition of a semiring, there exist $D_0, D_1, \dots, D_n \in \mathcal{J}$ with $A \cap B = D_0 \subset D_1 \subset \dots \subset D_n = A$, and $D_k \setminus D_{k-1} \in \mathcal{J}$, $\forall k \in \{1, \dots, n\}$. Then the equality

$$A \setminus B = \bigcup_{k=1}^n (D_k \setminus D_{k-1})$$

shows that $A \setminus B$ indeed belongs to $\mathcal{R}$.

Thirdly, we prove the implication:

- (III) $A, B \in \mathcal{R} \Rightarrow A \cap B \in \mathcal{R}$.

Write $A = A_1 \cup \dots \cup A_m$ and $B = B_1 \cup \dots \cup B_n$, with $(A_i)_{i=1}^m, (B_k)_{k=1}^n \subset \mathcal{J}$ pair-wise disjoint systems. If we define the sets $D_{ik} = A_i \cap B_k \in \mathcal{J}$, $(i, k) \in \{1, \dots, m\} \times \{1, \dots, n\}$ then it is obvious that

$$A \cap B = \bigcup_{i=1}^m \bigcup_{k=1}^n D_{ik},$$

and $(D_{ik})_{\substack{1 \leq i \leq m \\ 1 \leq k \leq n}} \subset \mathcal{J}$ is a pair-wise disjoint system, therefore $A \cap B$ indeed belongs to $\mathcal{R}$.

Finally, we show the implication:

- (IV) if $A, B \in \mathcal{R}$ and $A \supset B$, then $A \setminus B \in \mathcal{R}$.

Write $A = A_1 \cup \dots \cup A_m$, with $(A_i)_{i=1}^m \subset \mathcal{J}$ a pair-wise disjoint system. Notice that

$$A \setminus B = \bigcup_{i=1}^m (A_i \setminus B),$$

with $(A_i \setminus B)_{i=1}^m$ a pair-wise disjoint system, so by (I) it suffices to show that $A_i \setminus B \in \mathcal{R}$, $\forall i \in \{1, \dots, m\}$. To prove this, we fix i and we write $B = B_1 \cup \dots \cup B_n$, with $(B_k)_{k=1}^n \subset \mathcal{J}$ a pair-wise disjoint system. Then

$$A_i \setminus B = (A_i \setminus B_1) \cap \dots \cap (A_i \setminus B_n),$$

and the fact that $A_i \setminus B$ belongs to $\mathcal{R}$ follows from (II) and (III).

Having proven (I)-(IV), it we now prove that $\mathcal{R}$ is a ring. By (III), we only need to prove the implication

$$(*) \quad A, B \in \mathcal{R} \Rightarrow A \triangle B \in \mathcal{R}.$$

On the one hand, using (IV), it follows that the sets $A \setminus B = A \setminus (A \cap B)$ and $B \setminus A = B \setminus (A \cap B)$ both belong to $\mathcal{R}$. Since $A \triangle B = (A \setminus B) \cup (B \setminus A)$, and $(A \setminus B) \cap (B \setminus A) = \emptyset$, by (I) it follows that $A \triangle B$ indeed belongs to $\mathcal{R}$. $\square$

THEOREM 4.1 (Semiring-to-ring extension). *Let $\mathcal{J}$ be a semiring on X , and let $\mu : \mathcal{J} \rightarrow [0, \infty]$ be an additive map with $\mu(\emptyset) = 0$.*

- (i) *There exists a unique additive map $\bar{\mu} : \mathbf{R}(\mathcal{J}) \rightarrow [0, \infty]$, such that $\bar{\mu}|_{\mathcal{J}} = \mu$.*
- (ii) *If μ is σ -additive, then so is $\bar{\mu}$.*

PROOF. The key step is contained in the following

Claim: *If $(A_i)_{i=1}^m \subset \mathcal{J}$ and $(B_j)_{j=1}^n \subset \mathcal{J}$ are pair-wise disjoint systems, with*

$$A_1 \cup \dots \cup A_m = B_1 \cup \dots \cup B_n,$$

$$\text{then } \mu(A_1) + \dots + \mu(A_m) = \mu(B_1) + \dots + \mu(B_n).$$

To prove this fact, we define the pair-wise disjoint system $(D_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ by $D_{ij} = A_i \cap B_j$, $\forall (i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$. Since

$$\begin{aligned} \bigcup_{j=1}^n D_{ij} &= A_i, \quad \forall i \in \{1, \dots, m\}, \\ \bigcup_{i=1}^m D_{ij} &= B_j, \quad \forall j \in \{1, \dots, n\}, \end{aligned}$$

using additivity, we have the equalities

$$\begin{aligned} \sum_{j=1}^n \mu(D_{ij}) &= \mu(A_i), \quad \forall i \in \{1, \dots, m\}, \\ \sum_{i=1}^m \mu(D_{ij}) &= \mu(B_j), \quad \forall j \in \{1, \dots, n\}, \end{aligned}$$

and then we get

$$\sum_{i=1}^m \mu(A_i) = \sum_{i=1}^m \left[\sum_{j=1}^n \mu(D_{ij}) \right] = \sum_{j=1}^n \left[\sum_{i=1}^m \mu(D_{ij}) \right] = \sum_{j=1}^n \mu(B_j).$$

To prove (i), for any set $A \in \mathbf{R}(\mathcal{J})$ we choose (use Proposition 4.1) a finite pair-wise disjoint system $(A_i)_{i=1}^n \subset \mathcal{J}$, with $A = A_1 \cup \dots \cup A_n$, and we define

$$(1) \quad \bar{\mu}(A) = \mu(A_1) + \dots + \mu(A_n).$$

By the above Claim, the number $\bar{\mu}(A)$ is independent of the particular choice of the pair-wise disjoint system $(A_i)_{i=1}^n$. Also, it is clear that $\bar{\mu}|_{\mathcal{J}} = \mu$, and $\bar{\mu}$ is additive.

The uniqueness is also clear, because the equality $\bar{\mu}|_{\mathcal{J}} = \mu$ and additivity of $\bar{\mu}$ force (1)

(ii). Assume now that μ is σ -additive, and let us prove that $\bar{\mu}$ is again σ -additive. Start with a pair-wise disjoint sequence $(A_n)_{n=1}^{\infty} \subset \mathbf{R}(\mathcal{J})$, with $\bigcup_{n=1}^{\infty} A_n \in \mathbf{R}(\mathcal{J})$, and let us prove the equality

$$(2) \quad \bar{\mu}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \bar{\mu}(A_n).$$

Since $\bigcup_{n=1}^{\infty} A_n \in \mathbf{R}$, there exists a finite pair-wise disjoint system $(B_i)_{i=1}^p \subset \mathcal{J}$, such that $\bigcup_{n=1}^{\infty} A_n = B_1 \cup \dots \cup B_p$. With this choice we have

$$(3) \quad \bar{\mu}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{i=1}^p \mu(B_i).$$

For each $i \in \{1, \dots, p\}$, we have $B_i = \bigcup_{n=1}^{\infty} (B_i \cap A_n)$. Fix for the moment a pair $(n, i) \in \mathbb{N} \times \{1, \dots, p\}$. Since $B_i \cap A_n \in \mathbf{R}(\mathcal{J})$, it follows that there exist an integer $N_{ni} \geq 1$ and a finite pair-wise disjoint system $(C_k^{ni})_{k=1}^{N_{ni}} \subset \mathcal{J}$, such that $B_i \cap A_n = \bigcup_{k=1}^{N_{ni}} C_k^{ni}$.

Since, for each $i \in \{1, \dots, p\}$, the countable system $(C_k^{ni})_{\substack{n \in \mathbb{N} \\ 1 \leq k \leq N_{ni}}} \subset \mathcal{J}$ is pair-wise disjoint, and we have the equality

$$\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{N_{ni}} C_k^{ni} = \bigcup_{n=1}^{\infty} (B_i \cap A_n) = B_i \in \mathcal{J},$$

by the σ -additivity of μ , we have

$$(4) \quad \mu(B_i) = \sum_{n=1}^{\infty} \sum_{k=1}^{N_{ni}} \mu(C_k^{ni}), \quad \forall i \in \{1, \dots, p\}.$$

Since, for each $n \in \mathbb{N}$, the finite system $(C_k^{ni})_{\substack{1 \leq i \leq p \\ 1 \leq k \leq N_{ni}}} \subset \mathcal{J}$ is pair-wise disjoint, and we have the equality

$$\bigcup_{i=1}^p \bigcup_{k=1}^{N_{ni}} C_k^{ni} = \bigcup_{i=1}^p (B_i \cap A_n) = A_n \in \mathcal{J},$$

by the definition of $\bar{\mu}$, we have

$$\bar{\mu}(A_n) = \sum_{i=1}^p \sum_{k=1}^{N_{ni}} \mu(C_k^{ni}), \quad \forall i \in \{1, \dots, p\}.$$

Combining this with (4) yields

$$\sum_{n=1}^{\infty} \bar{\mu}(A_n) = \sum_{n=1}^{\infty} \sum_{i=1}^p \sum_{k=1}^{N_{ni}} \mu(C_k^{ni}) = \sum_{i=1}^p \mu(B_i),$$

and the equality (2) follows from (3). $\square$

DEFINITION. Let X be a non-empty set, and let $\mathcal{E} \subset \mathcal{P}(X)$ be a collection of sets. We say that a map $\mu : \mathcal{E} \rightarrow [0, \infty]$ is *sub-additive*, if

(ADD⁻) whenever $A \in \mathcal{E}$, and $(A_n)_{k=1}^n$ is a finite sequence in $\mathcal{E}$ with $A \subset \bigcup_{k=1}^n A_k$, it follows that $\mu(A) \leq \sum_{k=1}^n \mu(A_k)$.

Note that we do not require the A_k 's to be pair-wise disjoint. With this terminology, Theorem 4.1 has the following.

COROLLARY 4.1. *Let X be a non-empty set X , and let $\mathcal{J} \subset \mathcal{P}(X)$ be a semiring. Then any additive map $\mu : \mathcal{J} \rightarrow [0, \infty]$ is sub-additive.*

PROOF. Let $\bar{\mu} : \mathbf{R}(\mathcal{J}) \rightarrow [0, \infty]$ be the additive extension of μ to the ring generated by $\mathcal{J}$. It suffices to prove that $\bar{\mu}$ is sub-additive. Start with sets $A, A_1, \dots, A_n \in \mathbf{R}(\mathcal{J})$ such that $A \subset A_1 \cup \dots \cup A_n$. Define the sets $B_1 = A_1$, and

$$B_k = A_k \setminus (A_1 \cup \dots \cup A_{k-1}), \text{ for all } k \in \{1, \dots, n\}, k \geq 2.$$

Since we work in a ring, the sets $B_k, B_k \cap A, B_k \setminus A$, and $A_n \setminus B_n, n \in \mathbb{N}$, all belong to $\mathbf{R}(\mathcal{J})$. Moreover, the sequence $(B_k)_{k=1}^n$ is pair-wise disjoint and it satisfies

- $B_k \subset A_k, \forall k \in \{1, \dots, n\}$,
- $\bigcup_{k=1}^n B_k = \bigcup_{k=1}^n A_k \supset A$,

so by the additivity of $\bar{\mu}$, we get

$$\begin{aligned} \sum_{k=1}^n \bar{\mu}(A_k) &= \sum_{k=1}^n \bar{\mu}((A_k \setminus B_k) \cup B_k) = \sum_{k=1}^n [\bar{\mu}(A_k \setminus B_k) + \bar{\mu}(B_k)] \geq \\ &\geq \sum_{k=1}^n \bar{\mu}(B_k) = \sum_{k=1}^n \bar{\mu}((B_k \setminus A) \cup (B_k \cap A)) = \sum_{k=1}^n [\bar{\mu}(B_k \setminus A) + \bar{\mu}(B_k \cap A)] \geq \\ &\geq \sum_{k=1}^n \bar{\mu}(B_k \cap A) = \bar{\mu}\left(\bigcup_{k=1}^n [B_k \cap A]\right) = \bar{\mu}(A). \end{aligned} \quad \square$$

Exercise 5.* Let X_1, X_2 be non-empty sets, let $\mathcal{J}_k \subset \mathcal{P}(X_k), k = 1, 2$, be semirings, and let $\mu_k : \mathcal{J}_k \rightarrow [0, \infty]$ be additive maps. Consider the semiring (see Exercise 3)

$$\mathcal{J} = \{A_1 \times A_2 : A_1 \in \mathcal{J}_1, A_2 \in \mathcal{J}_2\} \subset \mathcal{P}(X_1 \times X_2).$$

Then the map $\mu : \mathcal{J} \rightarrow [0, \infty]$ defined by

$$\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2)$$

is additive. Here we use the convention $0 \cdot \infty = \infty \cdot 0 = 0$.

HINTS: One wants to show that, whenever $A_1 \times A_2 \in \mathcal{J}$ is written as a union

$$A_1 \times A_2 = \bigcup_{k=1}^n (A_1^k \times A_2^k),$$

with $(A_1^k \times A_2^k)_{k=1}^n \subset \mathcal{J}$ pair-wise disjoint, it follows that

$$\mu_1(A_1) \cdot \mu_2(A_2) = \sum_{k=1}^n \mu_1(A_1^k) \cdot \mu_2(A_2^k).$$

Analyze first the case of “strips,” that is, when $A_1^1 = \dots = A_1^n = A_1$ or $A_1^2 = \dots = A_1^n = A_2$. In the general case, use induction, by picking some k such that $A_1^k \subsetneq A_1$ and splitting $A_1 \times A_2$ into “strips” of the form $B_\ell \times A_2$, where $B_1, \dots, B_m \in \mathcal{J}_1$ are pairwise disjoint, with $B_1 = A_1^k$ and $B_1 \cup \dots \cup B_m = A_1$.

COMMENT. In connection with the above exercise, one can ask the following

Question: With the notations above, is it true that, if both μ_1 and μ_2 are measures, then μ is also a measure?

As we shall see a bit later in the course, that the answer is “yes.”

DEFINITION. Let X be a non-empty set, and let $\mathcal{E} \subset \mathcal{P}(X)$ be a collection of sets. We say that a map $\mu : \mathcal{E} \rightarrow [0, \infty]$ is σ -sub-additive, if

(ADD $_{\sigma}^{-}$) whenever $A \in \mathcal{E}$, and $(A_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{E}$ with $A \subset \bigcup_{n=1}^{\infty} A_n$, it follows that $\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n)$.

Note that we do not require the A_n 's to be pair-wise disjoint.

PROPOSITION 4.2 (characterization of semiring measures). *Let X be a non-empty set, let $\mathcal{J} \subset \mathcal{P}(X)$ be a semiring, and let $\mu : \mathcal{J} \rightarrow [0, \infty]$ be a map with $\mu(\emptyset) = 0$. The following are equivalent:*

- (i) μ is a measure on $\mathcal{J}$;
- (ii) μ is additive, and σ -sub-additive.

PROOF. (i) $\Rightarrow$ (ii). Assume μ is a measure on $\mathcal{J}$. It is clear that μ is additive, so we only need to prove σ -sub-additivity. Use Theorem 4.1 to find a measure $\bar{\mu}$ on the ring $\mathbf{R}(\mathcal{J})$ generated by $\mathcal{J}$, such that

$$\bar{\mu}(A) = \mu(A), \quad \forall A \in \mathcal{J}.$$

Then it suffices to show that $\bar{\mu}$ is σ -sub-additive. Start with a set $A \in \mathbf{R}(\mathcal{J})$, and a sequence $(A_n)_{n=1}^{\infty} \subset \mathbf{R}(\mathcal{J})$, such that $A \subset \bigcup_{n=1}^{\infty} A_n$. Define the sets $B_1 = A_1$, and

$$B_n = A_n \setminus (A_1 \cup \cdots \cup A_{n-1}), \quad \text{for all } n \geq 2.$$

Since we work in a ring, the sets B_n , $B_n \cap A$, $B_n \setminus A$, and $A_n \setminus B_n$, $n \in \mathbb{N}$, all belong to $\mathbf{R}(\mathcal{J})$. Moreover, the sequence $(B_n)_{n=1}^{\infty}$ is pair-wise disjoint and it satisfies

- $B_n \subset A_n$, $\forall n \in \mathbb{N}$,
- $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n \supset A$,

so by σ -additivity of $\bar{\mu}$, we get

$$\begin{aligned} \sum_{n=1}^{\infty} \bar{\mu}(A_n) &= \sum_{n=1}^{\infty} \bar{\mu}((A_n \setminus B_n) \cup B_n) = \sum_{n=1}^{\infty} [\bar{\mu}(A_n \setminus B_n) + \bar{\mu}(B_n)] \geq \\ &\geq \sum_{n=1}^{\infty} \bar{\mu}(B_n) = \sum_{n=1}^{\infty} \bar{\mu}((B_n \setminus A) \cup (B_n \cap A)) = \sum_{n=1}^{\infty} [\bar{\mu}(B_n \setminus A) + \bar{\mu}(B_n \cap A)] \geq \\ &\geq \sum_{n=1}^{\infty} \bar{\mu}(B_n \cap A) = \bar{\mu}\left(\bigcup_{n=1}^{\infty} [B_n \cap A]\right) = \bar{\mu}(A). \end{aligned}$$

(ii) $\Rightarrow$ (i). Assume $\mu : \mathcal{J} \rightarrow [0, \infty]$ is additive and σ -sub-additive, and let us show that μ is σ -additive. We again use Theorem 4.1, to find an additive map $\bar{\mu} : \mathbf{R}(\mathcal{J}) \rightarrow [0, \infty]$, such that $\bar{\mu}|_{\mathcal{J}} = \mu$. Start with a pair-wise disjoint sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{J}$, such that the union $A = \bigcup_{n=1}^{\infty} A_n$ belongs to $\mathcal{J}$. On the one hand, by σ -sub-additivity, we have the inequality

$$(5) \quad \mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

On the other hand, for any integer $N \geq 1$, we have

$$\begin{aligned}\mu(A) &= \bar{\mu}(A) = \bar{\mu}\left(\left[\bigcup_{n=1}^N A_n\right] \cup (A \setminus \left[\bigcup_{n=1}^N A_n\right])\right) \geq \\ &\geq \bar{\mu}\left(\bigcup_{n=1}^N A_n\right) = \sum_{n=1}^N \bar{\mu}(A_n) = \sum_{n=1}^N \mu(A_n),\end{aligned}$$

which then gives

$$\mu(A) \geq \sup_{N \in \mathbb{N}} \sum_{n=1}^N \mu(A_n) = \sum_{n=1}^{\infty} \mu(A_n),$$

so using (5) we immediately get $\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$. $\square$

The following technical result will be often employed in subsequent sections.

LEMMA 4.1 (Continuity). *Let $\mathcal{J}$ be a semiring, and let μ be a measure on $\mathcal{J}$.*

- (i) *If $(A_n)_{n=1}^{\infty} \subset \mathcal{J}$ is a sequence of sets, with $A_1 \subset A_2 \subset \dots$, and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{J}$, then*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

- (ii) *If $(B_n)_{n=1}^{\infty} \subset \mathcal{J}$ is a sequence of sets, with $B_1 \supset B_2 \supset \dots$, and $\bigcap_{n=1}^{\infty} B_n \in \mathcal{J}$, and $\mu(B_1) < \infty$, then*

$$\mu\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n \rightarrow \infty} \mu(B_n).$$

PROOF. Using Theorem 4.1, we can assume that $\mathcal{J}$ is already a ring. (Otherwise we replace $\mathcal{J}$ by $\mathbf{R}(\mathcal{J})$, and μ by its extension $\bar{\mu}$.)

- (i). Consider the sets $D_1 = A_1$, and $D_k = A_k \setminus A_{k-1}$, $\forall k \geq 2$. It is clear that $(D_k)_{k=1}^{\infty}$ is a pairwise disjoint sequence in $\mathcal{J}$, and we have the equality

$$(6) \quad \bigcup_{k=1}^n D_k = A_n, \quad \forall n \geq 1.$$

This gives of course the equality

$$\bigcup_{k=1}^{\infty} D_k = \bigcup_{n=1}^{\infty} A_n \in \mathcal{J}.$$

Using this equality, combined with the (σ) -additivity of μ , and with (6), we get

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{k=1}^{\infty} \mu(D_k) = \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \mu(D_k)\right] = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{k=1}^n D_k\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

- (ii). Consider the sets $B = \bigcap_{n=1}^{\infty} B_n$, and $A_n = B_1 \setminus B_n$, $\forall n \geq 1$. It is clear that $(A_n)_{n=1}^{\infty} \subset \mathcal{J}$, and we have $A_1 \subset A_2 \subset \dots$. Moreover, we have $\bigcup_{n=1}^{\infty} A_n = B_1 \setminus B$, so by part (i), we get

$$(7) \quad \mu(B_1 \setminus B) = \lim_{n \rightarrow \infty} \mu(B_1 \setminus B_n).$$

Using the fact that $\mu(B_1) < \infty$, it follows that

$$\mu(B) \leq \mu(B_n) \leq \mu(B_1) < \infty, \quad \forall n \geq 1.$$

This gives then the equalities

$$\mu(B_1 \setminus B) = \mu(B_1) - \mu(B) \text{ and } \mu(B_1 \setminus B_n) = \mu(B_1) - \mu(B_n), \quad \forall n \geq 1,$$

so the equality (7) immediately gives $\mu(B) = \lim_{n \rightarrow \infty} \mu(B_n)$. $\square$

The above result has a (minor) generalization, which we record for future use. To formulate it we introduce the following.

NOTATION. Let $\mathcal{R}$ be a ring, and let μ be a measure on $\mathcal{R}$. For two sets $A, B \in \mathcal{R}$, we write $A \subset_\mu B$, if $\mu(A \setminus B) = 0$.

Using this notation, we have the following generalization of Lemma 4.1.

PROPOSITION 4.3. *Let $\mathcal{R}$ be a ring, and let μ be a measure on $\mathcal{R}$.*

- (i) *If $(A_n)_{n=1}^\infty \subset \mathcal{R}$ is a sequence of sets, with $A_1 \subset_\mu A_2 \subset_\mu \dots$, and $\bigcup_{n=1}^\infty A_n \in \mathcal{R}$, then*

$$\mu\left(\bigcup_{n=1}^\infty A_n\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

- (ii) *If $(B_n)_{n=1}^\infty \subset \mathcal{R}$ is a sequence of sets, with $B_1 \supset_\mu B_2 \supset_\mu \dots$, and $\bigcap_{n=1}^\infty B_n \in \mathcal{R}$, and $\mu(B_1) < \infty$, then*

$$\mu\left(\bigcap_{n=1}^\infty B_n\right) = \lim_{n \rightarrow \infty} \mu(B_n).$$

PROOF. (i). Define the sequence of sets $(E_n)_{n=1}^\infty \subset \mathcal{R}$, by $E_n = \bigcup_{k=1}^n A_k$, $\forall n \geq 1$. Notice that, $A_1 = E_1$, and for each $n \geq 2$, we have $A_n \subset E_n$, as well as the equality

$$E_n \setminus A_n = \bigcup_{k=1}^{n-1} [A_n \setminus A_k].$$

Using sub-additivity, it follows that

$$\mu(E_n \setminus A_n) \leq \sum_{k=1}^{n-1} \mu(A_n \setminus A_k),$$

which forces $\mu(E_n \setminus A_n) = 0$. This gives

$$(8) \quad \mu(E_n) = \mu(A_n) + \mu(E_n \setminus A_n) = \mu(A_n), \quad \forall n \geq 1.$$

Since $\bigcup_{n=1}^\infty E_n = \bigcup_{n=1}^\infty A_n$, and we have the inclusions $E_1 \subset E_2 \subset \dots$, by Lemma 4.1, combined with (8), we get

$$\mu\left(\bigcup_{n=1}^\infty A_n\right) = \mu\left(\bigcup_{n=1}^\infty E_n\right) = \lim_{n \rightarrow \infty} \mu(E_n) = \lim_{n \rightarrow \infty} \mu(A_n).$$

Part (ii) is proven exactly as part (i) from Lemma 4.1. $\square$

Exercise 6. Let μ be a measure on a ring $\mathcal{R}$. Prove that, for $A, B \in \mathcal{R}$, one has the implication

$$A \subset_\mu B \Rightarrow \mu(A) \leq \mu(B).$$

EXAMPLE 4.3. Fix some integer $n \geq 1$. Consider the semiring of “half-open boxes” in $\mathbb{R}^n$

$$\mathcal{J}_n = \{\emptyset\} \cup \left\{ \prod_{j=1}^n [a_j, b_j) : a_1 < b_1, \dots, a_n < b_n \right\} \subset \mathcal{P}(\mathbb{R}^n).$$

For a non-empty box $A = [a_1, b_1) \times \dots \times [a_n, b_n) \in \mathcal{J}_n$, we define

$$\text{vol}_n(A) = \prod_{k=1}^n (b_k - a_k).$$

We also define $\text{vol}_n(\emptyset) = 0$.

THEOREM 4.2. *With the above notations, the map $\text{vol}_n : \mathcal{J} \rightarrow [0, \infty]$ is a measure on $\mathcal{J}_n$.*

PROOF. First we prove additivity. Using Exercise ?? (and induction on n) it suffices to analyze only the case $n = 1$, i.e. the case of half-open intervals in $\mathbb{R}$. We need to show the implication

$$(9) \quad \left. \begin{array}{l} [a, b) = \bigcup_{k=1}^p [a_k, b_k) \\ \{[a_k, b_k)\}_{k=1}^p \text{ pair-wise disjoint} \end{array} \right\} \implies b - a = \sum_{k=1}^p (b_k - a_k).$$

We can prove this using induction on p . The case $p = 1$ is trivial. Assuming that the above fact holds for $p = N$, let us prove it for $p = N + 1$. Pick $k_1 \in \{1, \dots, N + 1\}$ such that $a_{k_1} = a$. Then we clearly have

$$\bigcup_{\substack{1 \leq k \leq N+1 \\ k \neq k_1}} [a_k, b_k) = [b_{k_1}, b),$$

so by the inductive hypothesis we get

$$b - b_{k_1} = \sum_{\substack{1 \leq k \leq N+1 \\ k \neq k_1}} (b_k - a_k),$$

so we get

$$\sum_{k=1}^{N+1} (b_k - a_k) = (b_{k_1} - a_{k_1}) + (b - b_{k_1}) = b - a_{k_1} = b - a,$$

and we are done.

We now prove that vol_n is σ -sub-additive. Suppose we have $A \in \mathcal{J}_n$ and a sequence $(A_k)_{k=1}^\infty \subset \mathcal{J}_n$, such that $A \subset \bigcup_{k=1}^\infty A_k$, and let us prove the inequality

$$(10) \quad \text{vol}_n(A) \leq \sum_{k=1}^\infty \text{vol}_n(A_k).$$

It will be helpful to introduce the following notations. For every half-open box

$$B = [x_1, y_1) \times \dots \times [x_n, y_n),$$

and every $\delta > 0$, we define the boxes

$$B^\delta = [x_1 - \delta, y_1) \times \dots \times [x_n - \delta, y_n) \text{ and } B_\delta = [x_1, y_1 - \delta) \times \dots \times [x_n, y_n - \delta).$$

It is clear that, for any box $B \in \mathcal{J}_n$ we have

$$(11) \quad \overline{B_\delta} \subset B \subset \text{Int}(B^\delta),$$

$$(12) \quad \text{vol}_n(B) = \lim_{\delta \rightarrow 0^+} \text{vol}_n(B^\delta) = \lim_{\delta \rightarrow 0^+} \text{vol}_n(B_\delta).$$

To prove (10), we fix some $\varepsilon > 0$, and we choose positive numbers δ and $(\delta_k)_{k=1}^\infty$, such that

$$(13) \quad \text{vol}_n(A_\delta) > \text{vol}_n(A) - \varepsilon, \text{ and } \text{vol}_n((A_k)^{\delta_k}) < \frac{\varepsilon}{2^k} + \text{vol}_n(A_k), \quad \forall k \in \mathbb{N}.$$

Notice now that, using (11), we have the inclusions

$$\overline{A_\delta} \subset A \subset \bigcup_{k=1}^\infty A_k \subset \text{Int}((A_k)^{\delta_k}),$$

and using the compactness of $\overline{A_\delta}$, there exists some $N \geq 1$, such that

$$\overline{A_\delta} \subset \bigcup_{k=1}^N \text{Int}((A_k)^{\delta_k}).$$

This immediately gives the inclusion

$$A_\delta \subset \bigcup_{k=1}^N (A_k)^{\delta_k}.$$

Using sub-additivity (see Corollary 4.1) we now get

$$\text{vol}_n(A_\delta) \leq \sum_{k=1}^N \text{vol}_n((A_k)^{\delta_k}),$$

and using (13) we have

$$\text{vol}_n(A) - \varepsilon \leq \sum_{k=1}^N \left[\frac{\varepsilon}{2^k} + \text{vol}_n(A_k) \right] \leq \varepsilon + \sum_{k=1}^N \text{vol}_n(A_k) \leq \varepsilon + \sum_{k=1}^\infty \text{vol}_n(A_k).$$

This gives

$$\text{vol}_n(A) - 2\varepsilon \leq \sum_{k=1}^\infty \text{vol}_n(A_k).$$

But since this inequality holds for all $\varepsilon > 0$, the inequality (10) immediately follows. $\square$

LECTURE 22

5. Outer measures

Although measures can be defined on arbitrary collections of sets, the most natural domain of a measure is a σ -ring. In the previous section we dealt however only with (semi)rings. Therefore it is natural to ask the following

Question 1: Given a measure μ on a (semi)ring $\mathcal{J}$, is it possible to extend it to a measure on the σ -ring $\mathbf{S}(\mathcal{J})$ generated by $\mathcal{J}$?

As a particular case of the above question, we can specifically ask if there exists a measure on $\text{Bor}(\mathbb{R}^n)$, which agrees with vol_n on “half-open boxes.”

As a consequence of a remarkably clever construction, due to Caratheodory, we will be able to answer the above general question in the affirmative. Caratheodory’s approach is based on the following concept.

DEFINITION. Given a non-empty set X , an *outer measure on X* is simply a map $\nu : \mathcal{P}(X) \rightarrow [0, \infty]$ with the following properties.

- (0) $\nu(\emptyset) = 0$.
- (M) If $A, B \in \mathcal{P}(X)$ are such that $A \subset B$, then $\nu(A) \leq \nu(B)$.
- (ADD $_{\sigma}^{-}$) ν is σ -sub-additive, i.e. whenever $A \in \mathcal{P}(X)$, and $(A_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{P}(X)$ with $A \subset \bigcup_{n=1}^{\infty} A_n$, it follows that $\nu(A) \leq \sum_{n=1}^{\infty} \nu(A_n)$.

The property (M) is called *monotonicity*.

Remark that ν is automatically sub-additive, in the sense that, whenever $A, A_1, \dots, A_n \in \mathcal{P}(X)$ are such that $A \subset A_1 \cup \dots \cup A_n$, it follows that $\nu(A) \leq \nu(A_1) + \dots + \nu(A_n)$.

The following result explains how a measure on a semiring can be naturally extended to an outer measure on the ambient space.

PROPOSITION 5.1. *Let X be a non-empty set, let $\mathcal{J}$ be a semiring on X , and let $\mu : \mathcal{J} \rightarrow [0, \infty]$ be a measure on $\mathcal{J}$. Consider the collection*

$$\mathcal{P}_{\sigma}^{\mathcal{J}}(X) = \left\{ A \subset X : \text{there exists } (B_n)_{n=1}^{\infty} \subset \mathcal{J}, \text{ with } A \subset \bigcup_{n=1}^{\infty} B_n \right\}.$$

Define the map $\bar{\mu} : \mathcal{P}_{\sigma}^{\mathcal{J}}(X) \rightarrow [0, \infty]$ by

$$\bar{\mu}(A) = \inf \left\{ \sum_{n=1}^{\infty} \mu(B_n) : (B_n)_{n=1}^{\infty} \subset \mathcal{J}, A \subset \bigcup_{n=1}^{\infty} B_n \right\}, \quad \forall A \in \mathcal{P}_{\sigma}^{\mathcal{J}}(X).$$

Then the map $\mu^ : \mathcal{P}(X) \rightarrow [0, \infty]$, defined by*

$$\mu^*(A) = \begin{cases} \bar{\mu}(A) & \text{if } A \in \mathcal{P}_{\sigma}^{\mathcal{J}}(X) \\ \infty & \text{if } A \notin \mathcal{P}_{\sigma}^{\mathcal{J}}(X) \end{cases}$$

is an outer measure on X , and $\mu^|_{\mathcal{J}} = \mu$.*

PROOF. It is obvious that $\mu^*(\emptyset) = 0$. It is also clear that μ^* is monotone. To prove that μ^* is σ -sub-additive, start with $A \in \mathcal{P}(X)$ and a sequence $(A_n)_{n=1}^\infty \in \mathcal{P}(X)$, such that $A \subset \bigcup_{n=1}^\infty A_n$, and let us prove the inequality $\mu^*(A) \leq \sum_{n=1}^\infty \mu^*(A_n)$. If there exists some n with $A_n \notin \mathcal{P}_\sigma^\mathcal{J}(X)$, there is nothing to prove. Assume $A_n \in \mathcal{P}_\sigma^\mathcal{J}(X)$, for all n . Then it is clear that $A \in \mathcal{P}_\sigma^\mathcal{J}(X)$. Fix for the moment some $\varepsilon > 0$. For every $n \in \mathbb{N}$ choose a sequence $(B_k^n)_{k=1}^\infty \subset \mathcal{J}$, such that

$$\sum_{k=1}^\infty \mu(B_k^n) < \frac{\varepsilon}{2^n} + \bar{\mu}(A_n).$$

It is clear that, if we list the countable family $(B_k^n)_{n,k=1}^\infty$ as a sequence $(D_m)_{m=1}^\infty$, then $A \subset \bigcup_{m=1}^\infty D_m$, and

$$\bar{\mu}(A) \leq \sum_{m=1}^\infty \mu(D_m) = \sum_{n=1}^\infty \sum_{k=1}^\infty \mu(B_k^n) \leq \sum_{n=1}^\infty \left[\frac{\varepsilon}{2^n} + \bar{\mu}(A_n) \right] = \varepsilon + \sum_{n=1}^\infty \bar{\mu}(A_n).$$

Since the above inequality holds for all $\varepsilon > 0$, we conclude that

$$\mu^*(A) = \bar{\mu}(A) \leq \sum_{n=1}^\infty \bar{\mu}(A_n) = \sum_{n=1}^\infty \mu^*(A_n),$$

so μ^* is indeed σ -sub-additive.

Finally, we must show that $\mu^*|_{\mathcal{J}} = \mu$. Start with some $A \in \mathcal{J}$. On the one hand, since μ is a measure on $\mathcal{J}$, we know that μ is σ -subadditive (see Theorem 4.2). This means that, for any sequence $(B_n)_{n=1}^\infty \subset \mathcal{J}$ with $A \subset \bigcup_{n=1}^\infty B_n$, we have $\sum_{n=1}^\infty \mu(B_n) \geq \mu(A)$. Since A obviously belongs to $\mathcal{P}_\sigma^\mathcal{J}(X)$, this will force

$$\mu^*(A) = \bar{\mu}(A) \geq \mu(A).$$

On the other hand, if we consider the sequence $B_1 = A, B_2 = B_3 = \dots = \emptyset$, then we clearly have $\sum_{n=1}^\infty \mu(B_n) = \mu(A)$, which gives $\bar{\mu}(A) \leq \mu(A)$, so in fact we must have equality $\bar{\mu}(A) = \mu(A)$. $\square$

DEFINITION. The outer measure μ^* , defined in the above result, is called the *maximal outer extension* of μ . This terminology is justified by the following.

Exercise 1. Let $\mathcal{J}$ be a semiring on X , and let μ be a measure on $\mathcal{J}$. Prove that any outer measure ν on X , with $\nu|_{\mathcal{J}} = \mu$, then $\nu \leq \mu^*$, in the sense that

$$\nu(A) \leq \mu^*(A), \quad \forall A \subset X.$$

Exercise 2. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be semirings on X with $\mathcal{J}_1 \subset \mathcal{J}_2$, and let μ_1, μ_2 be respectively measures on $\mathcal{J}_1, \mathcal{J}_2$, such that $\mu_2|_{\mathcal{J}_1} \leq \mu_1$. Let μ_1^*, μ_2^* respectively be the maximal outer extensions of μ_1, μ_2 . Prove the inequality $\mu_2^* \leq \mu_1^*$.

Given a measure μ on a semiring $\mathcal{J}$ on X , one can ask whether there exists a unique outer measure on X , which extends μ . The answer is no, even in the most trivial cases.

EXAMPLE 5.1. Work on the set $X = \{1, 2\}$. Take the semiring $\mathcal{J} = \{\emptyset, X\}$ and define a measure μ on $\mathcal{J}$ by $\mu(\emptyset) = 0$ and $\mu(X) = 1$. Choose now any number $a \in (0, 1)$ and define $\nu_a : \mathcal{P}(X) \rightarrow [0, 1]$ by $\nu_a(A) = a\chi_A(1) + (1-a)\chi_A(2)$. Then ν_a is an outer measure on X - in fact ν_a is a measure on $\mathcal{P}(X)$ - and $\nu_a|_{\mathcal{J}} = \mu$. It is obvious that $\mu^*(\{1\}) = 1 \neq a = \nu_a(\{1\})$ and $\mu^*(\{2\}) = 1 \neq 1-a = \nu_a(\{2\})$.

We introduce now another concept, which is very important in our analysis.

DEFINITION. Let ν be an outer measure on a non-empty set X . A subset $A \subset X$ is said to be ν -measurable, if it satisfies the condition

$$(M) \quad \nu(S) = \nu(S \cap A) + \nu(S \setminus A), \quad \forall S \subset X.$$

For a given S , it is useful to think the equality $\nu(S) = \nu(S \cap A) + \nu(S \setminus A)$ in unorthodox terms as “ A sharply cuts S ,” so that saying that A is ν -measurable means that “ A sharply cut every set $S \subset X$.”

REMARKS 5.1. Let ν be an outer measure on X .

A. Since ν is (finitely) sub-additive, for any two sets $A, S \subset X$, one always has the inequality $\nu(S) \leq \nu(S \cap A) + \nu(S \setminus A)$. Therefore, a set $A \subset X$ is ν -measurable, if and only if

$$\nu(S) \geq \nu(S \cap A) + \nu(S \setminus A), \quad \forall S \subset X.$$

B. Any subset $N \subset X$, with $\nu(N) = 0$, is ν -measurable. Indeed, from the monotonicity of ν , we see that for every $S \subset X$, we have

$$\nu(S \cap N) + \nu(S \setminus N) \leq \nu(N) + \nu(S) = \nu(S),$$

so by the preceding remark, N is indeed ν -measurable. Such a set N is called ν -negligeable.

The first key result in this section is the following.

THEOREM 5.1. Let ν be an outer measure on a non-empty set X . Then the collection

$$\mathcal{M}_\nu(X) = \{A \subset X : A \text{ } \nu\text{-measurable}\}$$

is a σ -algebra on X . Moreover, the restriction

$$\nu|_{\mathcal{M}_\nu(X)} : \mathcal{M}_\nu(X) \rightarrow [0, \infty]$$

is a measure on $\mathcal{M}_\nu(X)$.

PROOF. The proof will be carried on in several steps.

Step 1: If $A \in \mathcal{M}_\nu(X)$, then $X \setminus A \in \mathcal{M}_\nu(X)$.

This is trivial, since for every $S \subset X$, one has the equalities

$$S \cap (X \setminus A) = S \setminus A \text{ and } S \setminus (X \setminus A) = S \cap A.$$

Step 2: If $A, B \in \mathcal{M}_\nu(X)$, then $A \cap B \in \mathcal{M}_\nu(X)$.

Start with some arbitrary $S \subset X$. Since B is ν -measurable, it “sharply cuts the set $S \setminus (A \cap B)$,” which means that

$$\nu(S \setminus (A \cap B)) = \nu([S \setminus (A \cap B)] \cap B) + \nu([S \setminus (A \cap B)] \setminus B).$$

Since we clearly have $[S \setminus (A \cap B)] \cap B = (S \cap B) \setminus A$, and $[S \setminus (A \cap B)] \setminus B = S \setminus B$, the above equality gives

$$\nu(S \setminus (A \cap B)) = \nu((S \cap B) \setminus A) + \nu(S \setminus B).$$

Adding $\nu((S \cap B) \cap A)$, and using the fact that A “sharply cuts $S \cap B$,” we now get

$$\begin{aligned} & \nu((S \cap (A \cap B))) + \nu(S \setminus (A \cap B)) = \\ & = \nu((S \cap B) \cap A) + \nu((S \cap B) \setminus A) + \nu(S \setminus B) = \nu(S \cap B) + \nu(S \setminus B). \end{aligned}$$

Finally, using the fact that B “sharply cuts S ,” we get

$$\nu((S \cap (A \cap B))) + \nu(S \setminus (A \cap B)) = \nu(S \cap B) + \nu(S \setminus B) = \nu(S),$$

so $A \cap B$ is indeed ν -measurable.

So far, Steps 1 and 2 prove that $\mathcal{M}_\nu(X)$ is an algebra on X .

Step 3: For any pair-wise disjoint finite sequence $(A_n)_{n=1}^N \subset \mathcal{M}_\nu(X)$, one has the equality

$$\nu(S \cap [A_1 \cup \dots \cup A_N]) = \sum_{n=1}^N \nu(S \cap A_n), \quad \forall S \subset X.$$

Since $\mathcal{M}_\nu(X)$ is an algebra, it suffices to prove the above equality only for $N = 2$. (The case of arbitrary N follows immediately by induction.) To prove that $\nu(S \cap (A_1 \cup A_2)) = \nu(S \cap A_1) + \nu(S \cap A_2)$, we simply use the fact that A_1 “sharply cuts $S \cap (A_1 \cup A_2)$,” which gives

$$\nu(S \cap (A_1 \cup A_2)) = \nu([S \cap (A_1 \cup A_2)] \cap A_1) + \nu([S \cap (A_1 \cup A_2)] \setminus A_1).$$

The desired equality then immediately follows from the obvious equalities

$$[S \cap (A_1 \cup A_2)] \cap A_1 = S \cap A_1 \text{ and } [S \cap (A_1 \cup A_2)] \setminus A_1 = S \cap A_2.$$

The preceding step can be in fact extended to infinite sequences.

Step 4: For any pair-wise disjoint sequence $(A_n)_{n=1}^\infty \subset \mathcal{M}_\nu(X)$, one has the equality

$$\nu\left(S \cap \left[\bigcup_{n=1}^\infty A_n\right]\right) = \sum_{n=1}^\infty \nu(S \cap A_n), \quad \forall S \subset X.$$

To prove this fact, we fix a sequence $(A_n)_{n=1}^\infty$ as above, as well as $S \subset X$. By σ -sub-additivity, we already know that

$$\nu\left(S \cap \left[\bigcup_{n=1}^\infty A_n\right]\right) = \nu\left(\bigcup_{n=1}^\infty [S \cap A_n]\right) \leq \sum_{n=1}^\infty \nu(S \cap A_n),$$

so the only thing we have to show is the inequality

$$\sum_{n=1}^N \nu(S \cap A_n) \leq \nu\left(S \cap \left[\bigcup_{n=1}^\infty A_n\right]\right), \quad \forall N \in \mathbb{N}.$$

This follows immediately from Step 3 and the monotonicity:

$$\sum_{n=1}^N \nu(S \cap A_n) = \nu\left(S \cap \left[\bigcup_{n=1}^N A_n\right]\right) \leq \nu\left(S \cap \left[\bigcup_{n=1}^\infty A_n\right]\right).$$

Step 5: $\mathcal{M}_\nu(X)$ is a monotone class.

We need to prove the properties:

- (i) whenever $(A_n)_{n=1}^\infty \subset \mathcal{M}_\nu(X)$ is a sequence with $A_n \subset A_{n+1}$, $\forall n \in \mathbb{N}$, it follows that $\bigcup_{n=1}^\infty A_n$ belongs to $\mathcal{M}_\nu(X)$;
- (ii) whenever $(A_n)_{n=1}^\infty \subset \mathcal{M}_\nu(X)$ is a sequence with $A_n \supset A_{n+1}$, $\forall n \in \mathbb{N}$, it follows that $\bigcap_{n=1}^\infty A_n$ belongs to $\mathcal{M}_\nu(X)$.

Since $\mathcal{M}_\nu(X)$ is an algebra, it suffices only to prove (i). Start with an arbitrary subset S , and a sequence $(A_n)_{n=1}^\infty \subset \mathcal{M}_\nu(X)$ with $A_n \subset A_{n+1}$, $\forall n \in \mathbb{N}$, and denote the union $\bigcup_{n=1}^\infty A_n$ simply by A . Define the sets $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$, $\forall n \geq 2$. It is obvious that $(B_n)_{n=1}^\infty$ is a pair-wise disjoint sequence. Since $\mathcal{M}_\nu(X)$

is an algebra, all the B_n 's belong to $\mathcal{M}_\nu(X)$. We have, $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n = A$, which, using Step 4 gives

$$(1) \quad \nu(S \cap A) = \nu\left(S \cap \left[\bigcup_{n=1}^{\infty} A_n\right]\right) = \nu\left(S \cap \left[\bigcup_{n=1}^{\infty} B_n\right]\right) = \sum_{n=1}^{\infty} \nu(S \cap B_n).$$

Using Step 3, combined with the equality $\bigcup_{n=1}^N B_n = A_N$, we also have

$$\sum_{n=1}^N \nu(S \cap B_n) = \nu\left(S \cap \left[\bigcup_{n=1}^N B_n\right]\right) = \nu(S \cap A_N), \quad \forall N \in \mathbb{N},$$

so by (1) we have

$$(2) \quad \nu(S \cap A) = \sum_{n=1}^{\infty} \nu(S \cap B_n) = \lim_{N \rightarrow \infty} \nu(S \cap A_N).$$

Notice now that, using the fact that A_N "sharply cuts S ," combined with the monotonicity of ν and the obvious inclusion $S \setminus A \subset S \setminus A_N$, we have

$$\nu(S \cap A_N) + \nu(S \setminus A) \leq \nu(S \cap A_N) + \nu(S \setminus A_N) = \nu(S), \quad \forall N \in \mathbb{N},$$

so using (2), we immediately get

$$\nu(S \cap A) + \nu(S \setminus A) \leq \nu(S).$$

Since the above inequality holds for all $S \subset X$, by Remark 5.1.A it follows that A indeed belongs to $\mathcal{M}_\nu(X)$.

By the results from Section 1, we know that the fact that $\mathcal{M}_\nu(X)$ is simultaneously an algebra, and a monotone class, implies the fact that $\mathcal{M}_\nu(X)$ is a σ -algebra.

We now show that $\nu|_{\mathcal{M}_\nu(X)}$ is a measure. If we start with a pair-wise disjoint sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{M}_\nu(X)$, then the equality

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n)$$

is an immediate consequence of Step 4, applied to the set $S = \bigcup_{n=1}^{\infty} A_n$, which clearly satisfies $S \cap A_n = A_n$, $\forall n \in \mathbb{N}$. $\square$

We are now in position to answer the Question 1.

THEOREM 5.2. *Let X be a non-empty set, let $\mathcal{J}$ be a semiring on X , let μ be a measure on $\mathcal{J}$, and let μ^* be the maximal outer extension of μ . Then $\mathcal{J} \subset \mathcal{M}_{\mu^*}(X)$. In particular, $\mathcal{M}_{\mu^*}(X)$ contains the σ -algebra $\Sigma(\mathcal{J})$ on X , generated by $\mathcal{J}$, and $\mu^*|_{\Sigma(\mathcal{J})}$ is a measure on $\Sigma(\mathcal{J})$.*

PROOF. What we need to prove is the fact that every set $A \in \mathcal{J}$ is μ^* -measurable. Start with an arbitrary set $S \subset X$. As noticed before (Remark 5.1.A), we only need to prove the inequality

$$(3) \quad \mu^*(S \cap A) + \mu^*(S \setminus A) \leq \mu^*(S).$$

If $\mu^*(S) = \infty$, there is nothing to prove, so we can assume that $\mu^*(S) < \infty$. In particular this means that $S \in \mathcal{P}_\sigma^{\mathcal{J}}(X)$. Fix for the moment $\varepsilon > 0$. By the definition

of $\mu^*(S) = \bar{\mu}(S)$, there exists a sequence $(B_n)_{n=1}^\infty \subset \mathcal{J}$, such that $S \subset \bigcup_{n=1}^\infty B_n$, and

$$(4) \quad \sum_{n=1}^\infty \mu(B_n) \leq \mu^*(S) + \varepsilon.$$

Since $\mathcal{J}$ is a semiring, for each $n \in \mathbb{N}$, we can find some integer $p_n \geq 1$, and a sequence $(D_j^n)_{j=0}^{p_n} \subset \mathcal{J}$, such that

- $B_n \cap A = D_0^n \subset D_1^n \subset \cdots \subset D_{p_n}^n = B_n$,
- $D_j^n \setminus D_{j-1}^n \in \mathcal{J}, \forall j \in \{1, \dots, p_n\}$.

Define the numbers $k_0 = 0$, and $k_n = \sum_{j=1}^n p_j, \forall n \in \mathbb{N}$, and the sequence $(C_m)_{m=1}^\infty \subset \mathcal{J}$, by

$$C_m = D_{m-k_{n-1}}^n \setminus D_{m-1-k_{n-1}}^n, \text{ if } k_{n-1} < m \leq k_n, \quad n \in \mathbb{N}.$$

By construction, for each $n \in \mathbb{N}$, we have

$$\bigcup_{m=k_{n-1}+1}^{k_n} C_m = \bigcup_{j=1}^{p_n} (D_j^n \setminus D_{j-1}^n) = B_n \setminus A_n.$$

Moreover, for each $n \in \mathbb{N}$ the system

$$(D_0^n, C_{k_{n-1}+1}, C_{k_{n-1}+2}, \dots, C_{k_n}) = (D_0^n, D_1^n \setminus D_0^n, D_2^n \setminus D_1^n, \dots, D_{p_n}^n \setminus D_{p_n-1}^n)$$

in $\mathcal{J}$ is pair-wise disjoint, and has

$$D_0^n \cup \bigcup_{m=k_{n-1}+1}^{k_n} C_m = B_n,$$

so we get the equality

$$\mu(D_0^n) + \sum_{m=k_{n-1}+1}^{k_n} \mu(C_m) = \mu(B_n).$$

Using (4) we now get

$$(5) \quad \begin{aligned} \sum_{n=1}^\infty \mu(D_0^n) + \sum_{m=1}^\infty \mu(C_m) &= \sum_{n=1}^\infty \mu(D_0^n) + \sum_{n=1}^\infty \left(\sum_{m=k_{n-1}+1}^{k_n} \mu(C_m) \right) = \\ &= \sum_{n=1}^\infty \left(\mu(D_0^n) + \sum_{m=k_{n-1}+1}^{k_n} \mu(C_m) \right) = \sum_{n=1}^\infty \mu(B_n) \leq \mu^*(S) + \varepsilon. \end{aligned}$$

On the one hand, we clearly have

$$\begin{aligned} \bigcup_{m=1}^\infty C_m &= \bigcup_{n=1}^\infty \left(\bigcup_{m=k_{n-1}+1}^{k_n} C_m \right) = \bigcup_{n=1}^\infty \left(\bigcup_{j=1}^{p_n} (D_j^n \setminus D_{j-1}^n) \right) = \\ &= \bigcup_{n=1}^\infty (D_{p_n}^n \setminus D_0^n) = \bigcup_{n=1}^\infty (B_n \setminus A) = \left(\bigcup_{n=1}^\infty B_n \right) \setminus A \supset S \setminus A, \end{aligned}$$

which gives the inequality

$$(6) \quad \sum_{m=1}^\infty \mu(C_m) \geq \mu^*(S \setminus A).$$

On the other hand, we also have

$$\bigcup_{n=1}^{\infty} D_0^n = \bigcup_{n=1}^{\infty} (B_n \cap A) = \left(\bigcup_{n=1}^{\infty} B_n \right) \cap A \supset S \cap A,$$

which gives the inequality

$$(7) \quad \sum_{n=1}^{\infty} \mu(D_0^n) \geq \mu^*(S \cap A).$$

Combining (6) and (7) with (5) immediately gives the desired inequality (3). $\square$

The construction

$$\left\{ \begin{array}{c} \mu \\ \text{measure on } \mathcal{J} \end{array} \right\} \xrightarrow{\text{maximal outer extension}} \left\{ \begin{array}{c} \mu^* \\ \text{outer measure on } X \end{array} \right\} \xrightarrow{\text{restriction}} \left\{ \begin{array}{c} \mu^*|_{\Sigma(\mathcal{J})} \\ \text{measure on } \Sigma(\mathcal{J}) \end{array} \right\}$$

is referred to as the *Caratheodory construction*.

DEFINITIONS. Let $\mathcal{J}$ be a semiring on X , and let μ be a measure on $\mathcal{J}$. The Caratheodory construction provides us with two measures. The first measure - $\mu^*|_{\mathbf{S}(\mathcal{J})}$ - is a measure on the σ -ring $\mathbf{S}(\mathcal{J})$ generated by $\mathcal{J}$, and is called the *maximal σ -ring extension of μ* . The second measure - $\mu^*|_{\Sigma(\mathcal{J})}$ - is a measure on the σ -algebra $\Sigma(\mathcal{J})$ generated by $\mathcal{J}$, and is called the *maximal σ -algebra extension of μ* .

The above terminology is justified by the following result.

PROPOSITION 5.2. *Let $\mathcal{J}$ be a semiring on X , and let μ be a measure on $\mathcal{J}$.*

- (i) *If ν is a measure on the σ -ring $\mathbf{S}(\mathcal{J})$ generated by $\mathcal{J}$, with $\nu|_{\mathcal{J}} = \mu$, then*

$$\nu \leq \mu^*|_{\mathbf{S}(\mathcal{J})}.$$
- (ii) *If ν is a measure on the σ -algebra $\Sigma(\mathcal{J})$ generated by $\mathcal{J}$, with $\nu|_{\mathcal{J}} = \mu$, then*

$$\nu \leq \mu^*|_{\Sigma(\mathcal{J})}.$$

PROOF. We prove both statements simultaneously. Let $\mathcal{J}_1$ denote either the σ -ring, or the σ -algebra generated by $\mathcal{J}$. In particular $\mathcal{J}_1$ is a semiring, and $\mathcal{J} \subset \mathcal{J}_1$. Since ν is a measure on $\mathcal{J}_1$ with $\nu|_{\mathcal{J}} = \mu$, if we denote by ν^* its maximal outer extension, then by Exercise 2 we know that $\nu^* \leq \mu^*$. In particular, by Proposition 5.1 and Theorem 5.2, we get $\nu = \nu^*|_{\mathcal{J}_1} \leq \mu^*|_{\mathcal{J}_1}$. $\square$

We now discuss the uniqueness of extensions of a semiring measure. In order to clarify this matter, we have to introduce a technical condition, which turns out to be very helpful not only here, but in many other situations.

DEFINITIONS. Let $\mathcal{J}$ be a semiring on X , and let μ be a measure on $\mathcal{J}$.

A. We say that a subset $A \subset X$ is *$\mathcal{J}$ - μ - σ -finite*, if there exists a sequence $(B_n)_{n=1}^{\infty} \subset \mathcal{J}$, such that $A \subset \bigcup_{n=1}^{\infty} B_n$, and $\mu(B_n) < \infty$, $\forall n \in \mathbb{N}$. (When there is no danger of confusion, we will use the terms “ μ - σ -finite,” or simply “ σ -finite.”)

B. We say that the measure μ is *σ -finite*, if every $A \in \mathcal{J}$ is σ -finite.

C. We say that the measure μ is *finite*, if $\mu(A) < \infty$, $\forall A \in \mathcal{J}$.

Clearly every finite measure on $\mathcal{J}$ is σ -finite.

REMARK 5.2. Let $\mathcal{J}$ be a semiring on X , let μ be a measure on $\mathcal{J}$, and let A be a set which belongs to the σ -algebra $\Sigma(\mathcal{J})$ generated by $\mathcal{J}$. If A is $\mathcal{J}$ - μ - σ -finite, then A in fact belongs to the semiring $\mathbf{S}(\mathcal{J})$ generated by $\mathcal{J}$. The only thing that is actually needed here is the existence of a sequence $(B_n)_{n=1}^{\infty} \subset \mathcal{J}$ with $A \subset \bigcup_{n=1}^{\infty} B_n$. This

gives the fact that A belongs to $\mathcal{P}_\sigma^\mathcal{J}(X)$, so by Proposition 2.3, the set A belongs to the intersection $\Sigma(\mathcal{J}) \cap \mathcal{P}_\sigma^\mathcal{J}(X) = \mathbf{S}(\mathcal{J})$.

Using the above terminology, we have the following uniqueness result.

THEOREM 5.3. *Let $\mathcal{J}$ be a semiring on X , let μ be a measure on $\mathcal{J}$, let μ^* be the maximal outer extension of μ , and let ν be a measure on the σ -ring $\mathbf{S}(\mathcal{J})$ generated by $\mathcal{J}$, with $\nu|_{\mathcal{J}} = \mu$. Then one has $\nu(A) = \mu^*(A)$, for all $\mathcal{J}$ - μ - σ -finite sets $A \in \mathbf{S}(\mathcal{J})$.*

PROOF. Fix a $\mathcal{J}$ - μ - σ -finite set $A \in \mathbf{S}(\mathcal{J})$.

Claim: *There exists a pair-wise disjoint sequence $(D_n)_{n=1}^\infty \subset \mathbf{S}(\mathcal{J})$ such that $A \subset \bigcup_{n=1}^\infty D_n$, and $\nu(D_n) = \mu^*(D_n) < \infty$, $\forall n \in \mathbb{N}$.*

To prove the above statement, start with a sequence $(B_n)_{n=1}^\infty \subset \mathcal{J}$ with $A \subset \bigcup_{n=1}^\infty B_n$ and $\mu(B_n) < \infty$, $\forall n \in \mathbb{N}$. Define the sets D_n , $n \in \mathbb{N}$ by $D_1 = B_1$, and $D_n = B_n \setminus (B_1 \cup \dots \cup B_{n-1})$, $\forall n \geq 2$. It is clear that the sequence $(D_n)_{n=1}^\infty$ is pair-wise disjoint, and

$$A \subset \bigcup_{n=1}^\infty B_n = \bigcup_{n=1}^\infty D_n.$$

Moreover, all the D_n 's belong to the ring $\mathbf{R}(\mathcal{J})$ generated by $\mathcal{J}$. The inclusions $D_n \subset B_n$ then prove that

$$\mu^*(D_n) \leq \mu^*(B_n) = \mu(B_n) < \infty, \quad \forall n \in \mathbb{N}.$$

Finally, since both $\mu^*|_{\mathbf{R}(\mathcal{J})}$ and $\nu|_{\mathbf{R}(\mathcal{J})}$ are measures on $\mathbf{R}(\mathcal{J})$, which have the same values on $\mathcal{J}$, using the Semiring-to-Ring Extension Theorem 4.1, it follows that

$$(8) \quad \mu^*|_{\mathbf{R}(\mathcal{J})} = \nu|_{\mathbf{R}(\mathcal{J})}.$$

In particular we have the equalities

$$\nu(D_n) = \mu^*(D_n), \quad \forall n \in \mathbb{N}.$$

Having proven the Claim, we now show that $\nu(A) = \mu^*(A)$. We choose a sequence $(D_n)_{n=1}^\infty \subset \mathbf{S}(\mathcal{J})$ as in the Claim. On the one hand, since the D_n 's are pair-wise disjoint, and both ν and $\mu^*|_{\mathbf{S}(\mathcal{J})}$ are measures on the σ -ring $\mathbf{S}(\mathcal{J})$, one has the equalities

$$\nu(A) = \sum_{n=1}^\infty \nu(A \cap D_n) \text{ and } \mu^*(A) = \sum_{n=1}^\infty \mu^*(A \cap D_n).$$

So, in order to prove the equality $\nu(A) = \mu^*(A)$, it suffices to prove that

$$(9) \quad \nu(A \cap D_n) = \mu^*(A \cap D_n), \quad \forall n \in \mathbb{N}.$$

Fix $n \in \mathbb{N}$. On the one hand, by Proposition 5.2(i), we have the inequalities

$$(10) \quad \nu(A \cap D_n) \leq \mu^*(A \cap D_n) < \infty \text{ and } \nu(D_n \setminus A) \leq \mu^*(D_n \setminus A) < \infty.$$

On the other hand, we have

$$\nu(A \cap D_n) + \nu(D_n \setminus A) = \nu(D_n) = \mu^*(D_n) = \mu^*(A \cap D_n) + \mu^*(D_n \setminus A).$$

Now if we go back to (10), we see that none of the two inequalities can be strict, because in that case we would get $\nu(D_n) < \mu^*(D_n)$. (The assumption that $\mu^*(D_n) < \infty$ is essential here.) So we must have (9), and we are done. $\square$

COROLLARY 5.1. *If μ is a σ -finite measure on a semiring $\mathcal{J}$, then there exists a unique measure ν on the σ -ring $\mathbf{S}(\mathcal{J})$ generated by $\mathcal{J}$, such that $\nu|_{\mathcal{J}} = \mu$. Moreover, ν is σ -finite.*

PROOF. The existence is given by the Caratheodory construction. The uniqueness follows from Theorem 5.3.

To prove σ -finiteness, start with some $A \in \mathbf{S}(\mathcal{J})$, and let us find a sequence $(B_n)_{n=1}^{\infty} \subset \mathbf{S}(\mathcal{J})$ with $A \subset \bigcup_{n=1}^{\infty} B_n$ and $\nu(B_n) < \infty$, $\forall n \in \mathbb{N}$. First of all, since $\mathcal{P}_{\sigma}^{\mathcal{J}}(X)$ is a σ -ring which contains $\mathcal{J}$, it follows that $\mathbf{S}(\mathcal{J}) \subset \mathcal{P}_{\sigma}^{\mathcal{J}}(X)$. In particular, there exists $(D_n)_{n=1}^{\infty} \subset \mathcal{J}$ such that $A \subset \bigcup_{n=1}^{\infty} D_n$. Using the fact that μ is σ -finite, we see that for each n we can find a sequence $(D_k^n)_{k=1}^{\infty} \subset \mathcal{J}$, with $D_n \subset \bigcup_{k=1}^{\infty} D_k^n$ and $\mu(D_k^n) < \infty$, $\forall k \in \mathbb{N}$. If we list all the sets D_k^n , $k, n \in \mathbb{N}$ as a sequence $(B_m)_{m=1}^{\infty}$, then we are done. $\square$

In the absence of the σ -finiteness condition the uniqueness of the σ -ring extension fails, as illustrated by the following.

EXAMPLE 5.2. Consider the set $X = \mathbb{Q}$, and the semiring of rational half-open intervals

$$\mathcal{J}_1 = \{\emptyset\} \cup \{[a, b) \cap \mathbb{Q} : a, b \in \mathbb{R}, a < b\}.$$

We equip $\mathcal{J}_1$ with the measure μ defined by

$$\mu(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \infty & \text{if } A \neq \emptyset \end{cases}$$

Notice that, if we look at the inclusion $\iota : \mathbb{Q} \hookrightarrow \mathbb{R}$, then $\mathcal{J}_1 = \mathcal{J}|_{\mathbb{Q}}$, where $\mathcal{J}$ is the semiring of half-open intervals in $\mathbb{R}$. By the Generating Theorem we then have

$$\mathbf{S}(\mathcal{J}_1) = \mathbf{S}(\mathcal{J}|_{\mathbb{Q}}) = \mathbf{S}(\mathcal{J})|_{\mathbb{Q}} = \text{Bor}(\mathbb{R})|_{\mathbb{Q}} = \mathcal{P}(\mathbb{Q}).$$

Define now the measures $\nu_1, \nu_2 : \mathbf{S}(\mathcal{J}) \rightarrow [0, \infty]$ by

$$\nu_1(A) = \begin{cases} \text{card } A & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite} \end{cases} \quad \nu_2(A) = \begin{cases} 2 \cdot \text{card } A & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite} \end{cases}$$

It is obvious that both ν_1 and ν_2 satisfy $\nu_1|_{\mathcal{J}_1} = \nu_2|_{\mathcal{J}_1} = \mu$, but obviously ν_1 and ν_2 are not equal.

COMMENT. In connection with the Caratheodory construction, it is legitimate to ask the following.

Question 2: *What happens if we do the Caratheodory construction twice?*

This problem has in fact two aspects.

Question 2A: *Suppose ω is an outer measure on X . Take $\mathcal{I} = \mathcal{M}_{\omega}(X)$ and $\nu = \omega|_{\mathcal{I}}$, so that $\mathcal{I}$ is a semiring (in fact it is a σ -algebra) on X , and ν is a measure on $\mathcal{I}$. Let ν^* be the maximal outer extension of ν . Is it true that $\nu^* = \omega$?*

By Exercise 2, we always have $\omega \leq \nu^*$. In general the answer to Question 2A is negative, as shown in Exercise ??? below. One can ask however the following

Question 2B: *Same question as 2A, but suppose $\omega = \mu^*$, the maximal outer extension of a measure μ on a semiring $\mathcal{J}$.*

The following result shows that Question 2B always has an affirmative answer.

PROPOSITION 5.3. Let X be a non-empty set, let $\mathcal{J}$ be a semiring on X , and let μ be a measure on $\mathcal{J}$. Let μ^* be the maximal outer extension of ν . Let $\mathcal{I}$ be a semiring, with $\mathcal{I} \supset \mathcal{J}$. Consider the measure $\nu = \mu|_{\mathcal{J}}$, and let ν^* be the maximal outer extension of ν . Then $\nu^* = \mu^*$.

PROOF. First of all, since $\nu|_{\mathcal{J}} = \mu|_{\mathcal{J}} = \mu$, by Exercise 2, we have the inequality $\nu^* \leq \mu^*$.

To prove the other inequality, we start with an arbitrary set $A \subset X$, and we prove that $\mu^*(A) \leq \nu^*(A)$. If $\nu^*(A) = \infty$, there is nothing to prove, so we may assume $\nu^*(A) < \infty$. In particular, $A \in \mathcal{P}_{\sigma}^{\mathcal{I}}(X)$, i.e. there exists at least one sequence $(B_n)_{n=1}^{\infty} \subset \mathcal{I}$, with $A \subset \bigcup_{n=1}^{\infty} B_n$, and we have

$$\nu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \nu(B_n) : (B_n)_{n=1}^{\infty} \subset \mathcal{I}, A \subset \bigcup_{n=1}^{\infty} B_n \right\}.$$

Fix for the moment a some $\varepsilon > 0$, and choose a sequence $(B_n^{\varepsilon})_{n=1}^{\infty} \subset \mathcal{I}$, such that

$$(11) \quad A \subset \bigcup_{n=1}^{\infty} B_n^{\varepsilon} \text{ and } \sum_{n=1}^{\infty} \nu(B_n^{\varepsilon}) \leq \nu^*(A) + \varepsilon.$$

By σ -subadditivity of μ^* , we have

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(B_n^{\varepsilon}).$$

Using the fact that $\nu = \mu^*|_{\mathcal{J}}$, the above inequality, combined with (11) yields

$$\mu^*(A) \leq \nu^*(A) + \varepsilon.$$

Since this inequality holds for all $\varepsilon > 0$, it forces the inequality $\mu^*(A) \leq \nu^*(A)$. $\square$

Exercise 3. Let X be an uncountable set, and define $\omega : \mathcal{P}(X) \rightarrow [0, \infty]$ by

$$\omega(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{if } 0 < \text{card } A \leq \aleph_0 \\ 2 & \text{if } A \text{ is uncountable} \end{cases}$$

- (i) Prove that ω is an outer measure on X .
- (ii) Take $\mathcal{I} = \mathcal{M}_{\omega}(X)$. Prove that $\mathcal{I} = \{\emptyset, X\}$.
- (iii) Consider the measure $\nu = \omega|_{\mathcal{J}}$, and let ν^* be the maximal outer extension of ν . Prove that there are sets $A \subset X$, with $\omega(A) < \nu^*(A)$.

HINTS: For (ii) start with some A with $\emptyset \subsetneq A \subsetneq X$. Prove that A is not ω -measurable, by showing that A does not “sharply cut” sets of the form $\{a, b\}$ with $a \in A$ and $b \in X \setminus A$.

COMMENT. Suppose $\mathcal{J}$ is a semiring on X , and μ is a measure on $\mathcal{J}$. We have used the maximal outer extension μ^* as a tool in defining measures on the σ -ring $\mathbf{S}(\mathcal{J})$ and the σ -algebra $\Sigma(\mathcal{J})$ generated by $\mathcal{J}$, by employing the Caratheodory construction, which uses the σ -algebra $\mathcal{M}_{\mu^*}(X)$ of μ^* -measurable sets. A legitimate question is then

Question 3: Is the inclusion $\Sigma(\mathcal{J}) \subset \mathcal{M}_{\mu^*}(X)$ strict?

In most cases this inclusion is indeed strict (see Examples ?? below, or the discussion in the next section). This can be seen by looking at μ^* -negligible sets $N \subset X$, which are automatically μ^* -measurable. The following result gives some useful information.

PROPOSITION 5.4. Suppose $\mathcal{J}$ is a semiring on X , and μ is a measure on $\mathcal{J}$. Let μ^* be the maximal outer extension of μ . For any set $A \in \mathcal{P}_\sigma^\mathcal{J}(X)$, there exists some set B in the σ -ring $\mathbf{S}(\mathcal{J})$ generated by $\mathcal{J}$, such that $A \subset B$, and $\mu^*(A) = \mu^*(B)$.

In particular, a subset $N \subset X$ is μ^* -negligible, i.e. $\mu^*(N) = 0$, if and only if there exists a μ^* -negligible set $B \in \mathbf{S}(\mathcal{J})$, such that $N \subset B$.

PROOF. Since $A \in \mathcal{P}_\sigma^\mathcal{J}(X)$, there exists a sequence $(D_n)_{n=1}^\infty \subset \mathcal{J}$ with $A \subset \bigcup_{n=1}^\infty D_n$. Moreover, we have

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^\infty \mu(D_n) : (D_n)_{n=1}^\infty \subset \mathcal{J}, A \subset \bigcup_{n=1}^\infty D_n \right\}.$$

For each integer $k \geq 1$, we can then choose a sequence $(B_n^k)_{n=1}^\infty \subset \mathcal{J}$ with $A \subset \bigcup_{n=1}^\infty B_n^k$ and $\sum_{n=1}^\infty \mu(B_n^k) \leq \mu^*(A) + 1/k$. For each integer $k \geq 1$, we define the set $B_k = \bigcup_{n=1}^\infty B_n^k$. It is clear that $A_k \in \mathbf{S}(\mathcal{J})$, and $B_k \supset A$, for all $k \in \mathbb{N}$. Moreover, by σ -sub-additivity of μ^* , and the equality $\mu^*|_{\mathcal{J}} = \mu$, we have the inequalities

$$\mu^*(B_k) \leq \sum_{n=1}^\infty \mu^*(B_n^k) = \sum_{n=1}^\infty \mu(B_n^k) \leq \mu^*(A) + \frac{1}{k}, \quad \forall k \in \mathbb{N}.$$

If we then form $B = \bigcap_{k=1}^\infty B_k$, then B still belongs to $\mathbf{S}(\mathcal{J})$, and we have $A \subset B \subset B_k$, which gives

$$\mu^*(A) \leq \mu^*(B) \leq \mu^*(B_k) \leq \mu^*(A) + \frac{1}{k} \quad \forall k \in \mathbb{N},$$

thus forcing $\mu^*(B) = \mu^*(A)$.

To prove the second assertion, we see that the “only if” part is a particular case of the first part. The “if” part is trivial, since the inclusion $N \subset B$ forces the inequality $\mu^*(N) \leq \mu^*(B)$. $\square$

In connection to Question 3, it is useful to introduce the following terminology.

DEFINITION. Let X be a non-empty set, and let $\mathcal{J}$ be a semiring on X . A measure μ on $\mathcal{J}$ is said to be *complete*, if it satisfies the condition

(c) whenever $N \in \mathcal{J}$ has $\mu(N) = 0$, it follows that $\mathcal{J}$ contains all the subsets of N .

REMARKS 5.3. A. Given an outer measure ν on a set X , the measure $\nu|_{\mathcal{M}_\nu(X)} : \mathcal{M}_\nu(X) \rightarrow [0, \infty]$ is always complete, as a consequence of monotonicity, and of Remark 5.1.B.

B. Given a semiring $\mathcal{J}$ on X , and a measure μ on $\mathcal{J}$, we now see that a sufficient condition, for having a strict inclusion $\Sigma(\mathcal{J}) \subsetneq \mathcal{M}_{\mu^*}(X)$, is the lack of completeness for the measure $\mu^*|_{\Sigma(\mathcal{J})}$. Later on (see Corollary 5.2) we shall see that in the case of σ -finite measures, defined on σ -total semirings, this condition is also necessary.

The lack of completeness of a σ -ring measure can be compensated by the following result.

THEOREM 5.4. Let X be a non-empty set, let $\mathcal{S}$ be a σ -ring on X , and let ν be a measure on $\mathcal{S}$.

(i) The collection

$$\mathcal{N}(\mathcal{S}, \nu) = \{N \subset X : \text{there exists } D \in \mathcal{S} \text{ with } N \subset D \text{ and } \nu(D) = 0\}$$

is a σ -ring on X . Moreover, if $N \in \mathcal{N}(\mathcal{S}, \nu)$, then $\mathcal{N}(\mathcal{S}, \nu)$ contains all subsets of N .

- (ii) For a subset $A \subset X$, the following are equivalent:
 (a) there exists $B \in \mathcal{S}$ and $N \in \mathcal{N}(\mathcal{S}, \nu)$, such that $A = B \setminus N$;
 (b) there exists $F \in \mathcal{S}$ and $M \in \mathcal{N}(\mathcal{S}, \nu)$, such that $A = F \cup M$.
 (iii) The collection $\bar{\mathcal{S}}$ of all subsets $A \subset X$, satisfying the equivalent conditions in (ii), is a σ -ring. We have the equality

$$\bar{\mathcal{S}} = \mathbf{S}(\mathcal{N}(\mathcal{S}, \nu) \cup \mathcal{S}).$$

- (iv) There exists a unique measure $\bar{\nu}$ on $\bar{\mathcal{S}}$, such that $\bar{\nu}|_{\mathcal{N}(\mathcal{S}, \nu)} = 0$ and $\bar{\nu}|_{\mathcal{S}} = \nu$. The measure $\bar{\nu}$ is complete.
 (v) If $\mathcal{E}$ is a σ -ring with $\mathcal{E} \supset \mathcal{S}$, and if λ is a complete measure on $\mathcal{E}$ with $\lambda|_{\mathcal{S}} = \nu$, then $\mathcal{E} \supset \bar{\mathcal{S}}$ and $\lambda|_{\bar{\mathcal{S}}} = \bar{\nu}$.

PROOF. (i). This is pretty clear. In fact, if one takes $\mathcal{E} = \{B \in \mathcal{S} : \nu(B) = 0\}$, then one has the equality $\mathcal{N}(\mathcal{S}, \nu) = \mathcal{P}_{\sigma}^{\mathcal{E}}(X)$.

(ii). (a) $\Rightarrow$ (b). Assume $A = B \setminus N$ with $B \in \mathcal{S}$ and $N \in \mathcal{N}(\mathcal{S}, \nu)$. Choose $D \in \mathcal{S}$ with $\nu(D) = 0$ and $N \subset D$. We now have

$$B \setminus D \subset B \setminus N = A,$$

so if we put $F = B \setminus D$, we have the equality $A = F \cup M$, where

$$M = A \setminus F = (B \setminus N) \setminus (B \setminus D) \subset D.$$

Notice that $F \in \mathcal{S}$, while the inclusion $M \subset D$ shows that $M \in \mathcal{N}(\mathcal{S}, \nu)$.

(b) $\Rightarrow$ (a). Assume $A = F \cup M$ with $F \in \mathcal{S}$ and $M \in \mathcal{N}(\mathcal{S}, \nu)$. Choose $D \in \mathcal{S}$ with $M \subset D$ and $\nu(D) = 0$. Define $B = F \cup D$. It is clear that $B \in \mathcal{S}$, and $A \subset B$. Define $N = B \setminus A$, so we clearly have $A = B \setminus N$. We have

$$N = (F \cup D) \setminus (F \cup M) \subset D \setminus M \subset D,$$

so N clearly belongs to $\mathcal{N}(\mathcal{S}, \nu)$.

(iii). We need to prove the following properties:

- (*) whenever A_1, A_2 are sets in $\bar{\mathcal{S}}$, it follows that the difference $A_1 \setminus A_2$ also belongs to $\bar{\mathcal{S}}$;
 (**) whenever $(A_n)_{n=1}^{\infty}$ is a sequence of sets in $\bar{\mathcal{S}}$, it follows that the union $\bigcup_{n=1}^{\infty} A_n$ also belongs to $\bar{\mathcal{S}}$.

To prove (*), we write $A_1 = B \setminus N$ and $A_2 = F \cup M$, with $B, F \in \mathcal{S}$ and $M, N \in \mathcal{N}(\mathcal{S}, \nu)$. Then we have

$$A_1 \setminus A_2 = (B \setminus N) \setminus (F \cup M) = B \setminus (F \cup M \cup N) = (B \setminus F) \setminus (M \cup N).$$

The difference $B \setminus F$ belongs to $\mathcal{S}$, and, using (i), the union $N \cup M$ belongs to $\mathcal{N}(\mathcal{S}, \nu)$. By (ii) it follows that $A_1 \setminus A_2$ belongs to $\bar{\mathcal{S}}$.

To prove (**), we write, for each $n \in \mathbb{N}$, the set A_n as $A_n = F_n \cup M_n$ with $F_n \in \mathcal{S}$ and $M_n \in \mathcal{N}(\mathcal{S}, \nu)$. Then

$$\bigcup_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} F_n \right) \cup \left(\bigcup_{n=1}^{\infty} M_n \right).$$

The union $\bigcup_{n=1}^{\infty} F_n$ belongs to $\mathcal{S}$, and, using (i), the union $\bigcup_{n=1}^{\infty} M_n$ belongs to $\mathcal{N}(\mathcal{S}, \nu)$. By (ii), the union $\bigcup_{n=1}^{\infty} A_n$ belongs to $\bar{\mathcal{S}}$.

Since $\bar{\mathcal{S}}$ is a σ -ring, which clearly contains both $\mathcal{N}(\mathcal{S}, \nu)$ and $\mathcal{S}$, it follows that $\bar{\mathcal{S}} \supset \mathbf{S}(\mathcal{N}(\mathcal{S}, \nu) \cup \mathcal{S})$. The other inclusion $\bar{\mathcal{S}} \subset \mathbf{S}(\mathcal{N}(\mathcal{S}, \nu) \cup \mathcal{S})$ is trivial, by the definition of $\bar{\mathcal{S}}$.

(iv). To prove the existence, we consider the maximal outer extension ν^* . When restricted to the σ -algebra $\mathcal{M}_{\nu^*}(X)$ of all ν^* -measurable sets, then we get a measure. Notice that $\nu^*(N) = 0$, $\forall N \in \mathcal{N}(\mathcal{S}, \nu)$, which gives the inclusion $\mathcal{N}(\mathcal{S}, \nu) \subset \mathcal{M}_{\nu^*}(X)$. In particular, since $\mathcal{M}_{\nu^*}(X)$ is a σ -algebra, which contains both $\mathcal{N}(\mathcal{S}, \nu)$ and $\mathcal{S}$, it follows that

$$\mathcal{M}_{\nu^*}(X) \supset \mathbf{S}(\mathcal{N}(\mathcal{S}, \nu) \cup \mathcal{S}) = \bar{\mathcal{S}}.$$

In particular, $\bar{\nu} = \nu^*|_{\bar{\mathcal{S}}}$ is a measure on $\bar{\mathcal{S}}$, which clearly satisfies the required properties.

To prove uniqueness, let μ be another measure on $\bar{\mathcal{S}}$, such that $\mu|_{\mathcal{N}(\mathcal{S}, \nu)} = 0$ and $\mu|_{\mathcal{S}} = \nu$. If we start with an arbitrary set $A \in \bar{\mathcal{S}}$, and we write it as $A = F \cup M$, with $F \in \mathcal{S}$ and $M \in \mathcal{N}(\mathcal{S}, \nu)$, then using the fact that $A \setminus F \subset M$, we see that $A \setminus F$ belongs to $\mathcal{N}(\mathcal{S}, \nu)$, so we have

$$\mu(A) = \mu(F) + \nu(A \setminus F) = \mu(F) = \nu(F) = \bar{\nu}(F) = \bar{\nu}(F) + \bar{\nu}(A \setminus F) = \bar{\nu}(A).$$

Finally, we prove that the measure $\bar{\nu}$ is complete. Let $A \in \bar{\mathcal{S}}$ be a set with $\bar{\nu}(A) = 0$, and let U be an arbitrary subset of A . Using (ii) we write $A = F \cup M$, with $F \in \mathcal{S}$ and $M \in \mathcal{N}(\mathcal{S}, \nu)$. Notice that we have

$$0 \leq \nu(F) = \bar{\nu}(F) \leq \bar{\nu}(F \cup M) = \bar{\nu}(A) = 0,$$

which forces $F \in \mathcal{N}(\mathcal{S}, \nu)$, so using (i), we see that A itself belongs to $\mathcal{N}(\mathcal{S}, \nu)$. By (i), it follows that $U \in \mathcal{N}(\mathcal{S}, \nu) \subset \bar{\mathcal{S}}$.

(v) Let $\mathcal{E}$ and λ be as indicated. In order to prove the inclusion $\mathcal{E} \supset \bar{\mathcal{S}}$, it suffices to prove the inclusion $\mathcal{N}(\mathcal{S}, \nu) \subset \mathcal{E}$. But this inclusion is pretty obvious. If we start with some $N \in \mathcal{N}(\mathcal{S}, \nu)$, then there exists $A \in \mathcal{S}$ with $N \subset A$ and $\nu(A) = 0$. In particular, we have $A \in \mathcal{E}$ and $\lambda(A) = 0$, and then the completeness of λ forces $N \in \mathcal{E}$. Notice that this also forces $\lambda(N) = \bar{\nu}(N) = 0$. Using (iv) it then follows that $\lambda|_{\bar{\mathcal{S}}} = \bar{\nu}$. $\square$

DEFINITION. Using the notations above, the σ -ring $\bar{\mathcal{S}}$ is called the *completion of $\mathcal{S}$ with respect to ν* . The correspondence $(\mathcal{S}, \nu) \mapsto (\bar{\mathcal{S}}, \bar{\nu})$ is referred to as the *measure completion*. Remark that, if ν is already complete, then $\bar{\mathcal{S}} = \mathcal{S}$ and $\bar{\nu} = \nu$.

Exercise 4. Using the notations from Theorem 5.4, prove that for a set $A \subset X$, the condition $A \in \bar{\mathcal{S}}$ is equivalent to any of the following:

- (a') there exists $B \in \mathcal{S}$ and $N \in \mathcal{N}(\mathcal{S}, \mu)$, with $A = B \setminus N$, and $N \subset B$;
- (b') there exists $F \in \mathcal{S}$ and $M \in \mathcal{N}(\mathcal{S}, \nu)$, such that $A = F \cup M$ and $F \cap M = \emptyset$;
- (c) there exists $E \in \mathcal{S}$ and $Z \in \mathcal{N}(\mathcal{S}, \nu)$, such that $A = E \Delta Z$.
- (d) there exist $B, F \in \mathcal{S}$ such that $F \subset A \subset B$, and $\mu(B \setminus F) = 0$.

The μ^* -measurable sets of a special type can be completely characterized using μ^* -negligible ones.

THEOREM 5.5. *Suppose $\mathcal{J}$ is a semiring on X , and μ is a measure on $\mathcal{J}$. Let μ^* be the maximal outer extension of μ . For a $\mathcal{J}$ - μ - σ -finite subset $A \subset X$, the following are equivalent;*

- (i) A is μ^* -measurable;

- (ii) *there exists B in the σ -ring $\mathbf{S}(\mathcal{J})$ generated by $\mathcal{J}$, and a μ^* -negligeable set $N \subset X$, such that $A = B \setminus N$.*

PROOF. (i) $\Rightarrow$ (ii). Start by choosing a sequence $(D_n)_{n=1}^\infty \subset \mathcal{J}$ with $A \subset \bigcup_{n=1}^\infty D_n$ and $\mu(D_n) < \infty$, $\forall n \in \mathbb{N}$. Since $\mathcal{M}_{\mu^*}(X)$ is an algebra, which contains $\mathcal{J}$, it follows that all the intersections $A_n = A \cap D_n$, $n \in \mathbb{N}$, belong to $\mathcal{M}_{\mu^*}(X)$. For each $n \in \mathbb{N}$, we use the previous result to find some set $B_n \in \mathbf{S}(\mathcal{J})$ such that $A_n \subset B_n$, and $\mu^*(B_n) = \mu^*(A_n)$. On the one hand, if we put $V_n = B_n \setminus A_n$, then $V_n \in \mathcal{M}_{\mu^*}(X)$, so we will have

$$\mu^*(B_n) = \mu^*(A_n) + \mu^*(V_n).$$

On the other hand, we know that $\mu^*(B_n) = \mu^*(A_n) \leq \mu^*(D_n) < \infty$, so the above equality forces $\mu^*(V_n) = 0$.

Since we have $B_n = A_n \cup V_n$, $\forall n \in \mathbb{N}$, we will get

$$\bigcup_{n=1}^\infty B_n = \left(\bigcup_{n=1}^\infty A_n \right) \cup \left(\bigcup_{n=1}^\infty V_n \right) = A \cup \left(\bigcup_{n=1}^\infty V_n \right),$$

so if we define $B = \bigcup_{n=1}^\infty B_n$ and $V = \bigcup_{n=1}^\infty V_n$, then B belongs to $\mathbf{S}(\mathcal{J})$, we have the equality $B = A \cup V$, and V is μ^* -negligeable, because of the inequalities

$$\mu^*(V) \leq \sum_{n=1}^\infty \mu^*(V_n).$$

The set $N = B \setminus A \subset V$ is clearly μ^* -negligeable, because $\mu^*(N) \leq \mu^*(V)$. Now we are done because $B \setminus N = A$.

(ii) $\Rightarrow$ (i). This part is trivial, since $\mathcal{M}_{\mu^*}(X)$ is an algebra. $\square$

REMARKS 5.4. A. The implication (ii) $\Rightarrow$ (i) holds without the assumption that A is $\mathcal{J}$ - μ - σ -finite. In fact, for any $A \subset X$, one has the implications (ii) $\Rightarrow$ (ii') $\Rightarrow$ (i), where

- (ii') *there exists B in the σ -algebra $\Sigma(\mathcal{J})$ generated by $\mathcal{J}$, and a μ^* -negligeable set $N \subset X$, such that $A = B \setminus N$.*

B. Consider the measure $\mu^*|_{\mathbf{S}(\mathcal{J})}$ on the σ -ring $\mathbf{S}(\mathcal{J})$. Using the notations from Theorem 5.4, by Proposition 5.3, we clearly have the equality

$$\{N \subset X : N \text{ } \mu^*\text{-negligeable}\} = \mathcal{N}(\mathbf{S}(\mathcal{J}), \mu^*|_{\mathbf{S}(\mathcal{J})}).$$

So, if we denote by $\overline{\mathbf{S}(\mathcal{J})}$ the completion of $\mathbf{S}(\mathcal{J})$ with respect to $\mu^*|_{\mathbf{S}(\mathcal{J})}$, condition (ii) from Theorem 5.5 reads: $A \in \overline{\mathbf{S}(\mathcal{J})}$. Similarly, if we denote by $\overline{\Sigma(\mathcal{J})}$ the completion of $\Sigma(\mathcal{J})$ with respect to the measure $\mu^*|_{\Sigma(\mathcal{J})}$, condition (ii') above reads: $A \in \overline{\Sigma(\mathcal{J})}$. With these notations, we have the inclusions

$$(12) \quad \overline{\mathbf{S}(\mathcal{J})} \subset \overline{\Sigma(\mathcal{J})} \subset \mathcal{M}_{\mu^*}(X).$$

With these notations, Theorem 5.5 states that

$$(13) \quad \overline{\mathbf{S}(\mathcal{J})} \cap \{A \subset X : A \text{ } \mathcal{J}\text{-}\mu\text{-}\sigma\text{-finite}\} = \mathcal{M}_{\mu^*}(X) \cap \{A \subset X : A \text{ } \mathcal{J}\text{-}\mu\text{-}\sigma\text{-finite}\}.$$

Theorem 5.5, written in the form (13) has the following.

COROLLARY 5.2. *If the semiring $\mathcal{J}$ is σ -total in X , and μ is a σ -finite measure on $\mathcal{J}$, then one has the equalities*

$$(14) \quad \overline{\mathbf{S}(\mathcal{J})} = \overline{\Sigma(\mathcal{J})} = \mathcal{M}_{\mu^*}(X).$$

PROOF. Indeed, under the given assumptions on $\mathcal{J}$ and μ , it follows that *every* set $A \subset X$ is $\mathcal{J}$ - μ - σ -finite. $\square$

EXAMPLES 5.3. A. The implication (i) $\Rightarrow$ (ii) from Theorem 5.5 may fail, if A is not σ -finite. Start with an arbitrary set X , consider the semiring $\mathcal{J} = \{\emptyset, X\}$ and the measure μ on $\mathcal{J}$ defined by $\mu(\emptyset) = 0$ and $\mu(X) = \infty$. Notice that $\mathcal{J}$ is a σ -algebra, so it is trivial that $\mathcal{J}$ is σ -total in X . The maximal outer extension μ^* of μ is defined by

$$\mu^*(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \infty & \text{if } A \neq \emptyset \end{cases}$$

It is clear that, since μ^* is a measure on $\mathcal{P}(X)$, we have the equality $\mathcal{M}_{\mu^*}(X) = \mathcal{P}(X)$, but the only μ^* -negligible set is the empty set $\emptyset$. This means that the sets satisfying condition (ii) in Theorem 5.4 are only the sets $\emptyset$ and X , so, if $\emptyset \neq A \subsetneq X$, the implication (i) $\Rightarrow$ (ii) fails, although $\mathcal{J}$ is σ -total in X . What occurs here is the total lack of $\mathcal{J}$ - μ - σ -finite sets.

B. Let X be an uncountable set, and let $\mathcal{J}$ be the semiring of all finite subsets of X . We have

$$\mathbf{S}(\mathcal{J}) = \{A \subset X : \text{card } A \leq \aleph_0\},$$

$$\Sigma(\mathcal{J}) = \{A \subset X : \text{either } \text{card } A \leq \aleph_0, \text{ or } \text{card}(X \setminus A) \leq \aleph_0\}.$$

Equip $\mathcal{J}$ with the trivial measure $\mu(A) = 0, \forall A \in \mathcal{J}$. The maximal outer extension μ^* is then defined by

$$\mu^*(A) = \begin{cases} 0 & \text{if } \text{card } A \leq \aleph_0 \\ \infty & \text{if } A \text{ is uncountable} \end{cases}$$

It is clear that μ^* is a measure on $\mathcal{P}(X)$, so we have $\mathcal{M}_{\mu^*}(X) = \mathcal{P}(X) \supsetneq \mathcal{J}$. Notice that both measures $\mu^*|_{\mathbf{S}(\mathcal{J})}$ and $\mu^*|_{\Sigma(\mathcal{J})}$ are complete, so using the notations from Remark 5.4.B, we have the equalities

$$\overline{\mathbf{S}(\mathcal{J})} = \mathbf{S}(\mathcal{J}) \text{ and } \overline{\Sigma(\mathcal{J})} = \Sigma(\mathcal{J}).$$

It is clear however that both inclusions in (12) are strict, although μ is finite. What happens here is the fact that $\mathcal{J}$ is not σ -total in X .

C. In the same setting as in Example B, if we take $\mathcal{I} = \Sigma(\mathcal{J})$, and $\nu = \mu^*|_{\mathcal{I}}$, then $\mathcal{I}$ is σ -total in X , simply because $\mathcal{I}$ is a σ -algebra. In this case, by Proposition 5.3, the maximal outer extension ν^* of ν coincides with μ^* . We have

$$\mathcal{I} = \mathbf{S}(\mathcal{I}) = \overline{\mathbf{S}(\mathcal{I})} = \Sigma(\mathcal{I}) = \overline{\Sigma(\mathcal{I})} \subsetneq \mathcal{M}_{\nu^*}(X),$$

the reason for the strict inclusion being this time the fact that ν is not σ -finite.

COMMENT. In the remainder of this section we take another look Question 3, trying to generalize the answer given by Corollary 5.2. To simplify matters a little bit, we start with a σ -algebra $\mathcal{B}$ on X (which is clearly σ -total in X), and a measure μ on $\mathcal{B}$. If we take μ^* to be the maximal outer extension of μ , and consider the completion $\overline{\mathcal{B}}$, we have the inclusion

$$(15) \quad \overline{\mathcal{B}} \subset \mathcal{M}_{\mu^*}(X),$$

so we can ask whether this inclusion is strict. Of course, if μ is σ -finite, then by Corollary 5.2 the inclusion (15) is not strict. As Example 5.3.C suggests, in the absence of the σ -finiteness assumption, the inclusion (15) may indeed be strict. As it turns out, the fact that the inclusion (15) is strict in Example 5.3.C is a consequence

of the fact that there are “new” measurable sets which are not necessarily of the form $B \setminus N$ with $B \in \mathcal{B}$ and N negligible. The existence of such sets is suggested by the following.

REMARK 5.5. Suppose ν is an outer measure on X . For a set $A \subset X$, the following are equivalent:

- (i) A is ν -measurable;
- (ii) $\nu(S) \geq \nu(S \cap A) + \nu(S \setminus A)$, for all $S \subset X$ with $\nu(S) < \infty$;

The implication (i) $\Rightarrow$ (ii) is trivial. To prove the converse, by Remark 5.1.A, we need to show that

$$\nu(S) \geq \nu(S \cap A) + \nu(S \setminus A), \quad \forall S \subset X.$$

But this is trivial, when $\nu(S) = \infty$. If $\nu(S) < \infty$, then this is exactly condition (ii).

The “new” sets, that were mentioned above, are of a type covered by the following.

DEFINITION. Let ν be an outer measure on X . A subset $N \subset X$ is said to be *locally ν -negligible*, if

$$\nu(N \cap A) = 0, \text{ for all } A \subset X \text{ with } \nu(A) < \infty.$$

It is clear that every subset of N is also locally ν -negligible.

The above observation shows that *every locally ν -negligible set is ν -measurable*.

The term “local” will be used in connection with properties that hold when the subject set is cut down by sets of finite measure. For example, one can formulate the following.

DEFINITIONS. Let $\mathcal{B}$ be a σ -algebra on X , and μ be a measure on $\mathcal{B}$. We say that a set $N \in \mathcal{B}$ is *locally μ -null*, if

$$(16) \quad \mu(F \cap N) = 0, \text{ for all } F \in \mathcal{B}, \text{ with } \mu(F) < \infty.$$

Remark that locally μ -null sets do not necessarily have zero measure (see Example 5.3.C)

We say that μ is *locally complete*, if it satisfies the condition

- (LC) *whenever $N \in \mathcal{B}$ is a locally μ -null set, it follows that $\mathcal{B}$ contains all subsets of N .*

REMARKS 5.6. Use the notations above.

A. If the measure μ is σ -finite, the local completeness of μ is equivalent to completeness. The reason is the fact that, in the σ -finite case, condition (16) is equivalent to $\mu(N) = 0$.

B. Given an outer measure ν on X , the measure $\nu|_{\mathcal{M}_\nu(X)}$ is locally complete.

COMMENT. If we look at Example 5.3.C, we now see that although the measure ν on $\mathcal{I}$ is complete, it is not locally complete, thus giving another explanation for the strict inclusion $\mathcal{I} \subsetneq \mathcal{M}_\nu^*(X)$.

We are now in position to analyze Question 3, in the simplified given setting. The following fact will be helpful.

LEMMA 5.1. *Let $\mathcal{B}$ be a σ -algebra on X , let μ be a measure on $\mathcal{B}$, and let μ^* be the maximal outer extension of μ . Then, for every subset $S \subset X$, one has the equality*

$$(17) \quad \mu^*(S) = \inf \{ \mu(B) : B \in \mathcal{B}, B \supset S \}.$$

PROOF. Since $\mathcal{B}$ is σ -total in X , by definition we have

$$(18) \quad \mu^*(S) = \inf \left\{ \sum_{n=1}^{\infty} \mu(B_n) : (B_n)_{n=1}^{\infty} \subset \mathcal{B}, S \subset \bigcup_{n=1}^{\infty} B_n \right\}.$$

If we denote the right hand side of (??) by $\nu(S)$, then using (18) we clearly have $\mu^*(S) \leq \nu(S)$. Conversely, if we start with any sequence $(B_n)_{n=1}^{\infty} \subset \mathcal{B}$ with $S \subset \bigcup_{n=1}^{\infty} B_n$, then we clearly have

$$\sum_{n=1}^{\infty} \mu(B_n) \geq \mu\left(\bigcup_{n=1}^{\infty} B_n\right) \geq \nu(S),$$

so taking the infimum yields $\mu^*(S) \geq \nu(S)$. $\square$

PROPOSITION 5.5. *Let $\mathcal{B}$ be a σ -algebra on X , and let μ be a measure on $\mathcal{B}$. Define the collection*

$$\mathcal{B}_{\text{fin}} = \{F \in \mathcal{B} : \mu(F) < \infty\}.$$

For every $F \in \mathcal{B}_{\text{fin}}$, denote by $\mathfrak{M}_F$ the completion of the σ -algebra $\mathcal{B}|_F$ (on F) with respect to the measure⁸ $\mu|_F$. Denote by μ^ the maximal outer extension of μ .*

A. *For a subset $A \subset X$, the following are equivalent*

- (i) *A is μ^* -measurable;*
- (ii) *$A \cap F \in \mathfrak{M}_F$, for each $F \in \mathcal{B}_{\text{fin}}$;*
- (iii) *$A \cap F$ is μ^* -measurable, for each $F \in \mathcal{B}_{\text{fin}}$.*

B. *For a subset $N \subset X$, the following are equivalent*

- (i) *N is locally μ^* -negligible;*
- (ii) *$\mu^*(N \cap F) = 0$, for all $F \in \mathcal{B}_{\text{fin}}$.*

PROOF. Let us fix some useful notations. By construction, for every $F \in \mathcal{B}_{\text{fin}}$, we have

$$\mathcal{B}|_F = \{A \cap F : A \in \mathcal{B}\} = \{B \in \mathcal{B} : B \subset F\}.$$

For each $F \in \mathcal{B}_{\text{fin}}$, we denote the σ -ring $\mathcal{N}(\mathcal{B}|_F, \mu|_F)$ simply by $\mathcal{N}_F$. With the above identification we have

$$\mathcal{N}_F = \{N \subset F : \text{there exists } D \in \mathcal{B} \text{ with } N \subset D \subset F \text{ and } \mu(D) = 0\},$$

so (see Theorem 5.4) the σ -algebra $\mathfrak{M}_F$ is given as

$$\mathfrak{M}_F = \{B \setminus N : B \in \mathcal{B}, B \subset F, N \in \mathcal{N}_F\}.$$

A. To prove the implication (i) $\Rightarrow$ (ii), start with a μ^* -measurable set A , and with some $F \in \mathcal{B}_{\text{fin}}$. Since F is μ^* -measurable, the intersection $A \cap F$ is μ^* -measurable. Since $\mu^*(A \cap F) \leq \mu^*(F) = \mu(F) < \infty$, by Theorem 5.5 there exist $B_0 \in \mathcal{B}$ and $N_0 \subset X$ with $\mu^*(N_0) = 0$ and $A \cap F = B_0 \setminus N_0$. If we then define $B = B_0 \cap F$ and $N = N_0 \cap F$, then we clearly have $B \in \mathcal{B}|_F$, $N \in \mathcal{N}_F$, and $A \cap F = B \setminus N$, so $A \cap F$ indeed belongs to $\mathfrak{M}_F$.

The implication (ii) $\Rightarrow$ (i) is trivial, since every set in $\mathfrak{M}_F$ is clearly μ^* -measurable.

To prove the implication (iii) $\Rightarrow$ (i), assume A has property (iii), and let us show that A is μ^* -measurable. We are going to use Remark 5.5, which means that it suffices to prove the inequality

$$(19) \quad \mu^*(S) \geq \mu^*(S \cap A) + \mu^*(S \setminus A),$$

⁸ Here $\mu|_F$ denotes the restriction of μ to the σ -algebra $\mathcal{B}|_F$.

only for those subsets $S \subset X$ with $\mu^*(S) < \infty$. Fix such a subset S . Since $\mu^*(S) < \infty$, Lemma 5.1 gives

$$(20) \quad \mu^*(S) = \inf \{ \mu(F) : F \in \mathcal{B}_{\text{fin}}, F \supset S \}.$$

Start with some arbitrary $\varepsilon > 0$, and choose some $F \in \mathcal{B}_{\text{fin}}$ with $F \supset S$ and $\mu(F) \leq \mu^*(S) + \varepsilon$. By (iii) the set $A \cap F$ is μ^* -measurable, so we have

$$\mu^*(F) = \mu^*(F \cap [A \cap F]) + \mu^*(F \setminus [A \cap F]) = \mu^*(F \cap A) + \mu^*(F \setminus A).$$

Since $F \cap A \supset S \cap A$, and $F \setminus A \supset S \setminus A$, we have the inequalities $\mu^*(F \cap A) \geq \mu^*(S \cap A)$ and $\mu^*(F \setminus A) \geq \mu^*(S \setminus A)$, so the above inequality gives

$$\mu^*(F) \geq \mu^*(S \cap A) + \mu^*(S \setminus A).$$

By the choice of F , this gives

$$\mu^*(S) + \varepsilon \geq \mu^*(S \cap A) + \mu^*(S \setminus A).$$

Since this inequality holds for all $\varepsilon > 0$, we immediately get the desired inequality (19).

B. The condition (i) says that

$$(21) \quad \mu^*(N \cap S) = 0, \text{ for all } S \subset X \text{ with } \mu^*(S) = 0.$$

It is obvious that we have the implication (i) $\Rightarrow$ (ii). Conversely, suppose N satisfies (ii), and let us prove (21). Start with some arbitrary subset $S \subset X$ with $\mu^*(S) < \infty$. Using (20), there exists some $F \in \mathcal{B}_{\text{fin}}$ with $S \subset F$. By (ii), and the monotonicity of μ^* we have $0 = \mu^*(N \cap F) \geq \mu^*(N \cap S)$, which clearly forces $\mu^*(N \cap S) = 0$. $\square$

The above result suggests that the σ -algebra $\mathcal{M}_{\mu^*}(X)$ can be regarded as some sort of “local” completion of $\mathcal{B}$. To simplify the exposition a little bit, we introduce the following.

NOTATION. Let $\mathcal{B}$ be a σ -algebra on X , let μ be a measure on $\mathcal{B}$, and let μ^* be the maximal outer extension of μ . The σ -algebra $\mathcal{M}_{\mu^*}(X)$, of all μ^* -measurable subsets of X , will be denoted by $\mathfrak{M}_{\mu}(\mathcal{B})$ (or just $\mathfrak{M}_{\mu}$, when there is no danger of confusion). The measure $\mu^*|_{\mathfrak{M}_{\mu}}$ will be denoted by $\tilde{\mu}$. The pair $(\mathfrak{M}_{\mu}, \tilde{\mu})$ will be called the *quasi-completion of $\mathcal{B}$ with respect to μ* .

Unfortunately, analogues of Theorem 5.4 are not available, unless some (otherwise natural) restrictions are imposed. The type of restrictions we have in mind also aimed at making the test conditions A.(iii) and B.(ii) easier to check. We would like to check them on a “small” sub-collection of $\mathcal{B}_{\text{fin}}$. This naturally suggests the following.

DEFINITION. Let $\mathcal{B}$ be a σ -algebra on X , and let μ be a measure on $\mathcal{B}$. A *sufficient μ -finite $\mathcal{B}$ -partition of X* is a collection $\mathcal{F}$ of non-empty subsets of X , with the following properties:

- (i) $\mathcal{F}$ is pairwise disjoint, and $\bigcup_{F \in \mathcal{F}} F = X$;
- (ii) $F \in \mathcal{B}$, and $\mu(F) < \infty$, for all $F \in \mathcal{F}$;
- (iii) for every set $B \in \mathcal{B}$, with $\mu(B) < \infty$, one has the equality

$$\mu(B) = \sum_{F \in \mathcal{F}} \mu(B \cap F).$$

Condition (iii) uses the summation convention from II.2. (The sum is defined as the supremum of all finite partial sums.)

REMARKS 5.7. A. Suppose $\mathcal{F}$ is a sufficient μ -finite $\mathcal{B}$ -partition of X . For every set $A \in \mathcal{B}$, we define the collection

$$S_{\mathcal{F}}^{\mu}(A) = \{F \in \mathcal{F} : \mu(A \cap F) > 0\}.$$

If $\mu(A) < \infty$, then

- (a) $S_{\mathcal{F}}^{\mu}(A)$ is at most countable, and
- (b) $\mu(A \setminus [\bigcup_{F \in S_{\mathcal{F}}^{\mu}(A)} (A \cap F)]) = 0$.

By condition (iii) in the definition, it follows that, the family $(\mu(A \cap F))_{F \in S_{\mathcal{F}}^{\mu}(A)}$ is summable, and

$$(22) \quad \sum_{F \in S_{\mathcal{F}}^{\mu}(A)} \mu(A \cap F) = \mu(A).$$

Since $\mu(A \cap F) > 0, \forall F \in S_{\mathcal{F}}^{\mu}(A)$, property (a) follows from Proposition II.2.2. If we denote the union $\bigcup_{F \in S_{\mathcal{F}}^{\mu}(A)} (A \cap F)$ by A_0 , then by the σ -additivity of μ (it is here where we use (a) in an essential way) the equality (22) gives

$$\mu(A_0) = \sum_{F \in S_{\mathcal{F}}^{\mu}(A)} \mu(A \cap F) = \mu(A),$$

which combined with $\mu(A) < \infty$ forces $\mu(A \setminus A_0) = 0$.

B. The existence of a sufficient μ -finite $\mathcal{B}$ -partition of X is a generalization of σ -finiteness. In fact the following are equivalent ($\mathcal{B}$ is a σ -algebra on X):

- μ is σ -finite;
- there exists a *countable* sufficient μ -finite $\mathcal{B}$ -partition of X .

In the presence of a sufficient μ -finite $\mathcal{B}$ -partition, the properties that appear in Proposition 5.5 are simplified.

PROPOSITION 5.6. *Let $\mathcal{B}$ be a σ -algebra on X , let μ be a measure on $\mathcal{B}$. Assume $\mathcal{F}$ is a sufficient μ -finite $\mathcal{B}$ -partition of X . Denote by μ^* the maximal outer extension of μ .*

A. *For a subset $A \subset X$, the following are equivalent*

- (i) *A is μ^* -measurable;*
- (ii) *$A \cap F$ is μ^* -measurable, for each $F \in \mathcal{F}$.*

B. *For a subset $N \subset X$, the following are equivalent*

- (i) *N is locally μ^* -negligible;*
- (ii) *$\mu^*(N \cap F) = 0$, for all $F \in \mathcal{F}$.*

C. *If $A \subset X$ is a subset with $\mu^*(A) < \infty$, then*

$$(23) \quad \mu^*(A) = \sum_{F \in \mathcal{F}} \mu^*(A \cap F).$$

PROOF. It will be useful to introduce the following notations (use also the notations from Proposition 5.5). For every $B \in \mathcal{B}_{\text{fin}}$, we define

$$B' = \bigcup_{F \in S_{\mathcal{F}}^{\mu}(B)} (B \cap F).$$

By Remark 5.7 we know that $\mu(B \setminus B') = 0$.

A. The implication (i) $\Rightarrow$ (ii) is trivial. To prove the implication (ii) $\Rightarrow$ (i), we start with a set $A \subset X$ satisfying condition (ii), and we show that A satisfies condition (iii) from Proposition 5.5.A. Start with some arbitrary set $B \in \mathcal{B}_{\text{fin}}$,

and let us show that $A \cap B$ is μ^* -measurable. Using the above notation, and the monotonicity of μ^* we have

$$\mu^*(A \cap [B \setminus B']) \leq \mu^*(B \setminus B) = \mu(B \setminus B') = 0,$$

which in particular shows that $A \cap [B \setminus B']$ is μ^* -measurable. Since we have $A \cap B = (A \cap B') \cup (A \cap [B \setminus B'])$, it then suffices to show that $A \cap B'$ is μ^* -measurable. Notice that

$$A \cap B' = \bigcup_{S_{\mathcal{F}}^{\mu}(B)} (A \cap F \cap B),$$

and since the indexing set $S_{\mathcal{F}}^{\mu}(B)$ is at most countable, it then suffices to show that $A \cap F \cap B$ is μ^* -measurable, for each F . But this is obvious, since $A \cap F$ is μ^* -measurable, by condition (ii), and $B \in \mathcal{B}$.

C. Let $A \subset X$ be a subset with $\mu^*(A)$. Using Lemma 5.1, we can find, for every $\varepsilon > 0$, some set $B_{\varepsilon} \in \mathcal{B}_{\text{fin}}$, such that $B_{\varepsilon} \supset A$, and $\mu(B_{\varepsilon}) \leq \mu^*(A) + \varepsilon$. Fix for the moment ε . Since the family $(\mu(B_{\varepsilon} \cap F))_{F \in \mathcal{F}}$ is summable, and $\mu^*(A \cap F) \leq \mu(B_{\varepsilon} \cap F)$, $\forall F \in \mathcal{F}$, it follows that the family $(\mu^*(A \cap F))_{F \in \mathcal{F}}$ is summable, and moreover one has the inequality

$$\sum_{F \in \mathcal{F}} \mu^*(A \cap F) \leq \sum_{F \in \mathcal{F}} \mu(B_{\varepsilon} \cap F) = \mu(B_{\varepsilon}) \leq \mu^*(A) + \varepsilon.$$

Since we have $\sum_{F \in \mathcal{F}} \mu^*(A \cap F) \leq \mu^*(A) + \varepsilon$, for all $\varepsilon > 0$, it follows that we have in fact the inequality

$$\sum_{F \in \mathcal{F}} \mu^*(A \cap F) \leq \mu^*(A).$$

To prove the reverse inequality, we fix $\varepsilon = 1$ and we define set

$$G = \bigcup_{F \in S_{\mathcal{F}}^{\mu}(B_1)} F.$$

Since $S_{\mathcal{F}}^{\mu}(B_1)$ is at most countable, the set G belongs to $\mathcal{B}$. With the above notation, we have the equality $B'_1 = B_1 \cap G$, and by Remark 5.7.A, we have $\mu(B_1 \setminus G) = \mu(B_1 \setminus B'_1) = 0$. Since $A \setminus G \subset B_1 \setminus G$, it follows that $\mu^*(A \setminus G) = 0$. Since G is μ^* -measurable, we get

$$\mu^*(A) = \mu^*(A \cap G) + \mu^*(A \setminus G) = \mu^*(A \cap G).$$

Since G is a countable union of F 's, by the σ -subadditivity of μ^* , we have

$$\mu^*(A) = \mu^*(A \cap G) = \mu^*\left(\bigcup_{F \in S_{\mathcal{F}}^{\mu}(B_1)} [A \cap F]\right) \leq \sum_{F \in S_{\mathcal{F}}^{\mu}(B_1)} \mu^*(A \cap F) \leq \sum_{F \in \mathcal{F}} \mu^*(A \cap F).$$

B. The implication (i) $\Rightarrow$ (ii) is trivial. To prove the implication (ii) $\Rightarrow$ (i), we must show that condition (ii) implies

$$\mu^*(N \cap B) = 0, \quad \forall B \in \mathcal{B}_{\text{fin}}.$$

But if we fix some $B \in \mathcal{B}_{\text{fin}}$, then of course we have $\mu^*N \cap B \leq \mu^*(B) = \mu(B) < \infty$, so using part C, we have

$$\mu^*(N \cap B) = \sum_{F \in \mathcal{F}} \mu^*(N \cap B \cap F) \leq \sum_{F \in \mathcal{F}} \mu^*(N \cap F) = 0,$$

and we are done. $\square$

COMMENTS. Let $\mathcal{B}$ be a σ -algebra on X , let μ be a measure on $\mathcal{B}$. Assume $\mathcal{F}$ is a sufficient μ -finite $\mathcal{B}$ -partition of X .

By Proposition 5.6.C, it follows that $\mathcal{F}$ is also a sufficient $\tilde{\mu}$ -finite $\mathfrak{M}_\mu$ -partition of X .

We see now that $\mathfrak{M}_\mu$ may contain more “new” sets, apart from the “natural candidates,” which are of the form $B \setminus N$, with $B \in \mathcal{B}$ and N locally μ^* -negligible. Such “new” sets are those which belong (see Section 2) to the σ -algebra $\bigvee_{F \in \mathcal{F}} (\mathcal{B}|_F)$. More precisely, we have the following.

COROLLARY 5.3. *Let $\mathcal{B}$ be a σ -algebra on X , let μ be a measure on $\mathcal{B}$. Assume $\mathcal{F}$ is a sufficient μ -finite $\mathcal{B}$ -partition of X .*

A. *One has the equality*

$$\mathfrak{M}_\mu = \bigvee_{F \in \mathcal{F}} (\mathfrak{M}_\mu|_F).$$

B. *For a subset $A \subset X$, the following are equivalent*

- (i) $A \in \mathfrak{M}_\mu$;
- (ii) *there exist a set $B \in \bigvee_{F \in \mathcal{F}} (\mathcal{B}|_F)$, and a locally μ^* -negligible set $N \subset X$, such that $A = B \setminus N$.*

PROOF. A. This is exactly property A from Proposition 5.6.

B. (i) $\Rightarrow$ (ii). Assume $A \in \mathfrak{M}_\mu$, i.e. A is μ^* -measurable. For every $F \in \mathcal{F}$, the set $A \cap F$ is μ^* -measurable. Since $\mu^*(A \cap F) < \infty$, by Theorem 5.5, it follows that $A \cap F = B_F \setminus N_F$, with $B_F \in \mathcal{B}$ and $\mu^*(N_F) = 0$. Replacing B_F with $B_F \cap F$, and N_F with $N_F \cap F$, we can assume that $B_F, N_F \subset F$. Form then the sets $B = \bigcup_{F \in \mathcal{F}} B_F$ and $N = \bigcup_{F \in \mathcal{F}} N_F$. On the one hand, we have $B \cap B = B_F \in \mathcal{B}$, $\forall F \in \mathcal{F}$, which means precisely that $B \in \bigvee_{F \in \mathcal{F}} (\mathcal{B}|_F)$. On the other hand, we also have $N \cap F = N_F$, so we get $\mu^*(N \cap F) = 0$, $\forall F \in \mathcal{F}$. By Proposition 5.6.B, it follows that N is locally μ^* -negligible. We clearly have $A = B \setminus N$.

The implication (ii) $\Rightarrow$ (i) is obvious. $\square$

There is yet another nicer consequence of Proposition 5.6, for which we are going to use the following terminology.

DEFINITION. Let $\mathcal{A}$ be a σ -algebra on X , and let μ be a measure on $\mathcal{A}$. A family $\mathcal{F}$ is called a μ -finite decomposition for $\mathcal{A}$, if

- (i) $\mathcal{F}$ is a sufficient μ -finite $\mathcal{A}$ -partition of X , and
- (ii) one has the equality $\bigvee_{F \in \mathcal{F}} (\mathcal{A}|_F) = \mathcal{A}$.

(Given a collection $\mathcal{F} \subset \mathcal{A}$, one always has the inclusion $\bigvee_{F \in \mathcal{F}} (\mathcal{A}|_F) \subset \mathcal{A}$.)

A measure μ on $\mathcal{A}$ is said to be *decomposable*, if there exists at least one μ -finite decomposition for $\mathcal{A}$.

REMARK 5.8. Decomposability is a generalization of σ -finiteness. This follows from Remark 5.6.B, combined with the fact that whenever $\mathcal{F} \subset \mathcal{A}$ is a *countable* sub-collection, one always has the equality $\bigvee_{F \in \mathcal{F}} (\mathcal{A}|_F) = \mathcal{A}$.

With this terminology, Corollary 5.3 states that *if $\mathcal{F}$ is a sufficient μ -finite $\mathcal{B}$ -partition of X , then $\mathcal{F}$ is a $\tilde{\mu}$ -finite decomposition for $\mathfrak{M}_\mu$.*

With the above terminology, Corollary 5.2 has the following generalization

THEOREM 5.6. *Let μ be a decomposable measure on the σ -algebra $\mathcal{B}$.*

A. *For a subset $A \subset X$, the following are equivalent*

- (i) A is μ^* -measurable;

(ii) *there exist $B \in \mathcal{B}$, and some locally μ^* -negligeable set N , such that $A = B \setminus N$.*

B. *For a subset $N \subset X$, the following are equivalent*

- (i) *N is locally μ^* -negligeable;*
- (ii) *there exists a locally μ -null set $D \in \mathcal{B}$ with $N \subset D$.*

PROOF. A. This is clear, by Corollary 5.3.

B. The implication (ii) $\Rightarrow$ (i) is trivial, because any locally μ -null set D is locally μ^* -negligeable, and so is every subset of D .

To prove the implication (i) $\Rightarrow$ (ii) start with a locally μ^* -negligeable set N , and we fix $\mathcal{F}$ a μ -finite decomposition of $\mathcal{B}$. We know that $\mu^*(N \cap F) = 0, \forall F \in \mathcal{F}$. In particular, using Remark 5.4.B, for each $F \in \mathcal{F}$, there exists some set $E_F \in \mathcal{B}$, with $N \cap F \subset E_F$, and $\mu(E_F) = 0$. Consider now the set $D = \bigcup_{F \in \mathcal{F}} (E_F \cap F)$. By construction, we have $D \cap F = E_F \cap F \in \mathcal{B}, \forall F \in \mathcal{F}$, which means that $D \in \bigvee_{F \in \mathcal{F}} (\mathcal{B}|_F)$. It is here where we use condition (i) in the definition of μ -finite decompositions, to conclude that D belongs to $\mathcal{B}$. Of course, we have

$$\mu(D \cap F) = \mu(E_F \cap F) \leq \mu(E_F) = 0, \forall F \in \mathcal{F},$$

which by Proposition 5.6 means that D is locally μ^* -negligeable. This means that

$$\mu(D \cap B) = \mu^*(D \cap B) = 0, \forall B \in \mathcal{B}_{\text{fin}},$$

which means that D is locally μ -null. Since $N \cap F \subset E_F \cap F \subset D, \forall F \in \mathcal{F}$, and $\mathcal{F}$ is a partition of X , we get $N \subset D$. $\square$

LECTURES 23-25

6. The Lebesgue measure

In this section we apply various results from the previous sections to a very basic example: the *Lebesgue measure on $\mathbb{R}^n$* .

NOTATIONS. We fix an integer $n \geq 1$. In Section 21 we introduced the semiring of “half-open boxes” in $\mathbb{R}^n$:

$$\mathcal{J}_n = \{\emptyset\} \cup \left\{ \prod_{j=1}^n [a_j, b_j) : a_1 < b_1, \dots, a_n < b_n \right\} \subset \mathcal{P}(\mathbb{R}^n).$$

For a non-empty box $A = [a_1, b_1) \times \dots \times [a_n, b_n) \in \mathcal{J}_n$, we defined its n -dimensional volume by

$$\text{vol}_n(A) = \prod_{k=1}^n (b_k - a_k).$$

We also defined $\text{vol}_n(\emptyset) = 0$.

By Theorem 4.2, we know that vol_n is a *finite measure on $\mathcal{J}_n$* .

DEFINITIONS. The maximal outer extension of vol_n is called the *n -dimensional outer Lebesgue measure*, and is denoted by λ_n^* .

The λ_n^* -measurable sets in $\mathbb{R}^n$ will be called *n -Lebesgue measurable*. The σ -algebra $\mathcal{M}_{\lambda_n^*}(\mathbb{R}^n)$ will be denoted simply by $\mathcal{M}(\mathbb{R}^n)$. The measure $\lambda_n^*|_{\mathcal{M}(\mathbb{R}^n)}$ is simply denoted by λ_n , and is called the *n -dimensional Lebesgue measure*. Although this notation may appear to be confusing, it turns out (see Proposition 5.3) that λ_n^* is indeed the maximal outer extension of λ_n . In the case when $n = 1$, the subscript will be omitted.

We know (see Section 21) that

$$\mathbf{S}(\mathcal{J}_n) = \mathbf{\Sigma}(\mathcal{J}_n) = \text{Bor}(\mathbb{R}^n).$$

Using the fact that the semiring $\mathcal{J}_n$ is σ -total in $\mathbb{R}^n$, by the definition of the outer Lebesgue measure, we have

$$(1) \quad \lambda_n^*(A) = \inf \left\{ \sum_{k=1}^{\infty} \text{vol}_n(B_k) : (B_k)_{k=1}^{\infty} \subset \mathcal{J}_n, \bigcup_{k=1}^{\infty} B_k \supset A \right\}, \quad \forall A \subset \mathbb{R}^n$$

Using Corollary 5.2, we have the equality

$$\mathcal{M}(\mathbb{R}^n) = \overline{\text{Bor}(\mathbb{R}^n)},$$

where $\overline{\text{Bor}(\mathbb{R}^n)}$ is the completion of $\text{Bor}(\mathbb{R}^n)$ with respect to the measure $\lambda_n|_{\text{Bor}(\mathbb{R}^n)}$. This means that a subset $A \subset \mathbb{R}^n$ is Lebesgue measurable, if and only if there exists a Borel set B and a negligible set N such that $A = B \cup N$. (The fact N is

negligible means that $\lambda_n^*(N) = 0$, and is equivalent to the existence of a Borel set $C \supset N$ with $\lambda_n(C) = 0$.)

Exercise 1. Let $A = [a_1, b_1) \times \cdots \times [a_n, b_n)$ be a half-open box in $\mathbb{R}^n$. Assume $A \neq \emptyset$ (which means that $a_1 < b_1, \dots, a_n < b_n$). Consider the open box $\text{Int}(A)$ and the closed box $\overline{A}$, which are given by

$$\text{Int}(A) = (a_1, b_1) \times \cdots \times (a_n, b_n) \text{ and } \overline{A} = [a_1, b_1] \times \cdots \times [a_n, b_n].$$

Prove the equalities

$$\lambda_n(\text{Int}(A)) = \lambda_n(\overline{A}) = \text{vol}_n(A).$$

REMARKS 6.1. If $D \subset \mathbb{R}^n$ is a non-empty open set, then $\lambda_n(D) > 0$. This is a consequence of the above exercise, combined with the fact that D contains at least one non-empty open box.

The Lebesgue measure of a countable subset $C \subset \mathbb{R}^n$ is zero. Using σ -additivity, it suffices to prove this only in the case of singletons $C = \{x\}$. If we write x in coordinates $x = (x_1, \dots, x_n)$, and if we consider half-open boxes of the form

$$J_\varepsilon = [x_1, x_1 + \varepsilon) \times \cdots \times [x_n, x_n + \varepsilon),$$

then the obvious inclusion $\{x\} \subset J_\varepsilon$ will force

$$0 \leq \lambda_n(\{x\}) \leq \lambda_n(J_\varepsilon) = \varepsilon^n,$$

so taking the limit as $\varepsilon \rightarrow 0$, we indeed get $\lambda_n(\{x\}) = 0$.

The (outer) Lebesgue measure is completely determined by its values on open sets. More explicitly, one has the following result.

PROPOSITION 6.1. *Let $n \geq 1$ be an integer. For every subset $A \subset \mathbb{R}^n$ one has:*

$$(2) \quad \lambda_n^*(A) = \inf\{\lambda_n(D) : D \text{ open subset of } \mathbb{R}^n, \text{ with } D \supset A\}.$$

PROOF. Throughout the proof the set A will be fixed. Let us denote, for simplicity, the right hand side of (2) by $\nu(A)$. First of all, since every open set is Lebesgue measurable (being Borel), we have $\lambda_n(D) = \lambda_n^*(D)$, for all open sets D , so by the monotonicity of λ_n^* , we get the inequality

$$\lambda_n^*(A) \leq \nu(A).$$

We now prove the inequality $\lambda_n^*(A) \geq \nu(A)$. Fix for the moment some $\varepsilon > 0$, and use (1). to get the existence of a sequence $(B_k)_{k=1}^\infty \subset \mathcal{J}_n$, such that $\bigcup_{k=1}^\infty B_k \supset A$, and

$$\sum_{k=1}^\infty \text{vol}_n(B_k) < \lambda_n^*(A) + \varepsilon.$$

For every $k \geq 1$, we write

$$B_k = [a_1^{(k)}, b_1^{(k)}) \times \cdots \times [a_n^{(k)}, b_n^{(k)}),$$

so that $\text{vol}_n(B_k) = \prod_{j=1}^n (b_j^{(k)} - a_j^{(k)})$. Using the obvious continuity of the map

$$\mathbb{R} \ni t \longmapsto \prod_{j=1}^n (b_j^{(k)} - a_j^{(k)} - t) \in \mathbb{R},$$

we can find, for each $k \geq 1$ some numbers $c_1^{(k)} < a_1^{(k)}, \dots, c_n^{(k)} < a_n^{(k)}$, with

$$(3) \quad \prod_{j=1}^n (b_j^{(k)} - c_j^{(k)}) < \frac{\varepsilon}{2^k} + \prod_{j=1}^n (b_j^{(k)} - a_j^{(k)}).$$

Notice that, if we define the half-open boxes

$$E_k = [c_1^{(k)}, b_1^{(k)}) \times \cdots \times [c_n^{(k)}, b_n^{(k)}),$$

then for every $k \geq 1$, we clearly have $B_k \subset \text{Int}(E_k)$, and by Exercise 1, combined with (3), we also have the inequality

$$\lambda_n(\text{Int}(E_k)) = \text{vol}_n(E_k) < \frac{\varepsilon}{2^k} + \text{vol}_n(B_k).$$

Summing up we then get

$$(4) \quad \sum_{k=1}^{\infty} \lambda_n(\text{Int}(E_k)) < \sum_{k=1}^{\infty} \left[\frac{\varepsilon}{2^k} + \text{vol}_n(B_k) \right] = \varepsilon + \sum_{k=1}^{\infty} \text{vol}_n(B_k) < 2\varepsilon + \lambda_n^*(A).$$

Now we observe that by σ -sub-additivity we have

$$\lambda_n\left(\bigcup_{k=1}^{\infty} \text{Int}(E_k)\right) \leq \sum_{k=1}^{\infty} \lambda_n(\text{Int}(E_k)),$$

so if we define the open set $D = \bigcup_{k=1}^{\infty} \text{Int}(E_k)$, then using (4) we get

$$(5) \quad \lambda_n(D) < 2\varepsilon + \lambda_n^*(A).$$

It is clear that we have the inclusions

$$A \subset \bigcup_{k=1}^{\infty} B_k \subset \bigcup_{k=1}^{\infty} \text{Int}(E_k) = D,$$

so by the definition of $\nu(A)$, combined with (5), we finally get

$$\nu(A) \leq \lambda_n(D) < 2\varepsilon + \lambda_n^*(A).$$

Up to this moment $\varepsilon > 0$ was fixed. Since the inequality $\nu(A) < 2\varepsilon + \lambda_n^*(A)$ holds for any $\varepsilon > 0$ however, we finally get the desired inequality $\nu(A) \leq \lambda_n^*(A)$. $\square$

The Lebesgue measure can also be recovered from its values on compact sets.

PROPOSITION 6.2. *Let $n \geq 1$ be an integer. For every Lebesgue measurable subset $A \subset \mathbb{R}^n$ one has:*

$$(6) \quad \lambda_n(A) = \sup\{\lambda_n(K) : K \text{ compact subset of } \mathbb{R}^n, \text{ with } K \subset A\}.$$

PROOF. Let us denote, for simplicity, the right hand side of (6) by $\mu(A)$. First of all, by the monotonicity we clearly have the inequality

$$\lambda_n(A) \geq \mu(A).$$

To prove the inequality $\lambda_n(A) \leq \mu(A)$, we shall first use a reduction to the bounded case. For each integer $k \geq 1$, we define the compact box

$$B_k = [-k, k] \times \cdots \times [-k, k].$$

Notice that we have $B_1 \subset B_2 \subset \cdots$, with $\bigcup_{k=1}^{\infty} B_k = \mathbb{R}^n$. We then have

$$B_1 \cap A \subset B_2 \cap A \subset \cdots,$$

with $\bigcup_{k=1}^{\infty} (B_k \cap A) = A$, so using the Continuity Lemma 4.1, we have

$$(7) \quad \lambda_n(A) = \lim_{k \rightarrow \infty} \lambda_n(B_k \cap A) = \sup\{\lambda_n(B_k \cap A) : k \geq 1\}.$$

Fix for the moment some $\varepsilon > 0$, and use the (7) to find some $k \geq 1$, such that $\lambda_n(A) \leq \lambda_n(B_k \cap A) + \varepsilon$. Apply Proposition 6.1 to the set $B_k \setminus A$, to find an open set D , with $D \supset B_k \setminus A$, and $\lambda_n(B_k \setminus A) \geq \lambda_n(D) - \varepsilon$. On the one hand, we have

$$(8) \quad \begin{aligned} \lambda_n(B_k) &= \lambda_n(B_k \cap A) + \lambda_n(B_k \setminus A) \geq \lambda_n(B_k \cap A) + \lambda_n(D) - \varepsilon \geq \\ &\geq \lambda_n(B_k \cap A) + \lambda_n(B_k \cap D) - \varepsilon. \end{aligned}$$

On the other hand, we have

$$\lambda_n(B_k) = \lambda_n(B_k \setminus D) + \lambda_n(B_k \cap D),$$

so using (8) we get the inequality

$$\lambda_n(B_k \setminus D) + \lambda_n(B_k \cap D) \geq \lambda_n(B_k \cap A) + \lambda_n(B_k \cap D) - \varepsilon,$$

and since all numbers involved in the above inequality are finite, we conclude that

$$\lambda_n(B_k \setminus D) \geq \lambda_n(B_k \cap A) - \varepsilon \geq \lambda_n(A) - 2\varepsilon.$$

Obviously the set $K = B_k \setminus D$ is compact, with $K \subset B_k \cap A \subset A$, so we have $\mu(A) \geq \lambda_n(K)$, hence we get the inequality

$$\mu(A) \geq \lambda_n(A) - 2\varepsilon.$$

Since this is true for all $\varepsilon > 0$, the desired inequality $\mu(A) \geq \lambda_n(A)$ follows. $\square$

COROLLARY 6.1. *For a set $A \subset \mathbb{R}^n$, the following are equivalent:*

- (i) *A is Lebesgue measurable;*
- (ii) *there exists a negligible set N and a sequence of $(K_j)_{j=1}^\infty$ of compact subsets of $\mathbb{R}^n$, such that*

$$A = N \cup \bigcup_{j=1}^{\infty} K_j.$$

PROOF. (i) $\Rightarrow$ (ii). Start by using the boxes

$$B_k = [-k, k] \times \cdots \times [-k, k]$$

which have the property that $\bigcup_{k=1}^{\infty} B_k = \mathbb{R}^n$, so we get $A = \bigcup_{k=1}^{\infty} (B_k \cap A)$. Fix for the moment k . Apply Proposition 6.2. to find a sequence $(C_r^k)_{r=1}^\infty$ of compact subsets of $B_k \cap A$, such that $\lim_{r \rightarrow \infty} \lambda_n(C_r^k) = \lambda_n(B_k \cap A)$. Consider the countable family $(C_r^k)_{k,r=1}^\infty$ of compact sets, and enumerate it as a sequence $(K_j)_{j=1}^\infty$, so that we have

$$\bigcup_{j=1}^{\infty} K_j = \bigcup_{k=1}^{\infty} \bigcup_{r=1}^{\infty} C_r^k.$$

If we define, for each $k \geq 1$, the sets $E_k = \bigcup_{r=1}^{\infty} C_r^k \subset B_k \cap A$ and $N_k = (B_k \cap A) \setminus E_k$, then, because of the inclusion $C_r^k \subset E_k \subset B_k \cap A$, we have the inequalities

$$(9) \quad 0 \leq \lambda_n(N_k) = \lambda_n(B_k \cap A) - \lambda_n(E_k) \leq \lambda_n(B_k \cap A) - \lambda_n(C_r^k), \quad \forall r \geq 1.$$

Using the fact that

$$\lim_{r \rightarrow \infty} \lambda_n(C_r^k) = \lambda_n(B_k \cap A) \leq \lambda_n(B_k) < \infty,$$

the inequalities (9) force $\lambda_n(N_k) = 0$, $\forall k \geq 1$. Now if we define the set $N = A \setminus (\bigcup_{j=1}^{\infty} K_j)$, we have

$$\begin{aligned} N &= \bigcup_{k=1}^{\infty} \left[(B_k \cap A) \setminus \left(\bigcup_{j=1}^{\infty} K_j \right) \right] = \bigcup_{k=1}^{\infty} \left[(B_k \cap A) \setminus \left(\bigcup_{p=1}^{\infty} E_p \right) \right] \subset \\ &\subset \bigcup_{k=1}^{\infty} [(B_k \cap A) \setminus E_k] = \bigcup_{k=1}^{\infty} N_k, \end{aligned}$$

which proves that $\lambda_n(N) = 0$.

The implication (ii) $\Rightarrow$ (i) is trivial. $\square$

Proposition 6.2 does not hold if $A \subset \mathbb{R}^n$ is non-measurable. In fact the equality (6), with λ_n replaced by λ_n^* , essentially forces A to be measurable, as shown by the following.

Exercise 2. Let $A \subset \mathbb{R}^n$ be an arbitrary subset, with $\lambda_n^*(A) < \infty$. Prove that the following are equivalent:

- (i) A is Lebesgue measurable;
- (ii) $\lambda_n^*(A) = \sup\{\lambda_n(K) : K \text{ compact subset of } \mathbb{R}^n, \text{ with } K \subset A\}$.

Propositions 6.1 and 6.2 are *regularity properties*. The following terminology is useful:

DEFINITIONS. Suppose $\mathcal{A}$ is a σ -algebra on X , and μ is a measure on $\mathcal{A}$. Suppose we have a sub-collection $\mathcal{F} \subset \mathcal{A}$.

- (i) We say that μ is *regular from below, with respect to* $\mathcal{F}$, if

$$\mu(A) = \sup\{\mu(F) : F \subset A, F \in \mathcal{F}\}.$$

- (ii) We say that μ is *regular from above, with respect to* $\mathcal{F}$, if

$$\mu(A) = \inf\{\mu(F) : F \supset A, F \in \mathcal{F}\}.$$

With this terminology, Proposition 6.1 gives the fact that the Lebesgue measure is regular from above with respect to open sets, while Proposition 6.2 gives the fact that the Lebesgue measure is regular from below with respect to compact sets.

Exercise 3. For a subset $A \subset \mathbb{R}^n$, prove that the following are equivalent:

- (i) A is Lebesgue measurable;
- (ii) There exist a sequence of compact sets $(K_j)_{j=1}^{\infty}$, and a sequence of open sets $(D_j)_{j=1}^{\infty}$, such that $\bigcup_{j=1}^{\infty} K_j \subset A \subset \bigcap_{j=1}^{\infty} D_j$, and the difference $(\bigcap_{j=1}^{\infty} D_j) \setminus (\bigcup_{j=1}^{\infty} K_j)$ is negligible.

HINT: For the implication (i) $\Rightarrow$ (ii) analyze first the case when $\lambda^*(A) < \infty$. Then write A as a countable union of sets of finite outer measure.

In the one-dimensional case $n = 1$, the Lebesgue measure of open sets can be computed with the aid of the following result.

PROPOSITION 6.3. *For every open set $D \subset \mathbb{R}$, there exists a countable (or finite) pair-wise disjoint collection $\{J_i\}_{i \in I}$ of open intervals with $D = \bigcup_{i \in I} J_i$.*

PROOF. For every point $x \in D$, we define

$$a_x = \inf\{a < x : (a, x) \subset D\} \text{ and } b_x = \sup\{b > x : (x, b) \subset D\}.$$

(The fact that D is open guarantees the fact that both sets above are non-empty.) It is clear that, for every $x \in D$, the open interval $J_x = (a_x, b_x)$ is contained in D , so

we have the equality $D = \bigcup_{x \in D} J_x$. The problem at this point is the fact that the collection $\{J_x\}_{x \in D}$ is not pair-wise disjoint. What we need to find is a countable (or finite) subset $X \subset D$, such that the sub-collection $\{J_x\}_{x \in X}$ is pair-wise disjoint, and we still have $D = \bigcup_{x \in X} J_x$. One way to do this is based on the following

Claim: For two points $x, y \in D$, the following are equivalent:

- (i) $x \in J_y$;
- (ii) $J_x \supset J_y$;
- (ii) $J_x \cap J_y \neq \emptyset$;
- (iii) $J_x = J_y$.

To prove the implication (i) $\Rightarrow$ (ii) we observe that if $x \in J_y$, then $a_y < x < b_y$, so we have $(a_y, x) \subset D$ and $(x, b_y) \subset D$, which means that $a_x \leq a_y$ and $b_x \geq b_y$, therefore we have the inclusion $J_x = (a_x, b_x) \supset (a_y, b_y) = J_y$. The implication (ii) $\Rightarrow$ (iii) is trivial. To prove (iii) $\Rightarrow$ (iv), assume $J_x \cap J_y \neq \emptyset$, and pick a point $z \in J_x \cap J_y$. Using the implication (i) $\Rightarrow$ (ii) we have the inclusions $J_z \supset J_x$ and $J_z \supset J_y$. In particular we have $x \in J_z$, so again using the implication (i) $\Rightarrow$ (ii) we get $J_x \supset J_z$, which means that we have in fact the equality $J_x = J_z$. Likewise we have the equality $J_y = J_z$, so (iv) follows. The implication (iv) $\Rightarrow$ (i) is trivial.

Going back to the proof of the Proposition, we now see that, using the fact that any open interval contains a rational number, if we put $X_0 = D \cap \mathbb{Q}$, then for any $y \in D$, there exists $x \in X_0$, such that $J_x = J_y$. This gives the equality $D = \bigcup_{x \in X_0} J_x$, this time with the indexing set X_0 countable. Finally, if we equip the set X_0 with the equivalence relation

$$x \sim y \iff J_x = J_y,$$

and we choose $X \subset X_0$ to be a list of all equivalence classes. This means that, for every $y \in X_0$, there exists a unique $x \in X$ with $J_x = J_y$. It is clear now that we still have $D = \bigcup_{x \in X} J_x$, but now if $x, x' \in X$ are such that $x \neq x'$, then $x \not\sim x'$, so we have $J_x \neq J_{x'}$, which by the Claim gives $J_x \cap J_{x'} = \emptyset$. $\square$

COMMENTS. When we want to compute the Lebesgue measure of an open set $D \subset \mathbb{R}$, we should first try to write $D = \bigcup_{i \in I} J_i$ with $(J_i)_{i \in I}$ a countable (or finite) pair-wise collection of open intervals. If we succeed, then we would have

$$\lambda(D) = \sum_{i \in I} \lambda(J_i).$$

For intervals (open or not) the Lebesgue measure is the same as the length.

There are instances when we can manage only to write a given open set D as a union $D = \bigcup_{k=1}^{\infty} J_k$, with the J 's not necessarily disjoint. In that case we can only get the estimate

$$\lambda(D) \leq \sum_{k=1}^{\infty} \lambda(J_k).$$

EXAMPLE 6.1. Consider the ternary Cantor set $K_3 \subset [0, 1]$, discussed in III.3. We know (see Remarks 3.5) that one can find a pair-wise sequence $(D_n)_{n=0}^{\infty}$ of open subsets of $(0, 1)$ such that $K_3 = [0, 1] \setminus \bigcup_{n=0}^{\infty} D_n$, and such that, for each $n \geq 0$, the open set D_n is a disjoint union of 2^n intervals of length $1/3^{n+1}$. In particular, this means that $\lambda(D_n) = 2^n/3^{n+1}$, so

$$\lambda(K_3) = \lambda([0, 1]) - \lambda\left(\bigcup_{n=0}^{\infty} D_n\right) = 1 - \sum_{n=0}^{\infty} \lambda(D_n) = 1 - \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = 0.$$

What is interesting here (see Remarks 3.5) is the fact that $\text{card } K_3 = \mathfrak{c}$.

REMARK 6.2. An interesting consequence of the above computation is the fact that *all subsets of K_3 are Lebesgue measurable*, i.e. one has the inclusion $\mathcal{P}(K_3) \subset \mathcal{M}(\mathbb{R})$. This gives the inequality

$$\text{card } \mathcal{M}(\mathbb{R}) \geq \text{card } \mathcal{P}(K_3) = 2^{\text{card } K_3} = 2^{\mathfrak{c}}.$$

Since we also have $\mathcal{M}(\mathbb{R}) \subset \mathcal{P}(\mathbb{R})$, we get

$$\text{card } \mathcal{M}(\mathbb{R}) \leq \text{card } \mathcal{P}(\mathbb{R}) = 2^{\text{card } \mathbb{R}} = 2^{\mathfrak{c}},$$

so using the Cantor-Bernstein Theorem we get the equality

$$\text{card } \mathcal{M}(\mathbb{R}) = 2^{\mathfrak{c}}.$$

We also know (see Corollary 2.5) that $\text{card } \text{Bor}(\mathbb{R}) = \mathfrak{c}$.

As a consequence of this difference in cardinalities, one gets the fact that we have a strict inclusion

$$(10) \quad \text{Bor}(\mathbb{R}) \subsetneq \mathcal{M}(\mathbb{R}).$$

Later on we shall construct (more or less) explicitly a Lebesgue measurable set which is not Borel.

Exercise 4. The strict inclusion (10) holds also if $\mathbb{R}$ is replaced with $\mathbb{R}^n$, with $n \geq 2$. In this case, instead of using Cantor sets, one can proceed as follows. Consider the set $S = \mathbb{R}^{n-1} \times \{0\}$. Prove that $\lambda_n(S) = 0$. Conclude that $\text{card } \mathcal{M}(\mathbb{R}^n) = 2^{\mathfrak{c}}$.

One key feature of the Lebesgue (outer) measure is the *translation invariance property*, described in the following result. To formulate it we introduce the following notation. For an integer $n \geq 1$, a point $x \in \mathbb{R}^n$, and a subset $A \subset \mathbb{R}^n$, we define the set

$$A + x = \{a + x : a \in A\}.$$

Remark that the map $\Theta_x : \mathbb{R}^n \ni a \mapsto a + x \in \mathbb{R}^n$ is a homeomorphism. In particular, both Θ_x and $\Theta_x^{-1} = \Theta_{-x}$ are Borel measurable, which means that, for a set $A \subset \mathbb{R}^n$, one has the equivalence

$$A \in \text{Bor}(\mathbb{R}^n) \iff A + x \in \text{Bor}(\mathbb{R}^n).$$

PROPOSITION 6.4. *Let $n \geq 1$ be an integer. For any set $A \subset \mathbb{R}^n$ one has the equality*

$$\lambda_n^*(A + x) = \lambda_n^*(A).$$

PROOF. Fix A and x . First remark that, for every half-open box $B \in \mathcal{J}_n$, its translation $B + x$ is again a half-open box, and we have the equality

$$\text{vol}_n(B + x) = \text{vol}_n(B).$$

Fix for the moment $\varepsilon > 0$, and choose a sequence $(B_k)_{k=1}^{\infty} \subset \mathcal{J}_n$, such that $A \subset \bigcup_{k=1}^{\infty} B_k$, and

$$\sum_{k=1}^{\infty} \text{vol}_n(B_k) \leq \lambda_n^*(A) + \varepsilon.$$

Then, using the obvious inclusion $A + x \subset \bigcup_{k=1}^{\infty} (B_k + x)$, by the remark made at the beginning of the proof, combined with the monotonicity of the outer Lebesgue measure, we have

$$\begin{aligned} \lambda_n^*(A + x) &\leq \lambda_n^*\left(\bigcup_{k=1}^{\infty} (B_k + x)\right) \leq \sum_{k=1}^{\infty} \lambda_n^*(B_k + x) = \\ &= \sum_{k=1}^{\infty} \text{vol}_n(B_k + x) = \sum_{k=1}^{\infty} \text{vol}_n(B_k) \leq \lambda_n^*(A) + \varepsilon. \end{aligned}$$

Since the inequality $\lambda_n^*(A + x) \leq \lambda_n^*(A) + \varepsilon$ holds for all $\varepsilon > 0$, we get

$$\lambda_n^*(A + x) \leq \lambda_n^*(A).$$

The other inequality follows from the above one applied to the set $A + x$ and the translation by $-x$. $\square$

COROLLARY 6.2. *For a subset $A \subset \mathbb{R}^n$, one has the equivalence*

$$A \in \mathcal{M}(\mathbb{R}^n) \iff A + x \in \mathcal{M}(\mathbb{R}^n).$$

PROOF. Write $A = B \cup N$, with B Borel, and N negligible. Then we have $A + x = (B + x) \cup (N + x)$. The set $B + x$ is Borel. By the above result we have $\lambda_n^*(N + x) = \lambda_n^*(N) = 0$, i.e. $N + x$ is negligible. Therefore $A + x$ is Lebesgue measurable. $\square$

As we have seen, the fact that there exist Lebesgue measurable sets that are not Borel is explained by the difference in cardinalities. Since $\text{card } \mathcal{M}(\mathbb{R}^n) = 2^{\mathfrak{c}} = \text{card } \mathcal{P}(\mathbb{R}^n)$, it is legitimate to ask whether the inclusion $\mathcal{M}(\mathbb{R}^n) \subset \mathcal{P}(\mathbb{R}^n)$ is strict. In other words, *do there exist sets that are not Lebesgue measurable?* The answer is affirmative, as discussed in the following.

EXAMPLE 6.2. Equip $\mathbb{R}$ with the equivalence relation

$$x \sim y \iff x - y \in \mathbb{Q}.$$

Denote by $\mathbb{R}/\mathbb{Q}$ the quotient space (this is in fact the quotient group of $(\mathbb{R}, +)$ with respect to the subgroup $\mathbb{Q}$), and denote by $\pi : \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Q}$ the quotient map. Since for every $x \in \mathbb{R}$, one can find some $y \sim x$, with $y \in [0, 1)$, it follows that the map $\pi|_{[0,1)} : [0, 1) \rightarrow \mathbb{R}/\mathbb{Q}$ is surjective. Choose then a map $\phi : \mathbb{R}/\mathbb{Q} \rightarrow [0, 1)$, such that $\phi \circ \pi = \text{Id}$, and put $E = \phi(\mathbb{R}/\mathbb{Q})$. The set E is a complete set of representatives for the equivalence relation $\sim$. In other words, $E \subset [0, 1)$ has the property that, for every $x \in \mathbb{R}$, there exists exactly one element $y \in E$, with $x \sim y$. In particular, the collection of sets $(E + q)_{q \in \mathbb{Q}}$ is pair-wise disjoint, and satisfies $\bigcup_{q \in \mathbb{Q}} (E + q) = \mathbb{R}$. Using σ -sub-additivity, we get

$$\infty = \lambda(\mathbb{R}) \leq \sum_{q \in \mathbb{Q}} \lambda^*(E + q).$$

Since (by Proposition 6.5) we have $\lambda^*(E + q) = \lambda^*(E)$, the above inequality forces $\lambda^*(E) > 0$.

Claim: The set E is not Lebesgue measurable

Assume E is Lebesgue measurable. If we define the set $X = \mathbb{Q} \cap [0, 1)$, then the sets $E + q$, $q \in X$ are pair-wise disjoint. On the one hand, the measurability of E , combined with the Corollary 6.2 would imply the measurability of the set $S = \bigcup_{q \in X} (E + q)$. On the other hand, the equalities $\lambda(E + q) = \lambda(E) > 0$ will

force $\lambda(S) = \infty$. But this is impossible, since we obviously have $S \subset [0, 2)$, which forces $\lambda(S) \leq 2$.

Exercise 5. Let $E \in \mathcal{M}(\mathbb{R}^n)$. Prove that the map

$$\mathbb{R}^n \ni x \longmapsto \lambda(E \cup (E + x)) \in [0, \infty]$$

is continuous.

HINT: Analyze first the case when E is compact. In this particular case, show that for every $x_0 \in \mathbb{R}^n$ and every open set $D \supset E \cup (E + x_0)$, there exists some neighborhood V of x_0 , such that

$$D \supset E \cup (E + x), \quad \forall x \in V.$$

Use then regularity from above, combined with the inequality⁹

$$|\lambda(A) - \lambda(B)| \leq \lambda(A \Delta B), \text{ for all } A, B \in \mathcal{M}(\mathbb{R}^n), \text{ with } \lambda(A), \lambda(B) < \infty.$$

In the general case, use regularity from below. (The case $\lambda(E) = \infty$ is trivial.)

Exercise 6. Let $E \in \mathcal{M}(\mathbb{R}^n)$, be such that $\lambda_n(E) > 0$. Prove that the set

$$E - E = \{x - y : x, y \in E\}$$

is a neighborhood of 0.

HINT: Assume the contrary, which means that there exists a sequence $(x_p)_{p=1}^\infty \subset \mathbb{R}^n \setminus (E - E)$, with $\lim_{p \rightarrow \infty} x_p = 0$. This will force $E \cap (E + x_p) = \emptyset$, $\forall p \geq 1$. Use the preceding Exercise to get a contradiction.

We are now in position to construct a Lebesgue measurable set which is not Borel.

EXAMPLE 6.3. In Section 3 we discussed the compact space $T = \{0, 1\}^{\aleph_0}$ and the maps

$$\phi_r : T \ni (\alpha_n)_{n=1}^\infty \longmapsto (r-1) \sum_{n=1}^\infty \frac{\alpha_n}{r^n} \in [0, 1].$$

For each $r \geq 2$ the map $\phi_r : T \rightarrow [0, 1]$ is continuous so the set $K_r = \phi_r(T)$ is compact. We have $K_2 = [0, 1]$, and K_3 is the ternary Cantor set. We also know (see Theorem 3.5) that, for a set $A \subset T$, one has the equivalence

$$(11) \quad A \in \text{Bor}(T) \iff \phi_r(A) \in \text{Bor}(K_r).$$

Choose now a set $E \subset [0, 1]$ which is not Lebesgue measurable. In particular, E is not Borel, so $E \notin \text{Bor}([0, 1])$. Since $\phi_2 : T \rightarrow [0, 1]$ is surjective, by (11) it follows that the set $A = \phi_2^{-1}(E)$ is not in $\text{Bor}(T)$. Again, by (11) it follows that the set $S = \phi_3(A)$ is not in $\text{Bor}(K_3)$. Since

$$\text{Bor}(K_3) = \text{Bor}(\mathbb{R})|_{K_3}$$

this gives $S \notin \text{Bor}(\mathbb{R})$. Notice however that since $S \subset K_3$, it follows that S is Lebesgue measurable.

COMMENT. When one wants to prove that a Lebesgue measurable set $M \subset \mathbb{R}$ has positive measure, a sufficient condition for this property is that $\text{Int}(M) \neq \emptyset$ (see Remark 6.1). It turns out however that this condition is not always necessary, as seen from the following:

⁹ This inequality holds for any additive map defined on a ring.

Exercise 7. Start with an arbitrary interval $[0, 1]$, and list all rational numbers in $[0, 1]$ as a sequence $\mathbb{Q} \cap [0, 1] = \{x_n\}_{n=1}^\infty$. Fix some $\varepsilon > 0$, and consider the open set

$$D = \bigcup_{n=1}^{\infty} \left(x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}}\right).$$

Consider the compact set $K = [0, 1] \setminus D$.

- (i) Prove that $\lambda(D) \leq \varepsilon$.
- (ii) Prove that $\lambda(K) \geq 1 - \varepsilon$.
- (iii) Prove that $\text{Int}(K) = \emptyset$.

HINT: For (iii) use the fact that $K \cap \mathbb{Q} = \emptyset$.

Exercise 8.* Prove that, for every non-empty open set $D \subset \mathbb{R}$, and any two positive numbers α, β with $\alpha + \beta < \lambda(D)$, there exist compact sets $A, B \subset D$, with $\lambda(A) > \alpha$, $\lambda(B) > \beta$, such that $A \cap B = \emptyset$ and $(A \cup B) \cap \mathbb{Q} = \emptyset$.

HINT: Write D as a union of a pair-wise disjoint sequence $(J_n)_{n=1}^\infty$ of open intervals, so that $\lambda(D) = \sum_{n=1}^\infty \lambda(J_n)$. Find then two sequences $(\alpha_n)_{n=1}^\infty$ and $(\beta_n)_{n=1}^\infty$ of positive numbers, such that $\sum_{n=1}^\infty \alpha_n > \alpha$, $\sum_{n=1}^\infty \beta_n > \beta$, and $\alpha_n + \beta_n < \lambda(J_n)$, for all $n \geq 1$. This reduces essentially the problem to the case when D is an open interval, for which one can use the construction outlined in Exercise 7.

Exercise 9.* Construct a Borel set $A \subset \mathbb{R}$, such that, for every open interval $I \subset \mathbb{R}$ one has $\lambda(I \cap A) > 0$ and $\lambda(I \setminus A) > 0$.

HINTS: List all open intervals with rational endpoints as a sequence $(I_n)_{n=1}^\infty$. Start (use exercise 8) off by choosing two compact sets $A_1, B_1 \subset I_1$, with $A_1 \cap B_1 = \emptyset$, $(A_1 \cup B_1) \cap \mathbb{Q} = \emptyset$, and $\lambda(A_1), \lambda(B_1) > 0$. Use Exercise 5 to construct two sequences $(A_n)_{n=1}^\infty$ and $(B_n)_{n=1}^\infty$ of compact sets, such that, for all $n \geq 1$ we have: (i) $A_n \cap B_n = \emptyset$; (ii) $(A_n \cup B_n) \cap \mathbb{Q} = \emptyset$; (iii) $\lambda(A_n), \lambda(B_n) > 0$; (iv) $A_{n+1} \cup B_{n+1} \subset I_{n+1} \setminus [\bigcup_{k=1}^n (A_k \cup B_k)]$. Put $A = \bigcup_{n=1}^\infty A_n$ and $B = \bigcup_{n=1}^\infty B_n$. Notice that $A \cap B = \emptyset$, $\lambda(A), \lambda(B) > 0$, and $\lambda(A \cap I_n), \lambda(B \cap I_n) > 0$, $\forall n \geq 1$.

In the remainder of this section we discuss some applications of the Lebesgue measure to the theory of Riemann integration. The following technical result will be very useful.

LEMMA 6.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a non-negative Riemann integrable function, let $A, B \subset [a, b]$ be two disjoint sets, with $A \cup B = [a, b]$. Then one has the estimates*

$$\lambda^*(A) \cdot \inf_{z \in A} f(z) \leq \int_a^b f(t) dt \leq (b-a) \cdot \sup_{x \in A} f(x) + \lambda^*(B) \cdot \sup_{y \in B} f(y).$$

PROOF. Define the numbers

$$\alpha = \sup_{x \in A} f(x), \quad \beta = \sup_{y \in B} f(y), \quad \text{and} \quad \gamma = \inf_{z \in A} f(z).$$

Recall first that, if for each partition $\Delta = (a = x_0 < x_1 < \cdots < x_n = b)$ of $[a, b]$, we define the lower and the upper Darboux sums of f with respect to Δ :

$$L(\Delta, f) = \sum_{k=1}^n (x_k - x_{k-1}) \cdot \inf_{t \in [x_{k-1}, x_k]} f(t),$$

$$U(\Delta, f) = \sum_{k=1}^n (x_k - x_{k-1}) \cdot \sup_{t \in [x_{k-1}, x_k]} f(t),$$

then one has the equalities

$$(12) \quad \int_a^b f(t) dt = \sup \{L(\Delta, f) : \Delta \text{ partition of } [a, b]\} = \\ = \inf \{U(\Delta, f) : \Delta \text{ partition of } [a, b]\}.$$

Fix now a partition $\Delta = (a = x_0 < x_1 < \cdots < x_n = b)$ of $[a, b]$, and define the set

$$S = \{k \in \{1, \dots, n\} : [x_{k-1}, x_k] \cap A \neq \emptyset\}.$$

It is clear that

$$\inf_{x \in [x_{k-1}, x_k]} f(x) \leq \alpha, \quad \sup_{x \in [x_{k-1}, x_k]} f(x) \geq \gamma, \quad \forall k \in S, \\ \inf_{y \in [x_{k-1}, x_k]} f(y) \leq \beta, \quad \sup_{y \in [x_{k-1}, x_k]} f(y) \geq 0, \quad \forall k \in \{1, \dots, n\} \setminus S,$$

so we get

$$(13) \quad L(\Delta, f) \leq \left[\sum_{k \in S} (x_k - x_{k-1}) \right] \cdot \alpha + \left[\sum_{k \notin S} (x_k - x_{k-1}) \right] \cdot \beta$$

$$(14) \quad U(\Delta, f) \geq \left[\sum_{k \in S} (x_k - x_{k-1}) \right] \cdot \gamma$$

Consider now the sets

$$M = \bigcup_{k \in S} [x_{k-1}, x_k] \text{ and } N = \bigcup_{k \notin S} [x_{k-1}, x_k].$$

Since the intervals involved in both M and N have at most singleton overlaps, it follows that we have the equalities

$$\sum_{k \in S} (x_k - x_{k-1}) = \lambda(M) \text{ and } \sum_{k \notin S} (x_k - x_{k-1}) = \lambda(N),$$

so the estimates (13) and (14) read

$$(15) \quad L(\Delta, f) \leq \lambda(M) \cdot \alpha + \lambda(N) \cdot \beta$$

$$(16) \quad U(\Delta, f) \geq \lambda(M) \cdot \gamma$$

Since we clearly have $A \subset M \subset [a, b]$ and $N \subset B$, we have the inequalities

$$\lambda^*(A) \leq \lambda(M) \leq b - a \text{ and } \lambda(N) \leq \lambda^*(B),$$

so the inequalities (15) and (16) give

$$L(\Delta, f) \leq (b - a) \cdot \alpha + \lambda^*(B) \cdot \beta \text{ and } U(\Delta, f) \geq \lambda^*(A) \cdot \gamma.$$

Since Δ is arbitrary, the desired inequality then follows from (12). $\square$

One application of the above result is the following.

PROPOSITION 6.5. *If $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable, and the set*

$$N = \{x \in [a, b] : f(x) \neq 0\}$$

is negligible, then

$$(17) \quad \int_a^b f(x) dx = 0.$$

PROOF. Since f is bounded, there exists some constant $C > 0$, such that the Riemann integrable functions $C + f$ and $C - f$ are both non-negative. Apply Lemma 6.1 to these two functions with $A = [a, b] \setminus N$ and $B = N$. Since $f|_{[a, b] \setminus N} = 0$, we get $(C \pm f)|_{[a, b] \setminus N} = C$, so we get

$$\int_a^b [C \pm f(x)] dx \leq (b - a) \cdot C,$$

which yields

$$\pm \int_a^b f(x) dx = \int_a^b \{[C \pm f(x)] - C\} dx = \int_a^b [C \pm f(x)] dx - (b - a) \cdot C \leq 0,$$

from which (17) immediately follows. $\square$

In order to make the exposition a bit easier to follow, it will be helpful to introduce the following

CONVENTION. Given two functions $f_1, f_2 : [a, b] \rightarrow \mathbb{R}$, and a relation R on $\mathbb{R}$ (in our case R will be either “=,” or “ $\geq$,” or “ $\leq$ ”), we write

$$f_1 R f_2, \text{ a.e.}$$

if the set

$$A = \{x \in [a, b] : f_1(x) R f_2(x)\}$$

has negligible complement in $[a, b]$, i.e. $\lambda^*([a, b] \setminus A) = 0$. The abbreviation “a.e.” stands for “almost everywhere.”

For example, using this convention, Proposition 6.6 reads: if $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable, and $f = 0$, a.e., then $\int_a^b f(x) dx = 0$.

Exercise 10. A. Prove that “= a.e.” is an equivalence relation, and “ $\geq$ a.e.” and “ $\leq$ a.e.” are transitive relations on the collection of all function $[a, b] \rightarrow \mathbb{R}$.

B. Prove that $f_1 \geq f_2$, a.e. and $f_1 \leq f_2$, a.e. imply $f_1 = f_2$, a.e.

C. Prove that these relations are compatible with the arithmetic operations, in the exact way as their “honest” versions. For example, if R is one of “=,” or “ $\geq$,” or “ $\leq$ ”, and if $f_1 R f_2$, a.e. and $g_1 R g_2$, a.e., then $(f_1 + g_1) R (f_2 + g_2)$, a.e.

Exercise 11. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous functions, such that $f \geq g$, a.e. Prove that $f \geq g$.

Exercise 12. Let $f : [a, b] \rightarrow \mathbb{R}$ be a non-negative Riemann integrable function, with $\int_a^b f(x) dx = 0$. Prove that $f = 0$, a.e.

COMMENT. Riemann integrability is quite a rigid condition. For example the characteristic function $\chi_{\mathbb{Q} \cap [a, b]}$ of the set of rational numbers in $[a, b]$ is not Riemann integrable. By the above result however, we can introduce a slightly weaker notion, which will make such functions integrable, in a weaker sense. This will be a first “improvement” of the Riemann integration theory. Eventually (see Chapter IV), a more sophisticated theory - the Lebesgue integral - will emerge.

DEFINITION. We say that a function $f : [a, b] \rightarrow \mathbb{R}$ is *almost Riemann integrable*, if there exists a Riemann integrable function $g : [a, b] \rightarrow \mathbb{R}$, with $f = g$, a.e. Of course, such a g is not unique. Notice however that, if $h : [a, b] \rightarrow \mathbb{R}$ is another

Riemann integrable function, with $f = h$, a.e., then $g = h$, a.e., so by Proposition 6.6, we immediately get the equality

$$\int_a^b g(x) dx = \int_a^b h(x) dx.$$

This observation shows that we can unambiguously define

$$\int_a^b f(x) dx = \int_a^b g(x) dx.$$

EXAMPLE 6.4. Consider the function $f = \chi_{\mathbb{Q} \cap [a, b]}$. Since $\mathbb{Q} \cap [a, b]$ is negligible, we have $f = 0$, a.e. So f is almost Riemann integrable (although it is not Riemann integrable), and we have

$$\int_a^b f(x) dx = 0.$$

We now focus our attention to (honest) Riemann integrability, with an eye on the role played by continuity. For a function $f : [a, b] \rightarrow \mathbb{R}$ we define the set

$$D_f = \{x \in [a, b] : f \text{ not continuous at } x\}.$$

It is well-known that continuous functions are Riemann integrable. There are discontinuous functions which are still Riemann integrable, for instance we know that

$$(18) \quad D_f \text{ finite} \implies f \text{ Riemann integrable.}$$

NOTATIONS. Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function. Suppose $\Delta = (a = x_0 < x_1 < \dots < x_n = b)$ is a partition. For each $k \in \{1, \dots, n\}$ we consider the numbers

$$M_k = \sup_{t \in [x_{k-1}, x_k]} f(t) \text{ and } m_k = \inf_{t \in [x_{k-1}, x_k]} f(t),$$

and we define the functions

$$\begin{aligned} f_\Delta &= m_1 \cdot \chi_{[x_0, x_1]} + m_2 \cdot \chi_{(x_1, x_2]} + \dots + m_n \cdot \chi_{(x_{n-1}, x_n]}, \\ f^\Delta &= M_1 \cdot \chi_{[x_0, x_1]} + M_2 \cdot \chi_{(x_1, x_2]} + \dots + M_n \cdot \chi_{(x_{n-1}, x_n]}. \end{aligned}$$

Clearly the functions f_Δ and f^Δ have only finitely many points of discontinuity, so they are Riemann integrable.

With these notations we have the following

PROPOSITION 6.6. *For a bounded function $f : [a, b] \rightarrow \mathbb{R}$, the following are equivalent:*

- (i) f is Riemann integrable;
- (ii) $\inf \left\{ \int_a^b [f^\Delta(x) - f_\Delta(x)] dx : \Delta \text{ partition of } [a, b] \right\} = 0$;
- (iii) there exists a sequence $(\Delta_p)_{p=1}^\infty$ of partitions of $[a, b]$, with $\Delta_1 \subset \Delta_2 \subset \dots$, and $\lim_{p \rightarrow \infty} \int_a^b [f^{\Delta_p}(x) - f_{\Delta_p}(x)] dx = 0$.

PROOF. From the definition of Riemann integrability, we know that (i) is equivalent to any of the following two conditions

- (ii') $\inf \{U(\Delta, f) - L(\Delta, f) : \Delta \text{ partition of } [a, b]\} = 0$;
- (iii') there exists a sequence $(\Delta_p)_{p=1}^\infty$ of partitions of $[a, b]$, with $\Delta_1 \subset \Delta_2 \subset \dots$, and $\lim_{p \rightarrow \infty} [U(\Delta_p, f) - L(\Delta_p, f)] = 0$.

Then the Proposition follows immediately from the fact that, for every partition Δ one has the equalities

$$\int_a^b f_{\Delta}(x) dx = L(\Delta, f) \text{ and } \int_a^b f^{\Delta}(x) dx = U(\Delta, f). \quad \square$$

The following result gives a complete description of the relationship between Riemann integrability and continuity.

THEOREM 6.1 (Lebesgue's criterion for Riemann integrability). *Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function. The following are equivalent:*

- (i) *f is Riemann integrable;*
- (ii) *the discontinuity set D_f is negligible.*

PROOF. (i) $\Rightarrow$ (ii). Assume f is Riemann integrable. Using Proposition 6.7, there exists a sequence $(\Delta_p)_{p=1}^{\infty}$ of partitions of $[a, b]$, such that $\Delta_1 \subset \Delta_2 \subset \dots$ and

$$\lim_{p \rightarrow \infty} \int_a^b [f^{\Delta_p}(x) - f_{\Delta_p}(x)] dx = 0.$$

Notice that

$$(19) \quad f^{\Delta_1} \geq f^{\Delta_2} \geq f^{\Delta_3} \geq \dots \geq f \geq \dots \geq f_{\Delta_3} \geq f_{\Delta_2} \geq f_{\Delta_1}.$$

Define the Riemann integrable functions $h_p = f^{\Delta_p} - f_{\Delta_p}$, $p \in \mathbb{N}$. We then clearly have

- (α) $h_p \geq h_{p+1} \geq 0, \forall p \in \mathbb{N}$;
- (β) $\lim_{p \rightarrow \infty} \int_a^b h_p(x) dx = 0$.

Using (α) we can define the function $h : [a, b] \rightarrow \mathbb{R}$ by

$$h(x) = \lim_{p \rightarrow \infty} h_p(x), \quad \forall x \in [a, b].$$

Claim 1: The set $N = \{x \in [a, b] : h(x) \neq 0\}$ is negligible.

First of all, the functions h_p are all Lebesgue measurable. Secondly, since h is a point-wise limit of a sequence of Lebesgue measurable functions, it follows (see Theorem 3.2) that h itself is Lebesgue measurable. In particular N is Lebesgue measurable. For every integer $j \geq 1$, define

$$N_j = \{x \in [a, b] : h(x) > \frac{1}{j}\},$$

so that the sets N_j , $j \geq 1$ are again Lebesgue measurable, and $N = \bigcup_{j=1}^{\infty} N_j$. In order to prove that N is negligible, it then suffices to prove that $\lambda(N_j) = 0$, for all $j \geq 1$. Fix for the moment $j \geq 1$. Since $h_p \geq h \geq 0$, it follows that

$$\inf_{x \in N_j} h_p(x) \geq \frac{1}{j}, \quad \forall p \geq 1,$$

so by Lemma 6.1 we get the inequality

$$\frac{\lambda(N_j)}{j} \leq \int_a^b h_p(x) dx, \quad \forall p \geq 1,$$

so by (β) we indeed get $\lambda(N_j) = 0$.

Define the set $S = \bigcup_{p=1}^{\infty} \Delta_p$.

Claim 2: If $y \in [a, b] \setminus (N \cup S)$, then f is continuous at y .

Fix $y \in [a, b] \setminus (N \cup S)$. In order to prove that f is continuous at y , we must find, for every $\varepsilon > 0$, some open interval $J_\varepsilon \ni y$, such that

$$(20) \quad |f(z) - f(y)| < \varepsilon, \quad \forall z \in J_\varepsilon \cap [a, b].$$

Since $y \notin N$, we have $\lim_{p \rightarrow \infty} h_p(y) = 0$. Fix ε and choose $p \geq 1$, such that $0 \leq h_p(y) < \varepsilon$. Write the partition Δ_p as

$$\Delta_p = (a = x_0 < x_1 < \cdots < x_n = b).$$

Using the fact that $y \notin \Delta_p$, if we define $k = \min \{j \in \{1, \dots, n\} : y < x_j\}$, we have $y \in (x_{k-1}, x_k)$. In particular, we get

$$f^{\Delta_p}(y) = \sup_{t \in [x_{k-1}, x_k]} f(t) \text{ and } f_{\Delta_p}(y) = \inf_{s \in [x_{k-1}, x_k]} f(s),$$

so the inequality $0 \leq h_p(y) < \varepsilon$ gives

$$\left[\sup_{t \in [x_{k-1}, x_k]} f(t) \right] - \left[\inf_{s \in [x_{k-1}, x_k]} f(s) \right] < \varepsilon,$$

so if we choose $J_\varepsilon = (x_{k-1}, x_k)$, we clearly have (20).

Now we are done, because using the fact that S is countable, it follows that S is negligible, so $N \cup S$ is also negligible. Since by Claim 2, we have $D_f \subset N \cup S$, it follows that D_f itself is negligible.

(ii) $\Rightarrow$ (i). Assume now the discontinuity set D_f is negligible, and let us prove that f is Riemann integrable. Fix a sequence $(\Delta_p)_{p=1}^\infty$ of partitions of $[a, b]$, with $\Delta_1 \subset \Delta_2 \subset \cdots$, and¹⁰ $\lim_{p \rightarrow \infty} |\Delta_p| = 0$. As before, we define the set $S = \bigcup_{p=1}^\infty \Delta_p$.

Claim 3: For any point $y \in [a, b] \setminus (D_f \cup S)$, one has the equalities

$$\lim_{p \rightarrow \infty} f^{\Delta_p}(y) = \lim_{p \rightarrow \infty} f_{\Delta_p}(y) = f(y).$$

Fix for the moment $\varepsilon > 0$. Since f is continuous at y , there exists some $\delta_\varepsilon > 0$, such that

$$(21) \quad |f(z) - f(y)| < \varepsilon, \quad \forall z \in (y - \delta_\varepsilon, y + \delta_\varepsilon) \cap [a, b].$$

Choose now $q \geq 1$, such that $|\Delta_q| < \delta_\varepsilon$. Write $\Delta_q = (a = x_0 < x_1 < \cdots < x_n = b)$. Using the fact that $y \notin \Delta_q$, we can find $k \in \{1, \dots, n\}$ such that $y \in (x_{k-1}, x_k)$. Since $x_k - x_{k-1} < \delta_\varepsilon$, we have the inclusion $[x_{k-1}, x_k] \subset (y - \delta_\varepsilon, y + \delta_\varepsilon)$, so by (21) we immediately get

$$\begin{aligned} f(y) &\leq f^{\Delta_q}(y) = \sup_{z \in [x_{k-1}, x_k]} f(z) \leq f(y) + \varepsilon; \\ f(y) &\geq f_{\Delta_q}(y) = \inf_{z \in [x_{k-1}, x_k]} f(z) \geq f(y) - \varepsilon. \end{aligned}$$

Since the sequence $(f^{\Delta_p}(y))_{p=1}^\infty$ is non-increasing, and the sequence $(f_{\Delta_p}(y))_{p=1}^\infty$ is non-decreasing, the above inequalities give

$$|f^{\Delta_p}(y) - f(y)| \leq \varepsilon \text{ and } |f_{\Delta_p}(y) - f(y)| \leq \varepsilon, \text{ for all } p \geq q,$$

and the Claim follows.

Going back to the proof of the Theorem, we will now prove that f satisfies condition (iii) in Proposition 6.6. Fix $\varepsilon > 0$. Since $D_f \cup S$ is also negligible, using regularity from above with respect to open sets, we can find an open set $E \subset \mathbb{R}$

¹⁰ Recall that, for a partition $\Delta = (a = x_0 < \cdots < x_n = b)$, the number $|\Delta|$ is defined as $|\Delta| = \max \{x_k - x_{k-1} : 1 \leq k \leq n\}$.

such that $E \supset D_f \cup S$, and $\lambda(E) < \varepsilon$. Define the compact set $A = [a, b] \setminus E$, and put $B = [a, b] \cap E$. We clearly have

$$(22) \quad \lambda(B) \leq \lambda(E) < \varepsilon.$$

Define the sequence $(h_p)_{p=1}^\infty$ by $h_p = f^{\Delta_p} - f_{\Delta_p}$. Since $A \cap \Delta_p = \emptyset$, it follows that $h_p|_A$ is continuous, for each $p \geq 1$. Since $A \cap (D_f \cup S) = \emptyset$, by Claim 3, we know that $\lim_{p \rightarrow \infty} h_p(y) = 0$, $\forall y \in A$. Since $(h_p)_{p=1}^\infty$ is monotone, by Dini's Theorem (see ??) it follows that

$$\lim_{p \rightarrow \infty} [\max_{y \in A} h_p(y)] = 0.$$

In particular, there exists $p_\varepsilon \geq 1$, such that

$$(23) \quad h_{p_\varepsilon}(y) \leq \varepsilon, \quad \forall y \in A.$$

Let

$$M = \sup_{x \in [a, b]} f(x) \text{ and } m = \inf_{x \in [a, b]} f(x).$$

Using Lemma 6.1 for h_{p_ε} and the sets A and B , combined with (22), we have

$$\begin{aligned} \int_a^b h_{p_\varepsilon}(x) dx &\leq (b-a) \cdot \sup_{y \in A} h_{p_\varepsilon}(y) + \lambda^*(B) \cdot \sup_{z \in B} h_{p_\varepsilon}(z) \leq \\ &\leq \varepsilon(b-a) + \lambda^*(B)(M-m) \leq \varepsilon(b-a + M-m). \end{aligned}$$

Since $h_{p_\varepsilon} \geq h_p \geq 0$, for all $p \geq p_\varepsilon$, we get the inequalities

$$0 \leq \int_a^b h_p(x) dx \leq \varepsilon(b-a + M-m), \quad \forall p \geq p_\varepsilon.$$

The above argument proves that $\lim_{p \rightarrow \infty} \int_a^b h_p(x) dx = 0$, i.e.

$$\lim_{p \rightarrow \infty} \int_a^b [f^{\Delta_p}(x) - f_{\Delta_p}(x)] dx = 0.$$

By Proposition 6.6, it follows that f is Riemann integrable. $\square$

Exercise 13. Prove that a Riemann integrable function $f : [a, b] \rightarrow \mathbb{R}$ is Lebesgue measurable.

HINT: Use a sequence of partitions $(\Delta_p)_{p=1}^\infty$, with $\Delta_1 \subset \Delta_2 \subset \dots$, and $\lim_{p \rightarrow \infty} |\Delta_p| = 0$. Use the arguments given in the proof of the implication (ii) $\Rightarrow$ (i), to find a negligible set $N \subset [a, b]$, such that

$$\lim_{p \rightarrow \infty} f_{\Delta_p}(x) = f(x), \quad \forall x \in [a, b] \setminus N.$$

The sequence $(f_{\Delta_p})_{p=1}^\infty$ is non-decreasing, so it has a point-wise limit, say g , which is Lebesgue measurable. Use the fact that

$$f(x) = g(x) \quad \forall x \in [a, b] \setminus N,$$

to show that f itself is Lebesgue measurable.

Exercise 14. Let $K \subset [0, 1]$ be a compact set with $K \cap \mathbb{Q} = \emptyset$, and $\lambda(K) > 0$ (see Exercise 7 for the existence of such sets). Prove that the characteristic function $\chi_K : [0, 1] \rightarrow \mathbb{R}$ is not Riemann integrable. In fact, f cannot be almost Riemann integrable either.

HINT: Examine the discontinuity set D_f , and prove that $K \subset D_f$.

Exercise 15. Let $f_n : [a, b] \rightarrow \mathbb{R}$, $n \geq 1$ be a sequence of Riemann integrable functions. Consider the product space $P = \prod_{n=1}^{\infty} \text{Ran } f_n$, equipped with the product topology (the sets $\text{Ran } f_n$, $n \geq 1$, are equipped with the topology induced from $\mathbb{R}$), and the function $F : [a, b] \rightarrow P$, defined by $F(x) = (f_n(x))_{n=1}^{\infty}$. Prove that, for every bounded continuous function $g : P \rightarrow \mathbb{R}$, the composition $g \circ F : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable. In other words, *the result of a bounded continuous operation, involving a sequence of Riemann integrable functions, is again a Riemann integrable function.*

HINT: Examine the relationship between the discontinuity set $D_{g \circ F}$ and the discontinuity sets D_{f_n} , $n \geq 1$.

Exercise 16. Let M be an arbitrary subset of $[a, b]$, and let $f : [a, b] \rightarrow \mathbb{R}$ be a Riemann integrable function, such that $f \leq \chi_M$. Prove the inequality

$$\int_a^b f(x) dx \leq \lambda^*(M).$$

HINT: Consider the function $g : [a, b] \rightarrow \mathbb{R}$ defined by $g(x) = \max\{f(x), 1\}$. Then $f \geq g \geq \chi_M$, and g is still Riemann integrable. Apply Lemma 6.1 (the first inequality) to the function $1 - g$.

Exercise 17.* Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function. Prove that the following are equivalent:

- (i) f is Riemann integrable;
- (ii) for every $\varepsilon > 0$, there exist continuous functions $g, h : [a, b] \rightarrow \mathbb{R}$ with $g \geq f \geq h$, and $\int_a^b [g(x) - h(x)] dx < \varepsilon$;
- (iii) for every $\varepsilon > 0$, there exist Riemann integrable functions $g, h : [a, b] \rightarrow \mathbb{R}$ with $g \geq f \geq h$, and $\int_a^b [g(x) - h(x)] dx < \varepsilon$.

HINTS: For the implication (i) $\Rightarrow$ (ii) analyze first the particular case when $f = \chi_J$, with J a sub-interval of $[a, b]$. Then analyze the functions of the type f^Δ and f_Δ . For the implication (iii) $\Rightarrow$ (i), analyze the relationship among lower/upper Darboux sums of f , g and h .

COMMENT. The statement of Theorem 6.1 shows that, apart from trivial cases, the problem of checking that a function $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable, is a rather difficult one. The main difficulty arises from the fact that, if $N \subset [a, b]$ is a negligible set, and $f|_{[a, b] \setminus N}$ is continuous, then f need not be continuous at all points in $[a, b] \setminus N$. For instance, if we consider the characteristic function $f = \chi_{\mathbb{Q} \cap [a, b]}$ of the rationals in $[a, b]$, and $N = \mathbb{Q} \cap [a, b]$, then clearly N is negligible, $f|_{[a, b] \setminus N}$ is continuous (because it is constant zero), but $D_f = [a, b]$.

As earlier suggested, in the hope that such an anomaly can be eliminated, it is reasonable to consider the slightly weaker notion of almost Riemann integrability. In the remainder of this section, we take a closer look at this notion, and we will eventually show (see Theorem 6.2) that this indeed removes the above anomaly.

We begin with an “almost” version of Exercise 17.

LEMMA 6.2. *For a function $f : [a, b] \rightarrow \mathbb{R}$, the following are equivalent:*

- (i) f is almost Riemann integrable;
- (ii) for every $\varepsilon > 0$, there exist continuous functions $g, h : [a, b] \rightarrow \mathbb{R}$ with $g \geq f \geq h$ a.e., and $\int_a^b [g(x) - h(x)] dx < \varepsilon$;
- (iii) for every $\varepsilon > 0$, there exist Riemann integrable functions $g, h : [a, b] \rightarrow \mathbb{R}$ with $g \geq f \geq h$ a.e., and $\int_a^b [g(x) - h(x)] dx < \varepsilon$.

PROOF. The implication $(i) \Rightarrow (iii)$ is trivial.

The implication $(iii) \Rightarrow (ii)$ follows from Exercise 17.

We now prove $(ii) \Rightarrow (i)$. Assume f has property (ii). For each integer $n \geq 1$, choose continuous functions $g_n, h_n : [a, b] \rightarrow \mathbb{R}$, such that $g_n \geq f \geq h_n$, a.e., and $\int_a^b [g_n(x) - h_n(x)] dx \leq 1/n$. Define the functions $G_n, H_n : [a, b] \rightarrow \mathbb{R}$, $n \geq 1$, by

$$G_n(x) = \min \{g_1(x), \dots, g_n(x)\},$$

$$H_n(x) = \max \{h_1(x), \dots, h_n(x)\}.$$

It is clear that

- (α) $G_m \geq f \geq H_n$, a.e., $\forall m, n \geq 1$;
- (β) $G_1 \geq G_2 \geq \dots$ and $H_1 \leq H_2 \leq \dots$;
- (γ) $\int_a^b [G_n(x) - H_n(x)] dx \leq \int_a^b [g_n(x) - h_n(x)] dx \leq 1/n$, $\forall n \geq 1$.

Notice that, since the G_m 's and the H_n 's are continuous, by Exercise ??, we also have

- (α') $G_m \geq H_n$ (everywhere!), $\forall m, n \geq 1$.

Use (β) to define the functions $G, H : [a, b] \rightarrow \mathbb{R}$, by

$$G(x) = \lim_{n \rightarrow \infty} G_n(x) \text{ and } H(x) = \lim_{n \rightarrow \infty} H_n(x), \quad \forall x \in [a, b],$$

so by (α') we clearly have $G_n \geq G \geq H \geq H_n$, $\forall n \geq 1$. Using then (γ), by Exercise 17 it follows that both G and H are Riemann integrable. Moreover, we have $G - H \geq 0$ and

$$0 \leq \int_a^b [G(x) - H(x)] dx \leq \int_a^b [G_n(x) - H_n(x)] dx \leq 1/n, \quad \forall n \geq 1,$$

Which forces $\int_a^b [G(x) - H(x)] dx = 0$, so by Exercise ??, we get $G = H$, a.e. By (α) it follows that $f = G$, a.e., so f is indeed almost Riemann integrable. $\square$

We are now in position to prove the “almost” version of Theorem 6.1.

THEOREM 6.2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function. The following are equivalent:*

- (i) f is almost Riemann integrable;
- (ii) there exists a negligible set $N \subset [a, b]$ such that $f|_{[a, b] \setminus N}$ is continuous.

PROOF. $(i) \Rightarrow (ii)$. Assume f is almost Riemann integrable, so there exists a Riemann integrable function $g : [a, b] \rightarrow \mathbb{R}$, such that $f = g$, a.e. By Theorem 6.1, the discontinuity set D_g is negligible. Take

$$M = \{x \in [a, b] : f(x) \neq g(x)\}.$$

Since $f = g$, a.e., the set M is negligible, and so is the set $N = M \cup D_g$. On the one hand, since $D_g \subset N$, the restriction $g|_{[a, b] \setminus N}$ is continuous. On the other hand, since $M \subset N$, we have $f|_{[a, b] \setminus N} = g|_{[a, b] \setminus N}$, so (ii) follows.

$(ii) \Rightarrow (i)$. We are going to imitate the proof of Theorem 6.1, with some minor modifications. Fix $N \subset [a, b]$ negligible, such that $f|_{[a, b] \setminus N}$ is continuous. Fix also a sequence $(\Delta_p)_{p=1}^\infty$ of partitions, with $\Delta_1 \subset \Delta_2 \subset \dots$, and $\lim_{p \rightarrow \infty} |\Delta_p| = 0$. Put $S = \bigcup_{p=1}^\infty \Delta_p$. Since S is countable, the set $N \cup S$ is still negligible. We put $T = [a, b] \setminus (N \cup S)$, and we define the analogues of the functions f^{Δ_p} and f_{Δ_p} as

follows. Write each partition as $\Delta_p = (a = x_0^p < x_1^p < \cdots < x_{n_p}^p = b)$, and define, for each $k \in \{1, \dots, n_p\}$, the numbers

$$M_k^p = \sup \{f(t) : t \in [x_{k-1}^p, x_k^p] \cap T\} \text{ and } m_k^p = \inf \{f(t) : t \in [x_{k-1}^p, x_k^p] \cap T\}.$$

We then define, for each $p \geq 1$, the functions

$$\begin{aligned} g_p &= m_1^p \cdot \chi_{[x_0^p, x_1^p]} + m_2^p \cdot \chi_{(x_1^p, x_2^p]} + \cdots + m_{n_p}^p \cdot \chi_{(x_{n_p-1}^p, x_{n_p}^p]}, \\ g^p &= M_1^p \cdot \chi_{[x_0^p, x_1^p]} + M_2^p \cdot \chi_{(x_1^p, x_2^p]} + \cdots + M_{n_p}^p \cdot \chi_{(x_{n_p-1}^p, x_{n_p}^p]}. \end{aligned}$$

Note that we have the inequalities $g^p(x) \geq f(x) \geq g_p(x)$, $\forall x \in T$, which give

$$(24) \quad g^p \geq f \geq g_p, \text{ a.e., } \forall p \geq 1.$$

It is obvious that g^p and g_p , $p \geq 1$, are all Riemann integrable. We are now going to estimate the integrals $\int_a^b [g^p(x) - g_p(x)] dx$. Put $h_p = g^p - g_p$, $p \geq 1$. First we observe that, since $f|_T$ is continuous, and $T \cap \Delta_p = \emptyset$, $\forall p \geq 1$, we clearly have the equalities $\lim_{p \rightarrow \infty} g^p(x) = \lim_{p \rightarrow \infty} g_p(x) = f(x)$, $\forall x \in T$, which give

$$(25) \quad \lim_{p \rightarrow \infty} h_p(x) = 0, \quad \forall x \in T.$$

Fix some $\varepsilon > 0$, and use regularity from above, to find an open set D with $D \supset N \cup S$ and $\lambda(D) < \varepsilon$. Take the compact set $A = [a, b] \setminus D$. Note that $f|_A$ is continuous, since $A \subset [a, b] \setminus N$. Note also that, since $A \subset [a, b] \setminus S$, the functions $g^p|_A$ and $g_p|_A$ are also continuous, and so will be $h_p|_A$, for every $p \geq 1$. Since $(g^p(x))_{p=1}^\infty$ is non-increasing, and $(g_p(x))_{p=1}^\infty$ is non-decreasing, for all x , it follows that the sequence $(h_p)_{p=1}^\infty$ is monotone, so by Dini's Theorem, (25) gives

$$\lim_{p \rightarrow \infty} [\max_{x \in A} h_p(x)] = 0.$$

In particular, there exists some $p_\varepsilon \geq 1$, such that

$$(26) \quad h_p(x) \leq \varepsilon, \quad \forall p \geq p_\varepsilon, \quad x \in A.$$

Put $B = [a, b] \setminus A$, and take $M = \sup_{x \in [a, b]} f(x)$ and $m = \inf_{x \in [a, b]} f(x)$. Using the inclusion $B \subset D$, we get $\lambda^*(B) \leq \lambda(D) \leq \varepsilon$, so by Lemma 6.1, (the functions h_p , $p \geq 1$, are clearly non-negative), combined with (26), we get

$$\begin{aligned} \int_a^b h_p(x) dx &\leq (b-a) \cdot \sup_{x \in A} h_p(x) + \lambda^*(B) \cdot \sup_{x \in B} h_p(x) \leq \\ &\leq (b-a)\varepsilon + \lambda^*(B)(M-m) \leq \varepsilon(b-a+M-m), \quad \forall p \geq p_\varepsilon. \end{aligned}$$

This estimate then proves that $\lim_{p \rightarrow \infty} \int_a^b h_p(x) dx = 0$, i.e.

$$\lim_{p \rightarrow \infty} \int_a^b [g^p(x) - g_p(x)] dx = 0.$$

Combining this with (24), and applying Lemma 6.2, yields the fact that f is almost Riemann integrable. $\square$

COMMENT. The hypothesis that f is bounded can be replaced with a slightly weaker one, which assumes that f is *almost* bounded, meaning that there exists a negligible set $U \subset [a, b]$, such that $f|_{[a, b] \setminus U}$ is bounded.

Exercise 18. Let $f_n : [a, b] \rightarrow \mathbb{R}$, $n \geq 1$, be almost Riemann integrable functions, such that

- (i) $f_n \geq f_{n+1} \geq 0$, a.e., $\forall n \geq 1$;
- (ii) $\lim_{n \rightarrow \infty} f_n(x) = 0$, for “almost all” $x \in [a, b]$, i.e. there exists a negligible set $N \subset [a, b]$, such that $\lim_{n \rightarrow \infty} f_n(x) = 0$, $\forall x \in [a, b] \setminus N$.

Prove that

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = 0.$$

LECTURES 26-29

7. Measure theory on locally compact spaces

Earlier in this chapter we discussed the construction of (outer) measures, starting with more primitive objects: semiring measures. The main application was the construction of the (outer) Lebesgue measure on $\mathbb{R}^n$. In this section we describe an alternative construction, which has as its starting point another primitive object: a *regular content*. The idea is again to start with the measure defined on a “small” class of sets, extend it to an outer measure, and then use the Caratheodory construction. Among other applications, we will get an alternative construction of the (outer) Lebesgue measure on $\mathbb{R}^n$.

DEFINITION. Let X be a locally compact space. Denote by $\mathcal{C}_X$ the collection of all compact subsets of X . A *content on X* , is a map $\omega : \mathcal{C}_X \rightarrow [0, \infty)$, with the following properties:

- (i) $\omega(\emptyset) = 0$;
- (ii) if $K, L \in \mathcal{C}_X$ are such that $K \subset L$, then $\omega(K) \leq \omega(L)$;
- (iii) $\omega(K \cup L) \leq \omega(K) + \omega(L)$, for all $K, L \in \mathcal{C}_X$;
- (iv) $\omega(K \cup L) = \omega(K) + \omega(L)$, for all $K, L \in \mathcal{C}_X$, with $K \cap L = \emptyset$.

COMMENTS. Note that ω takes *finite values*. The collection $\mathcal{C}_X$ does not have any nice set-arithmetic properties, except for the following: (i) the union of any finite collection of sets in $\mathcal{C}_X$ is again in $\mathcal{C}_X$; (ii) an arbitrary intersection of sets in $\mathcal{C}_X$ is again in $\mathcal{C}_X$.

EXAMPLES 7.1. A. If μ is a measure on $Bor(X)$, then $\mu|_{\mathcal{C}_X}$ is a content.

B. Take $X = \mathbb{R}$, and for a compact subset $K \subset \mathbb{R}$, define

$$\omega(K) = \begin{cases} 1 & \text{if } 0 \in \text{Int}(K) \\ 0 & \text{if } 0 \notin \text{Int}(K) \end{cases}$$

It is obvious that ω is a content on $\mathbb{R}$. Notice however that if we consider the compact sets $K_n = [-\frac{1}{n}, \frac{1}{n}]$, then $\omega(\bigcap_{n=1}^{\infty} K_n) = 0$, but $\omega(K_n) = 1$, $\forall n \geq 1$. This shows that, in general, a content cannot be extended to a measure on $Bor(X)$.

One useful property, which will be invoked several times in this section, is contained in the following:

Exercise 1. Let X be a locally compact space, let $K \subset X$ be compact, and let $D_1, D_2 \subset X$ be open subsets, with $K \subset D_1 \cup D_2$. Show there exist compact sets K_1 and K_2 , such that $K_1 \subset D_1$, $K_2 \subset D_2$, and $K = K_1 \cup K_2$.

As Example 7.1.B suggests, one obstruction for the extendability of a content on X , to a measure on $Bor(X)$, is its behaviour with respect to interiors. The following notion isolates an important property, which will be shown to be sufficient for the extendability property.

DEFINITION. Let X be a locally compact space. A content ω on X is said to be *regular*, if for any $K \in \mathcal{C}_X$, one has the equality

$$\omega(K) = \inf \{ \omega(L) : L \in \mathcal{C}_X, \text{Int}(L) \supset K \}.$$

The following exercise shows how the lack of regularity can always be repaired.

Exercise 2. Let X be a locally compact space, and let ω be a content on X . Define $\tilde{\omega} : \mathcal{C}_X \rightarrow [0, \infty)$, by

$$\tilde{\omega}(K) = \inf \{ \omega(L) : L \in \mathcal{C}_X, \text{Int}(L) \supset K \}, \quad \forall K \in \mathcal{C}_X.$$

Prove that:

- (i) $\tilde{\omega}$ is a regular content on X ;
- (ii) $\tilde{\omega}(K) \geq \omega(K)$, $\forall K \in \mathcal{C}_X$;
- (iii) if η is a regular content on X , with $\eta(K) \geq \omega(K)$, $\forall K \in \mathcal{C}_X$, then $\eta(K) \geq \tilde{\omega}(K)$, $\forall K \in \mathcal{C}_X$;
- (iv) ω is regular, if and only if $\tilde{\omega} = \omega$.

DEFINITION. With the notations from Exercise 2, the regular content $\tilde{\omega}$ is called the *regularization* of ω .

THEOREM 7.1. Let X be a locally compact space, and let ω be a content on X . Denote by $\mathcal{T}_X$ the collection of all open subsets of X . Define the map $\hat{\omega} : \mathcal{T}_X \rightarrow [0, \infty]$ by

$$\hat{\omega}(D) = \sup \{ \omega(K) : K \in \mathcal{C}_X, K \subset D \}, \quad \forall D \in \mathcal{T}_X,$$

and define the map $\omega^* : \mathcal{P}(X) \rightarrow [0, \infty]$, by

$$\omega^*(A) = \inf \{ \hat{\omega}(D) : D \in \mathcal{T}_X, D \supset A \}, \quad \forall A \subset X.$$

Then ω^* is an outer measure on X .

PROOF. We begin by collecting the useful properties of the map $\hat{\omega}$.

Claim: The map $\hat{\omega}$ has the following properties

- (i) $\hat{\omega}(\emptyset) = 0$;
- (ii) $\hat{\omega}$ is monotone, i.e. whenever $D, E \in \mathcal{T}_X$ satisfy $D \subset E$, it follows that $\hat{\omega}(D) \leq \hat{\omega}(E)$;
- (iii) $\hat{\omega}$ is σ -sub-additive, i.e., for any sequence $(D_n)_{n=1}^\infty \subset \mathcal{T}_X$, one has the inequality $\hat{\omega}(\bigcup_{n=1}^\infty D_n) \leq \sum_{n=1}^\infty \hat{\omega}(D_n)$.

Properties (i) and (ii) are trivial.

To prove property (iii), let us start with some sequence $(D_n)_{n=1}^\infty$ of open sets, and let us denote for simplicity the union $\bigcup_{n=1}^\infty D_n$ by D . Start with some arbitrary compact set $K \subset D$. Using compactness, there exists some index $p \geq 1$, such that $K \subset D_1 \cup D_2 \cup \dots \cup D_p$. Use Exercise 1 (and induction) to find compact sets $K_1 \subset D_1, K_2 \subset D_2, \dots, K_p \subset D_p$, such that $K = K_1 \cup K_2 \cup \dots \cup K_p$. We then clearly have the inequalities

$$\omega(K) \leq \sum_{n=1}^p \omega(K_n) \leq \sum_{n=1}^p \hat{\omega}(D_n) \leq \sum_{n=1}^\infty \hat{\omega}(D_n).$$

Since we have

$$\omega(K) \leq \sum_{n=1}^\infty \hat{\omega}(D_n), \text{ for all } K \in \mathcal{C}_X \text{ with } K \subset D,$$

by the definition of $\hat{\omega}$, we immediately get

$$\hat{\omega}(D) \leq \sum_{n=1}^{\infty} \hat{\omega}(D_n).$$

Having proven the Claim, we now check the conditions in the definition of an outer measure. It is clear that $\omega^*(\emptyset) = 0$. It is also clear, from the definition, and property (ii) from the Claim, that

$$A \subset B \implies \omega^*(A) \leq \omega^*(B).$$

Finally, we need to show σ -sub-additivity, i.e.

$$(1) \quad \omega^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \omega^*(A_n).$$

Start with some sequence $(A_n)_{n=1}^{\infty}$ of subsets of X . Of course, if one of the terms in the right hand side of (1) is infinite, there is nothing to prove. Assume that $\omega^*(A_n) < \infty$, $\forall n \geq 1$. Fix some $\varepsilon > 0$, and choose, for each $n \geq 1$, an open set $D_n \supset A_n$, such that $\hat{\omega}(D_n) \leq \omega^*(A_n) + \frac{\varepsilon}{2^n}$. Put $D = \bigcup_{n=1}^{\infty} D_n$. Using part (iii) of the Claim, we have

$$\hat{\omega}(D) \leq \sum_{n=1}^{\infty} \hat{\omega}(D_n) \leq \sum_{n=1}^{\infty} \left[\omega^*(A_n) + \frac{\varepsilon}{2^n}\right] = \varepsilon + \sum_{n=1}^{\infty} \omega^*(A_n).$$

Since we obviously have the inclusion $D \supset \bigcup_{n=1}^{\infty} A_n$, the above inequality gives

$$\omega^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \hat{\omega}(D) \leq \varepsilon + \sum_{n=1}^{\infty} \omega^*(A_n).$$

Now we have

$$\omega^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \varepsilon + \sum_{n=1}^{\infty} \omega^*(A_n),$$

for all $\varepsilon > 0$, so the inequality (1) follows. $\square$

DEFINITION. Let X be a locally compact space, and let ω be a content on X . The outer measure ω^* on X , defined in Theorem 7.1, is called the *outer measure induced by ω* .

REMARKS 7.1. Let X be a locally compact space, let ω be a content on X , and let ω^* be the outer measure induced by ω .

A. The map $\hat{\omega} : \mathcal{T}_X \rightarrow [0, \infty]$, defined in the statement of Theorem 7.1, is given by $\hat{\omega} = \omega^*|_{\mathcal{T}_X}$. To see that this is the case, start with some open set D . On the one hand, by the definition of ω^* , we know that

$$\omega^*(D) = \inf \{ \hat{\omega}(E) : E \in \mathcal{T}_X, E \supset D \},$$

which (using $E = D$) immediately gives the inequality $\omega^*(D) \leq \hat{\omega}(D)$. On the other hand, using property (ii) from the Claim stated in the proof, we also know that

$$\hat{\omega}(E) \geq \hat{\omega}(D), \text{ for all } E \in \mathcal{T}_X \text{ with } E \supset D,$$

which gives the reverse inequality, $\omega^*(D) \geq \hat{\omega}(D)$.

B. As a consequence of the above remark, we get the fact that ω^* is *regular from above, with respect to the collection $\mathcal{T}_X$ of all open sets in X* , i.e.

$$\omega^*(A) = \inf \{ \omega^*(D) : D \in \mathcal{T}_X, D \supset A \}, \quad \forall A \subset X.$$

C. If one denotes by $\check{\omega}$ the regularization of ω (see Exercise 2), then $\check{\omega}^* = \omega^*$. In fact, using the notations from Theorem 7.1, we have the equality $\hat{\omega} = \hat{\check{\omega}}$. Indeed, on the one hand, since we have the inequality

$$\check{\omega}(K) \geq \omega(K), \quad \forall K \in \mathcal{C}_X,$$

it follows immediately by the definitions, that

$$\hat{\check{\omega}}(D) \geq \hat{\omega}(D), \quad \forall D \in \mathcal{T}_X.$$

To prove the other inequality, fix some open set $D \subset X$. Suppose $K \subset D$ is some compact subset. Using the well-known properties of locally compact spaces, there exists some compact set L , with

$$K \subset \text{Int}(L) \subset L \subset D,$$

so by the definitions of $\hat{\omega}$ and $\check{\omega}$, we get

$$\hat{\omega}(D) \geq \omega(L) \geq \check{\omega}(K).$$

Since we have the inequality

$$\hat{\omega}(D) \geq \check{\omega}(K), \text{ for all } K \in \mathcal{C}_X \text{ with } K \subset D,$$

taking supremum in the right hand side yields

$$\hat{\omega}(D) \geq \sup \{ \check{\omega}(K) : K \in \mathcal{C}_X, K \subset D \} = \hat{\check{\omega}}(D).$$

PROPOSITION 7.1. *Let X be a locally compact space, let ω be a content on X , and let ω^* be the outer measure induced by ω . If we denote by $\check{\omega}$ the regularization of ω , then one has the equality*

$$\omega^*|_{\mathcal{C}_X} = \check{\omega}.$$

PROOF. Using Remark 7.1.C, we can assume that ω is regular, and in this case we need to prove that $\omega^*|_{\mathcal{C}_X} = \omega$. Start with some compact set $K \subset X$. By the definition of ω^* , using the notations from Theorem 7.1, we know that

$$(2) \quad \omega^*(K) = \inf \{ \hat{\omega}(D) : D \in \mathcal{T}_X, D \supset K \}.$$

It is clear that, for every open set $D \supset K$, we have the inequality

$$\hat{\omega}(D) \geq \omega(K),$$

so taking infimum in the left hand side, and using (2), immediately gives the inequality

$$\omega^*(K) \geq \omega(K).$$

To prove the reverse inequality, we start by fixing $\varepsilon > 0$, and we use regularity to find some compact set L with $K \subset \text{Int}(L)$, and $\omega(L) \leq \omega(K) + \varepsilon$. Consider the open set $D = \text{Int}(L)$. On the one hand, for every compact set $F \subset D$, we have the obvious inclusion $F \subset L$, which gives $\omega(F) \leq \omega(L)$. Taking supremum over all compact sets $F \subset D$ then gives $\hat{\omega}(D) \leq \omega(L)$. By the choice of L , by the definition of ω^* , and using the inclusion $D \supset K$, we then get

$$\omega^*(K) \leq \hat{\omega}(D) \leq \omega(L) \leq \omega(K) + \varepsilon.$$

Since the inequality

$$\omega^*(K) \leq \omega(K) + \varepsilon,$$

holds for all $\varepsilon > 0$, we then must have $\omega^*(K) \leq \omega(K)$. □

The above result gives a nice characterization for the regularity of a content, in terms of the induced outer measure.

COROLLARY 7.1. *Let X be a locally compact space. A content ω is regular, if and only if $\omega^*|_{\mathcal{C}_X} = \omega$.*

PROOF. Immediate from Proposition 7.1 and exercise 2. $\square$

THEOREM 7.2. *Let X be a locally compact space, let ω be a content on X , and let ω^* be the outer measure induced by ω . Then every open set $D \subset X$ is ω^* -measurable.*

PROOF. Fix an open set $D \subset X$. We need to prove (see Section 5) that D “sharply cuts” every subset of X , which is equivalent to the fact that, for every $A \subset X$, one has the inequality:

$$(3) \quad \omega^*(A) \geq \omega^*(A \cap D) + \omega^*(A \setminus D).$$

This will be shown in several steps.

Claim 1: *For any open set $E \subset X$, and any compact set $K \subset E$, one has the inequality*

$$\omega^*(E) \geq \omega(K) + \omega^*(E \setminus K).$$

To prove this inequality, we first note that, since both E and $E \setminus K$ are open, by Remark 7.1.A, we have the equalities $\omega^*(E) = \hat{\omega}(E)$ and $\omega^*(E \setminus K) = \hat{\omega}(E \setminus K)$, where $\hat{\omega} : \mathcal{T}_X \rightarrow [0, \infty]$ is the map defined in the statement of Theorem 7.1. If $L \subset E \setminus K$ is an arbitrary compact set, then we obviously have $K \cap L = \emptyset$, so using the inclusion $K \cup L \subset E$, we get

$$\omega(K) + \omega(L) = \omega(K \cup L) \leq \hat{\omega}(E) = \omega^*(E),$$

which then gives

$$\omega^*(E) - \omega(K) \geq \omega(L), \text{ for all } L \in \mathcal{C}_X \text{ with } L \subset E \setminus K.$$

Taking supremum in the right hand side then gives

$$\omega^*(E) - \omega(K) \geq \sup \{ \omega(L) : L \in \mathcal{C}_X, L \subset E \setminus K \} = \hat{\omega}(E \setminus K) = \omega^*(E \setminus K),$$

and the Claim follows.

Claim 2: *The inequality (3) holds for all open subsets $A \subset X$.*

Assume A is open. If the left hand side of (3) is infinite, there is nothing to prove. Assume $\omega^*(A) < \infty$, so both $\omega^*(A \cap D)$ and $\omega^*(A \setminus D)$ are also finite. Since $A \cap D$ is open, we have

$$(4) \quad \omega^*(A \cap D) = \hat{\omega}(A \cap D) = \sup \{ \omega(K) : K \in \mathcal{C}_X, K \subset A \cap D \}.$$

Fix for the moment a compact subset $K \subset A \cap D$. Using Claim 1 we have the inequality

$$\omega^*(A) \geq \omega(K) + \omega^*(A \setminus K).$$

Since we obviously have the inclusion $A \setminus K \supset A \setminus D$, the above inequality gives $\omega^*(A) \geq \omega(K) + \omega^*(A \setminus D)$, which can be rewritten as

$$\omega^*(A) - \omega^*(A \setminus D) \geq \omega(K), \text{ for all } K \in \mathcal{C}_X \text{ with } K \subset A \cap D.$$

Taking supremum in the right hand side, and using (4), we immediately get the desired inequality (3).

We now proceed with the proof of (3) for arbitrary A 's. Fix A , and consider an arbitrary open set $E \supset A$. By Claim 2, we have

$$\omega^*(E) \geq \omega^*(E \cap D) + \omega^*(E \setminus D).$$

Using the obvious inclusions $E \cap D \supset A \cap D$ and $E \setminus D \supset A \setminus D$, we then get

$$\omega^*(E) \geq \omega^*(A \cap D) + \omega^*(A \setminus D).$$

The desired inequality (3) follows now by taking infimum in the left hand side, and using Remark 7.1.B. $\square$

The most important consequence of Theorem 7.2 is the following.

COROLLARY 7.2. *Let X be a locally compact space, and let ω be a regular content on X . Then ω can be extended uniquely to a measure μ_ω on $\text{Bor}(X)$, with the following properties.*

- (i) μ is regular from above, with respect to the collection $\mathcal{T}_X$ of all open sets, that is

$$\mu_\omega(B) = \inf \{ \mu_\omega(D) : D \in \mathcal{T}_X, D \supset B \}, \quad \forall B \in \text{Bor}(X);$$

- (ii) for every open set $D \subset X$, one has the equality

$$\mu_\omega(D) = \sup \{ \mu_\omega(K) : K \in \mathcal{C}_X, K \subset D \}.$$

Conversely, if μ is a measure on $\text{Bor}(X)$ with properties (i) and (ii), and such that $\mu(K) < \infty$, $\forall K \in \mathcal{C}_X$, then $\mu|_{\mathcal{C}_X}$ is regular content.

PROOF. If we denote by $\mathcal{M}_{\omega^*}(X)$ the σ -algebra of ω^* -measurable sets, then Theorem 7.2 gives the inclusion $\text{Bor}(X) \subset \mathcal{M}_{\omega^*}(X)$, so the existence follows by taking $\mu_\omega = \omega^*|_{\text{Bor}(X)}$. The fact that μ_ω has properties (i) and (ii) is trivial, by construction and by Remarks 7.1 and Proposition 7.1.

The uniqueness is trivial, since property (ii) uniquely defines μ_ω on open sets, and (i) then uniquely defines μ_ω on all Borel sets.

To prove the second assertion, assume μ is a measure on $\text{Bor}(X)$ with properties (i) and (ii), and let us show that $\omega = \mu|_{\mathcal{C}_X}$ is a regular content. The fact that ω is a content is trivial, so the only thing we must show is regularity. Fix some compact set $K \subset X$. It is clear that

$$\omega(K) \leq \inf \{ \omega(L) : L \in \mathcal{C}_X, K \subset \text{Int}(L) \}.$$

To prove the converse, we use property (i), to find, for each $\varepsilon > 0$, an open set $D_\varepsilon \supset K$, such that $\mu(D_\varepsilon) \leq \mu(K) + \varepsilon$. If we choose, for each ε ., a compact set L_ε , such that

$$K \subset \text{Int}(L_\varepsilon) \subset L_\varepsilon \subset D_\varepsilon,$$

then we obviously have

$$\mu(K) \leq \mu(L_\varepsilon) \leq \mu(D_\varepsilon) \leq \mu(K) + \varepsilon,$$

so we get the inequality

$$\inf \{ \omega(L) : L \in \mathcal{C}_X, K \subset \text{Int}(L) \} \leq \omega(K) + \varepsilon.$$

Since this holds for all $\varepsilon > 0$, we get in fact the inequality

$$\inf \{ \omega(L) : L \in \mathcal{C}_X, K \subset \text{Int}(L) \} \leq \omega(K),$$

and we are done. $\square$

DEFINITION. Let X be a locally compact space. A *Radon measure* on X is a measure μ on $Bor(X)$ with the following properties:

- (i) $\mu(K) < \infty$, for all compact sets $K \subset X$;
- (ii) for every open set D one has

$$\mu(D) = \sup \{ \mu(K) : K \subset D, K \text{ compact} \};$$

- (iii) for every Borel set B one has

$$\mu(B) = \inf \{ \mu(D) : D \supset B, D \text{ open} \}.$$

By Corollary 7.2, the map $\omega \mapsto \mu_\omega$ establishes a bijective correspondence between the set of all regular contents on X , and the set of all Radon measures on X . For a regular content ω , the measure μ_ω is called the *Radon measure extension* of ω .

PROPOSITION 7.2. *Let X be a locally compact space.*

- (a) *If μ is a Radon measure on X , and $t \in [0, \infty)$, then $t\mu$ is also a Radon measure on X .*
- (b) *If μ_1 and μ_2 are Radon measures on X , then $\mu_1 + \mu_2$ is also a Radon measure on X .*

PROOF. Property (a) is trivial.

To prove property (b) let us denote $\mu_1 + \mu_2$ simply by μ . We first observe that μ is indeed a measure on $Bor(X)$, and we clearly have

$$\mu(K) = \mu_1(K) + \mu_2(K) < \infty, \quad \forall K \in \mathcal{C}_X.$$

Let us show that μ satisfies condition (ii). Fix some open set $D \subset X$, and let us prove that

$$(5) \quad \mu(D) = \sup \{ \mu(K) : K \in \mathcal{C}_X, K \subset D \}.$$

If $\mu(D) = \infty$, then either $\mu_1(D) = \infty$ or $\mu_2(D) = \infty$, so we get

$$\sup \{ \max(\mu_1(K), \mu_2(K)) : K \in \mathcal{C}_X, K \subset D \} = \infty,$$

and since $\mu(K) \geq \max(\mu_1(K), \mu_2(K))$, $\forall K \in \mathcal{C}_X$, the equality (5) immediately follows. Suppose now $\mu(D) < \infty$, which is equivalent to the fact that $\mu_1(D), \mu_2(D) < \infty$. Denote the right hand side of (5) by $\nu(D)$. For every $\varepsilon > 0$, using the fact that μ_1 and μ_2 are Radon measures, we can find two compact sets $K_1^\varepsilon, K_2^\varepsilon \subset D$, such that $\mu_1(K_1^\varepsilon) \geq \mu_1(D) - \frac{\varepsilon}{2}$ and $\mu_2(K_2^\varepsilon) \geq \mu_2(D) - \frac{\varepsilon}{2}$. Of course, the compact set $K_\varepsilon = K_1^\varepsilon \cup K_2^\varepsilon$ is still a subset of D , and satisfies

$$\begin{aligned} \mu_1(K_\varepsilon) &\geq \mu_1(K_1^\varepsilon) \geq \mu_1(D) - \frac{\varepsilon}{2}, \\ \mu_2(K_\varepsilon) &\geq \mu_2(K_2^\varepsilon) \geq \mu_2(D) - \frac{\varepsilon}{2}, \end{aligned}$$

so we get $\mu(K_\varepsilon) = \mu_1(K_\varepsilon) + \mu_2(K_\varepsilon) \geq \mu_1(D) + \mu_2(D) - \varepsilon = \mu(D) - \varepsilon$. This proves that $\nu(D) \geq \mu(D) - \varepsilon$, and since this inequality is true for all $\varepsilon > 0$, we get $\nu(D) \geq \mu(D)$. The inequality $\nu(D) \leq \mu(D)$ is trivial.

We now show that μ satisfies condition (iii). Fix some set $A \in Bor(X)$, and let us prove that

$$(6) \quad \mu(A) = \inf \{ \mu(D) : D \in \mathcal{T}_X, D \supset A \}.$$

If $\mu(A) = \infty$, there is nothing to prove. Suppose now $\mu(A) < \infty$, which is equivalent to the fact that $\mu_1(A), \mu_2(A) < \infty$. Denote the right hand side of (6) by $\lambda(A)$. For every $\varepsilon > 0$, using the fact that μ_1 and μ_2 are Radon measures, we can find two

open sets $D_1^\varepsilon, D_2^\varepsilon \supset A$, such that $\mu_1(D_1^\varepsilon) \leq \mu_1(A) + \frac{\varepsilon}{2}$ and $\mu_2(D_2^\varepsilon) \leq \mu_2(A) + \frac{\varepsilon}{2}$. Then open set $D_\varepsilon = D_1^\varepsilon \cap D_2^\varepsilon$ still contains A , and satisfies

$$\begin{aligned}\mu_1(D_\varepsilon) &\leq \mu_1(D_1^\varepsilon) \leq \mu_1(A) + \frac{\varepsilon}{2}, \\ \mu_2(D_\varepsilon) &\leq \mu_2(D_2^\varepsilon) \leq \mu_2(A) + \frac{\varepsilon}{2},\end{aligned}$$

so we get $\mu(D_\varepsilon) = \mu_1(D_\varepsilon) + \mu_2(D_\varepsilon) \leq \mu_1(A) + \mu_2(A) + \varepsilon = \mu(A) + \varepsilon$. This proves that $\lambda(A) \leq \mu(A) + \varepsilon$, and since this inequality is true for all $\varepsilon > 0$, we get $\lambda(A) \leq \mu(A)$. The inequality $\lambda(A) \geq \mu(A)$ is trivial. $\square$

Radon measures are also functorial with respect to proper maps, in the following sense.

PROPOSITION 7.3. *Let X and Y be locally compact spaces, let $\Phi : X \rightarrow Y$ be a proper continuous map, and let μ be a Radon measure on X . Then the map $\nu : \text{Bor}(Y) \rightarrow [0, \infty]$, defined by*

$$\nu(B) = \mu(\Phi^{-1}(B)), \quad \forall B \in \text{Bor}(Y),$$

is a Radon measure on Y .

PROOF. First of all, remark that since Φ is continuous, it is Borel measurable, which means that

$$\Phi^{-1}(B) \in \text{Bor}(X), \quad \forall B \in \text{Bor}(Y).$$

Secondly, by the well known properties of measures, the map ν is a measure.

We now check that ν is a Radon measure. First of all, if $K \subset Y$ is compact, then using the fact that Φ is proper, it means that $\Phi^{-1}(K)$ is compact in X , so we clearly get

$$\nu(K) = \mu(\Phi^{-1}(K)) < \infty.$$

To prove that ν satisfies condition (ii), start with some open set $D \subset Y$, and let us find a sequence $(L_n)_{n=1}^\infty$ of compact subsets of D , such that $\nu(D) = \lim_{n \rightarrow \infty} \nu(L_n)$. The set $\Phi^{-1}(D)$ is open, so there exists a sequence $(K_n)_{n=1}^\infty$ of compact subsets of $\Phi^{-1}(D)$, with

$$(7) \quad \nu(D) = \mu(\Phi^{-1}(D)) = \lim_{n \rightarrow \infty} \mu(K_n).$$

If we define the subsets $L_n = \Phi(K_n)$, then $(L_n)_{n \geq 1}$ is a sequence of compact subsets of D , and the inclusion $K_n \subset \Phi^{-1}(L_n)$ immediately gives $\nu(D) \geq \nu(L_n) \geq \mu(K_n)$, so by (7) we also get $\nu(D) = \lim_{n \rightarrow \infty} \nu(L_n)$.

To prove condition (iii) start with some arbitrary subset $B \in \text{Bor}(Y)$, and let us a sequence $(E_n)_{n=1}^\infty$ of open subset of Y , such that $\nu(B) = \lim_{n \rightarrow \infty} \nu(E_n)$, and $E_n \supset B$, $\forall n \geq 1$. Use the fact that μ is a Radon measure, to find a sequence $(D_n)_{n=1}^\infty$ of open subset of X , such that

$$(8) \quad \nu(B) = \mu(\Phi^{-1}(B)) = \lim_{n \rightarrow \infty} \mu(D_n),$$

and $D_n \supset \Phi^{-1}(B)$, $\forall n \geq 1$. Put $T_n = X \setminus D_n$, so that T_n is closed, for each $n \geq 1$. By Proposition I.5.2, the sets $\Phi(T_n)$ are closed in Y , hence their complements $E_n = Y \setminus \Phi(T_n)$, $n \geq 1$ are open. Remark that we have the inclusions $B \subset E_n$, $\forall n \geq 1$. Otherwise, we would have $B \cap \Phi(T_n) \neq \emptyset$, forcing $T_n \cap \Phi^{-1}(B) \neq \emptyset$, which is impossible, since $\Phi^{-1}(B) \subset D_n = X \setminus T_n$. Moreover, we also have the inclusions

$$\Phi^{-1}(B) \subset \Phi^{-1}(E_n) \subset D_n, \quad \forall n \geq 1,$$

which then force

$$\nu(B) \leq \nu(E_n) \leq \mu(D_n), \quad \forall n \geq 1.$$

Using by (8) this gives the equality $\lim_{n \rightarrow \infty} \nu(E_n) = \nu(B)$. $\square$

Of course, if X is a compact Hausdorff space, then every Radon measure μ on X is finite. The following gives an interesting converse of this property, which also shows that sometimes functoriality can be present beyond the proper case described above.

PROPOSITION 7.4. *Let X be a locally compact space, let μ be a Radon measure on X , and let (θ, T) be a compactification of X . The following are equivalent:*

- (i) $\mu(X) < \infty$;
- (ii) the map $\nu : \text{Bor}(T) \rightarrow [0, \infty)$, defined by

$$\nu(B) = \mu(\theta^{-1}(B)), \quad \forall B \in \text{Bor}(T),$$

is a Radon measure on T .

PROOF. Recall that the fact that (θ, T) is a compactification of X means that

- T is a compact Hausdorff space;
- $\theta : X \rightarrow T$ is continuous;
- $\theta(X)$ is open and dense in T ;
- $\theta : X \rightarrow \theta(X)$ is a homeomorphism.

Without any loss of generality, we can assume that X is a dense open subset of T , and θ is the inclusion map. With this convention, the map ν is defined by

$$(9) \quad \nu(B) = \mu(B \cap X), \quad \forall B \in \text{Bor}(T).$$

(i) $\Rightarrow$ (ii). Assume $\mu(X) < \infty$. It is clear that ν is a finite measure on $\text{Bor}(T)$, and in fact we have $\nu(T \setminus X) = 0$.

The fact that $\nu(K) < \infty$, for every compact subset $K \subset T$ is of course trivial.

We now check the second condition in the definition. Fix some open subset $D \subset T$, and let us show that

$$\nu(D) = \sup \{ \nu(K) : K \text{ compact}, K \subset D \}.$$

All we need is a sequence $(K_n)_{n=1}^\infty$ of compact subsets of D , with $\lim_{n \rightarrow \infty} \nu(K_n) = \nu(D)$. To get this sequence we simply use the fact that $D \cap X$ is open (in X), so we can find a sequence $(K_n)_{n=1}^\infty$ of compact subsets of $D \cap X$, with $\lim_{n \rightarrow \infty} \mu(K_n) = \mu(D \cap X) = \nu(D)$. Now we are done, because the fact that $K_n \subset X$, gives $\mu(K_n) = \nu(K_n)$, $\forall n \geq 1$.

We now check the third condition in the definition. Fix some set $B \in \text{Bor}(T)$, and let us show that

$$\nu(B) = \inf \{ \nu(D) : D \subset T \text{ open}, D \supset B \}.$$

All we need is a sequence $(D_n)_{n=1}^\infty$ of open subsets of T , with $D_n \supset B$, $\forall n \geq 1$, and $\lim_{n \rightarrow \infty} \nu(D_n) = \nu(B)$. Start off by choosing a sequence $(K_n)_{n=1}^\infty$ of compact subsets of X , such that $\lim_{n \rightarrow \infty} \mu(K_n) = \mu(X)$, we will get $\lim_{n \rightarrow \infty} \mu(X \setminus K_n) = 0$ (the condition that $\mu(X) < \infty$ is essential here). If we define then the open sets $A_n = T \setminus K_n$, then we will have $\nu(A_n) = \mu(A_n \cap X) = \mu(X \setminus K_n)$, $\forall n \geq 1$, so we have

$$(10) \quad \lim_{n \rightarrow \infty} \nu(A_n) = 0.$$

Notice also that

$$(11) \quad A_n \supset T \setminus X, \quad \forall n \geq 1.$$

Use now the fact that μ is a Radon measure on X , and the fact that $B \cap X \in \text{Bor}(X)$, to find a sequence $(E_n)_{n=1}^\infty$ of open subsets of X , with $E_n \supset B \cap X$, $\forall n \geq 1$, and

$$(12) \quad \lim_{n \rightarrow \infty} \nu(E_n) = \mu(B \cap X).$$

Since X is open in T , it follows that all the E_n 's are open in T . If we define $D_n = E_n \cup A_n$, then using (11) we have the inclusions

$$D_n = E_n \cup A_n \supset (B \cap X) \cup (T \setminus X) \supset B, \quad \forall n \geq 1,$$

as well as the inequalities

$$\begin{aligned} \mu(B \cap X) = \nu(B) &\leq \nu(D_n) \leq \nu(E_n) + \nu(A_n) = \\ &= \mu(E_n \cap X) + \nu(A_n) = \mu(E_n) + \nu(A_n), \quad \forall n \geq 1, \end{aligned}$$

which, combined with (10) and (12), clearly give $\lim_{n \rightarrow \infty} \nu(D_n) = \mu(B \cap X) = \nu(B)$.

(ii) $\Rightarrow$ (i). This implication is trivial, because the fact that ν is a Radon measure forces $\mu(X) = \nu(X) \leq \nu(T) < \infty$. $\square$

COMMENT. Assume μ is a Radon measure on a locally compact space X . Although the measure μ is regular from above with respect to open sets by (iii), in general, one cannot conclude that it is regular from below with respect to compact sets. The following example illustrates such an anomaly.

Exercise 3.* Equip the space $X = \mathbb{R}^2$ with the *disjoint union topology* defined by the decomposition $X = \bigcup_{y \in \mathbb{R}} (\mathbb{R} \times \{y\})$. More explicitly, if we define, for each $A \subset X$, and each $y \in \mathbb{R}$, the set

$$A_y = \{x \in \mathbb{R} : (x, y) \in A\},$$

then a set $D \subset X$ is declared to be open, if and only if all subsets $D_y \subset \mathbb{R}$, $y \in \mathbb{R}$ are open (in the usual topology on $\mathbb{R}$). For each subset $A \subset X$, define its support

$$S_A = \{y \in \mathbb{R} : A_y \neq \emptyset\}.$$

Prove the following.

- (i) A set $K \subset X$ is compact, if and only if its support S_K is finite and, for each $y \in S_K$, the set $K_y \subset \mathbb{R}$ is compact (in the usual topology on $\mathbb{R}$).
- (ii) X is a locally compact space.
- (iii) If we define, for every compact subset $K \subset X$, the number

$$\omega(K) = \sum_{y \in S_K} \lambda(K_y),$$

where λ is the Lebesgue measure on $\mathbb{R}$, then ω is a regular content on X .

- (iv) Let μ denote the Radon measure extension of ω . Then for every open set $D \subset X$, one has the equality

$$\mu(D) = \sum_{y \in \mathbb{R}} \lambda(D_y),$$

where one uses the summation conventions discussed in II.2. (The sum in the right hand side is defined as the supremum of all finite sums.)

- (v) If $B \in \text{Bor}(X)$ has uncountable support S_B , then $\mu(B) = \infty$.

- (vi) Consider the y -axis $Y = \{0\} \times \mathbb{R} \subset X$. Show that F is closed in X (hence Borel), it has infinite measure $\mu(F) = \infty$, but $\mu(K) = 0$, for all compact subsets $K \subset F$.

HINTS: Using regularity from above, it suffices to prove (v) only when B is open. In this case use the fact that if a map $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ is summable, then the set $\{t \in \mathbb{R} : \alpha(t) \neq 0\}$ is countable. For (vi), the equality $\mu(F) = \infty$ is a consequence of (v). To get the fact that all compact subsets of F have measure zero, use part (i).

REMARK 7.2. Let X be a locally compact space, and let μ be a Radon measure on X . We define the maximal outer extension of μ (see Section 5) by

$$\mu^*(A) = \inf \{ \mu(B) : B \in \text{Bor}(X), B \supset A \}, \quad \forall A \subset X.$$

By the regularity from above, one has the equality

$$(13) \quad \mu^*(A) = \inf \{ \mu(D) : D \in \mathcal{T}_X, D \supset A \}, \quad \forall A \subset X.$$

If one considers the regular content $\omega = \mu|_{\mathcal{C}_X}$, then $\mu^* = \omega^*$, the outer measure induced by ω . We also know that if we consider the σ -algebra $\mathcal{M}_{\mu^*}(X)$ of all μ^* -measurable subsets of X , we have the inclusion $\text{Bor}(X) \subset \mathcal{M}_{\mu^*}(X)$, and $\mu^*|_{\text{Bor}(X)} = \mu$.

Exercise 4. Consider the collection $\mathcal{D}$ of all subsets of $\mathbb{R}^n$, of the form

$$D = (a_1, b_1) \times \cdots \times (a_n, b_n), \quad a_1 < b_1, \dots, a_n < b_n.$$

For every such D we define

$$\text{vol}_n = \prod_{j=1}^n (b_j - a_j).$$

We define, for every bounded subset $B \subset \mathbb{R}^n$, the number

$$(14) \quad \mathbf{v}(B) = \inf \left\{ \sum_{p=1}^N \text{vol}_n(D_p) : (D_p)_{p=1}^N \subset \mathcal{D}, B \subset \bigcup_{p=1}^N D_p \right\}.$$

- (i) If we define $\mathcal{B} = \{B \subset \mathbb{R}^n : B \text{ bounded}\}$, then $\mathcal{B}$ is a ring, the map $\mathbf{v} : \mathcal{B} \rightarrow [0, \infty)$ is sub-additive, but not σ -sub-additive. In particular, $\mathbf{v}$ does not extend to an outer measure on $\mathbb{R}^n$.
- (ii) If we consider the unit square $S = [0, 1]^n$, then the collection $\mathcal{N}_{\mathbf{v}}(S) = \{N \subset S : \mathbf{v}(N) = 0\}$ is a ring, but not a σ -ring.
- (iii) When restricted to the collection $\mathcal{C}_{\mathbb{R}^n}$, of all compact subsets of $\mathbb{R}^n$, the map $\omega = \mathbf{v}|_{\mathcal{C}_{\mathbb{R}^n}}$ defines a regular content on $\mathbb{R}^n$.
- (iv) The outer measure ω^* , defined by ω , is precisely the outer Lebesgue measure λ_n^* .

The above construction somehow belongs to the “prehistory” of measure theory. The map $\mathbf{v} : \mathcal{B} \rightarrow [0, \infty)$ is called the *Jordan content*. Bounded sets $N \subset \mathbb{R}^n$, with $\mathbf{v}(N) = 0$ are called *Jordan negligible*. The theory of Riemann integration (especially for functions of several variables) relies heavily on the use of Jordan negligible sets. Part (ii) shows that, when restricted to $\text{Bor}(S)$, the map $\mathbf{v}$ fails to be a measure. Parts (iii) and (iv) explain how the construction can be “fixed.” The regular content $\omega = \mathbf{v}|_{\mathcal{C}_{\mathbb{R}^n}}$ is called the *Lebesgue content*. The correspondence $\omega \mapsto \omega^*$ gives an alternative construction of the outer Lebesgue measure, which starts with its definition on compact sets as the Jordan content.

For Radon measures, the lack of regularity from below, with respect to compact sets, in somehow compensated by the following result (compare with Exercise 2 from Section 6).

LEMMA 7.1. *Let X be a locally compact space, let μ be a Radon measure on X , and let μ^* be the maximal outer extension of μ . For a subset $A \subset X$, with $\mu^*(A) < \infty$, the following are equivalent*

- (i) A is μ^* -measurable;
- (ii) $\mu^*(A) = \sup\{\mu(K) : K \in \mathcal{C}_X, K \subset A\}$;
- (iii) there exists a sequence $(K_n)_{n=1}^\infty$ of compact subsets of A , such that

$$\mu^*(A \setminus [\bigcup_{n=1}^\infty K_n]) = 0.$$

PROOF. (i) $\Rightarrow$ (ii). Suppose A is μ^* -measurable, and let us prove the equality (ii). Denote the right hand side of (ii) simply by $\nu(A)$. It is obvious, by the monotonicity of μ^* , and the fact that $\mu^*|_{\text{Bor}(X)} = \mu$, that we have the inequality $\mu^*(A) \geq \nu(A)$. To prove the other inequality we fix for the moment some $\varepsilon > 0$. Using (13), there exists an open set $D \supset A$, such that $\mu(D) \leq \mu^*(A) + \varepsilon$. Use property (ii) in the definition of Radon measures, to find some compact set $L \subset D$ such that

$$\mu(D) \leq \mu(L) + \varepsilon.$$

Since $\mu(D) = \mu(D \setminus L) + \mu(L)$, and $\mu(L) \leq \mu(D) < \infty$, this inequality gives

$$\mu(D \setminus L) \leq \varepsilon,$$

which, combined with the obvious inclusion $A \setminus L \subset D \setminus L$, yields

$$(15) \quad \mu^*(A \setminus L) \leq \mu^*(D \setminus L) = \mu(D \setminus L) \leq \varepsilon.$$

Using (13) we can also find an open set $E \supset L \setminus A$, such that

$$(16) \quad \mu(E) \leq \mu^*(L \setminus A) + \varepsilon.$$

Since $L \setminus A$ is μ^* -measurable, we have $\mu(E) = \mu^*(E) = \mu^*(E \setminus (L \setminus A)) + \mu^*(L \setminus A)$. Since $\mu^*(L \setminus A) \leq \mu^*(E) = \mu(E) < \infty$, the inequality (16) gives

$$(17) \quad \mu^*(E \setminus (L \setminus A)) \leq \varepsilon.$$

Consider the set $K = L \setminus E$. It is obvious that K is compact, and we have the inclusion

$$K \subset L \setminus (L \setminus A) = L \cap A \subset A.$$

Moreover, we have

$$(L \cap A) \setminus K \subset E \setminus (L \setminus A).$$

Using the inequality (17), we then get

$$\mu^*((L \cap A) \setminus K) \leq \varepsilon.$$

Finally, the above inequality, combined with (15), gives

$$\mu^*(A \setminus K) \leq \mu^*((L \cap A) \setminus K) + \mu^*((A \setminus L) \setminus K) \leq \varepsilon + \mu^*(A \setminus L) \leq 2\varepsilon.$$

Since $K \subset A$, we get

$$\mu^*(A) \leq \mu^*(A \setminus K) + \mu^*(K) \leq 2\varepsilon + \mu(K) \leq 2\varepsilon + \nu(A).$$

Since the inequality $\mu^*(A) \leq 2\varepsilon + \nu(A)$ holds for all $\varepsilon > 0$, we get $\mu^*(A) \leq \nu(A)$, so (ii) follows.

(ii) $\Rightarrow$ (iii). Assume A satisfies (ii), and let us show that A has property (iii). For every integer $n \geq 1$, we use (ii) to find a compact set $K_n \subset A$, such that

$$(18) \quad \mu^*(A) \leq \mu(K_n) + \frac{1}{n}.$$

On the one hand, we have the inclusions $A \setminus [\bigcup_{n=1}^{\infty} K_n] \subset A \setminus K_p$, which give

$$(19) \quad \mu^*(A \setminus [\bigcup_{n=1}^{\infty} K_n]) \leq \mu^*(A \setminus K_p), \quad \forall p \geq 1.$$

On the other hand, since K_p is measurable, we have the equality

$$\mu^*(A) = \mu^*(A \setminus K_p) + \mu^*(K_p) = \mu^*(A \setminus K_p) + \mu(K_p),$$

and then the fact that $\mu^*(A) < \infty$, combined with (18), will force

$$\mu^*(A \setminus K_p) \leq \frac{1}{p}, \quad \forall p \geq 1.$$

Using (19), this forces $\mu^*(A \setminus [\bigcup_{n=1}^{\infty} K_n]) = 0$.

(iii) $\Rightarrow$ (i). This is pretty obvious. We define the sets $B = \bigcup_{n=1}^{\infty} K_n \subset A$, and $N = A \setminus B$. Then $\mu^*(N) = 0$, so in particular, N is μ^* -measurable. Since B is Borel, it is also μ^* -measurable, so $A = B \cup N$ is indeed μ^* -measurable. $\square$

The following result generalizes Lemma 7.1 to the σ -finite case.

THEOREM 7.3. *Let X be a locally compact space, let μ be a Radon measure on X , and let μ^* be the maximal outer extension of μ . For a set $A \subset X$, the following are equivalent*

- (i) A is μ^* -measurable, and μ^* - σ -finite.
- (ii) There exists sequences $(K_n)_{n=1}^{\infty} \subset \mathcal{C}_X$ and $(D_n)_{n=1}^{\infty} \subset \mathcal{T}_X$, such that

$$\bigcup_{n=1}^{\infty} K_n \subset A \subset \bigcap_{n=1}^{\infty} D_n \quad \text{and} \quad \mu([\bigcap_{n=1}^{\infty} D_n] \setminus [\bigcup_{n=1}^{\infty} K_n]) = 0.$$

(The condition that A is μ^* - σ -finite means that there exists a sequence $(A_n)_{n=1}^{\infty}$ of subsets of X , with $A = \bigcup_{n=1}^{\infty} A_n$, and $\mu^*(A_n) < \infty$, for all $n \geq 1$.)

PROOF. (i) $\Rightarrow$ (ii). Assume A is μ^* -measurable and μ^* - σ -finite.

Claim 1: *There exists a sequence $(A_n)_{n=1}^{\infty}$ of μ^* -measurable sets, such that $A = \bigcup_{n=1}^{\infty} A_n$, and $\mu^*(A_n) < \infty$, $\forall n \geq 1$.*

A priori, we only know that there exists a sequence $(A_n^0)_{n=1}^{\infty}$ of subsets of X (not assumed to be μ^* -measurable), with $A = \bigcup_{n=1}^{\infty} A_n^0$, and $\mu^*(A_n^0) < \infty$, $\forall n \geq 1$. Using (13), we can choose however, for each $n \geq 1$, an open set E_n , with $A_n^0 \subset E_n$, and $\mu(E_n) < \infty$. In particular, E_n is μ^* -measurable, and so will be $A_n = A \cap E_n$. We clearly have $A = \bigcup_{n=1}^{\infty} A_n$, and $\mu^*(A_n) \leq \mu^*(E_n) < \infty$, $\forall n \geq 1$.

Using Claim 1, we start off by writing $A = \bigcup_{n=1}^{\infty} A_n$, with the A_n 's μ^* -measurable, and $\mu^*(A_n) < \infty$. For each $n \geq 1$, we use Lemma 7.1 to find a sequence $(L_n^p)_{p=1}^{\infty}$ of compact subsets of A_n , such that

$$\mu^*(A_n \setminus [\bigcup_{p=1}^{\infty} L_n^p]) = 0.$$

Let us list the countable collection $\{L_n^p : p, n \geq 1\}$ as a sequence $(K_n)_{n=1}^\infty$, so that we have

$$\bigcup_{n=1}^\infty K_n = \bigcup_{n=1}^\infty \bigcup_{p=1}^\infty L_n^p \subset \bigcup_{n=1}^\infty A_n = A.$$

Claim 2: The set $M = A \setminus (\bigcup_{n=1}^\infty K_n)$ is μ^* -negligible, i.e. $\mu^*(M) = 0$.

Indeed, if we define, for each $k \geq 1$ the set $M_k = A_k \setminus [\bigcup_{n=1}^\infty K_n]$, then we have the obvious equality $M = \bigcup_{k=1}^\infty M_k$, and the inclusions

$$M_k = A_k \setminus [\bigcup_{p=1}^\infty \bigcup_{n=1}^\infty L_n^p] \subset A_k \setminus [\bigcup_{p=1}^\infty L_k^p], \quad \forall k \geq 1,$$

which, by the choice of the L 's, prove that $\mu^*(M_k) = 0, \forall k \geq 1$.

We proceed now with the construction of the D 's. For each pair of integers (p, n) , we use (13) to find an open set $E_n^p \supset A_n$, such that $\mu(E_n^p) \leq \mu^*(A_n) + \frac{1}{2^{p+n}}$. Since the A_n 's are μ^* -measurable, we have

$$\mu(E_n^p) = \mu^*(E_n^p) = \mu^*(A_n) + \mu^*(E_n^p \setminus A_n).$$

Since $\mu^*(A_n) < \infty$, by the choice of the E 's, we will get

$$(20) \quad \mu^*(E_n^p \setminus A_n) \leq \frac{1}{2^{p+n}}, \quad \forall p, n \geq 1.$$

We then define, for each $p \geq 1$, the open set $D_p = \bigcup_{n=1}^\infty E_n^p$. Notice that, for each $p \geq 1$, we have the inclusion $A = \bigcup_{n=1}^\infty A_n \subset \bigcup_{n=1}^\infty E_n^p = D_p$, and

$$D_p \setminus A = \bigcup_{n=1}^\infty [E_n^p \setminus A] \subset \bigcup_{n=1}^\infty [E_n^p \setminus A_n].$$

Using (20), we then get

$$(21) \quad \mu^*(D_p \setminus A) \leq \sum_{n=1}^\infty \mu^*(E_n^p \setminus A_n) \leq \sum_{n=1}^\infty \frac{1}{2^{p+n}} = \frac{1}{2^p}, \quad \forall p \geq 1.$$

Since $A \subset D_p, \forall p \geq 1$, we get $A \subset \bigcap_{p=1}^\infty D_p$. Moreover, if we define the set $N = [\bigcap_{p=1}^\infty D_p] \setminus A$, we obviously have the inclusions $N \subset D_p \setminus A, \forall p \geq 1$, and then (21) clearly forces $\mu^*(N) = 0$.

Now we have $\bigcup_{n=1}^\infty K_n \subset A \subset \bigcap_{p=1}^\infty D_p$, and $[\bigcap_{p=1}^\infty D_p] \setminus [\bigcup_{n=1}^\infty K_n] = N \cup M$, with $\mu^*(M) = \mu^*(N) = 0$, so we indeed have (ii).

The implication (ii) $\Rightarrow$ (i) is pretty obvious. If there exist sequences $(K_n)_{n=1}^\infty$ and $(D_n)_{n=1}^\infty$ as in (ii), then the sets $B = \bigcup_{n=1}^\infty K_n$ and $G = \bigcap_{n=1}^\infty D_n$ are Borel. Moreover, the inclusions $B \subset A \subset G$, give $A \setminus B \subset G \setminus B$, so we have $\mu^*(A \setminus B) \leq \mu^*(G \setminus B)$. By the second feature in (ii) we know that $\mu^*(G \setminus B) = 0$, therefore the set $P = A \setminus B$ is μ^* -negligible, hence μ^* -measurable. Since $A = B \cup P$, it follows that A is indeed μ^* -measurable. $\square$

COMMENT. The implication (ii) $\Rightarrow$ (i) in Theorem 7.3 holds without the μ^* - σ -finiteness assumption on A . In fact, condition (ii) actually forces A to be μ^* - σ -finite.

COROLLARY 7.3. If μ is a Radon measure on X , and the set A is μ^* -measurable, and μ^* - σ -finite, then one has the equality

$$\mu^*(A) = \sup \{ \mu(K) : K \in \mathcal{C}_X, K \subset A \}.$$

PROOF. Follow the first part of the proof of (i) $\Rightarrow$ (ii) to find a sequence $(K_n)_{n=1}^\infty$ of compact subsets of A , such that

$$\mu^*(A \setminus \bigcup_{n=1}^\infty K_n) = 0.$$

Since $\bigcup_{n=1}^\infty K_n$ is μ^* -measurable, this forces the equality

$$\mu^*(A) = \mu^*\left(\bigcup_{n=1}^\infty K_n\right) = \lim_{n \rightarrow \infty} \mu^*(K_1 \cup \cdots \cup K_n). \quad \square$$

Exercise 5.* Let X be a locally compact space, and let μ be a Radon measure on X . Suppose $\nu : \text{Bor}(X) \rightarrow [0, \infty]$ is a measure satisfying the following conditions:

- (a) $\nu(B) \leq \mu(B)$, $\forall B \in \text{Bor}(X)$;
- (b) for every $B \in \text{Bor}(X)$, one has the implication $\nu(B) < \infty \Rightarrow \mu(B) < \infty$.

Prove that ν is a Radon measure on X . (Notice that, in the case when μ is finite, the condition (b) is superfluous.)

HINTS: To prove condition (ii) in the definition of Radon measures, start with some open set $D \subset X$, and choose a sequence $K_1 \subset K_2 \subset \cdots \subset D$ of compact subsets, such that

$$\lim_{n \rightarrow \infty} \mu(K_n) = \mu(D),$$

and define the Borel set $B = \bigcup_{n=1}^\infty K_n \subset D$. Notice that we have the equalities $\mu(B) = \lim_{n \rightarrow \infty} \mu(K_n)$ and $\nu(B) = \lim_{n \rightarrow \infty} \nu(K_n)$. Argue that, when $\nu(D) = \infty$, we must have $\nu(B) = \infty$. When $\nu(D) < \infty$, show that $\mu(D \setminus B) = 0$. In either case we get $\nu(B) = \nu(D)$.

The next result explains somehow the anomaly illustrated by Exercise 3.

PROPOSITION 7.5. *If μ is a Radon measure on X , and let μ^* denote its maximal outer extension. For a subset $N \subset X$, the following are equivalent*

- (i) N is μ^* -measurable, and for every compact subset $K \subset N$, one has the equality $\mu(K) = 0$;
- (ii) $\mu^*(D \cap N) = 0$, for all open subsets $D \subset X$ with $\mu(D) < \infty$;
- (iii) N is locally μ^* -negligible, i.e.

$$\mu^*(A \cap N) = 0, \text{ for all subsets } A \subset X \text{ with } \mu^*(A) < \infty.$$

PROOF. (i) $\Rightarrow$ (ii). Assume N satisfies condition (i). Fix some open set $D \subset X$, with $\mu(D) < \infty$. Then the set $D \cap N$ is measurable, and $\mu^*(D \cap N) \leq \mu(D) < \infty$. The equality $\mu^*(D \cap N) = 0$ then follows from (i), combined with Corollary 7.3.

(ii) $\Rightarrow$ (iii). Assume N satisfies condition (ii). Fix some arbitrary subset $A \subset X$, with $\mu^*(A) < \infty$. Using (13), there exists some open set $D \supset A$ with $\mu(D) < \infty$. Then we have the inequality $\mu^*(A \cap N) \leq \mu^*(D \cap N)$, so condition (ii) will force $\mu^*(A \cap N) = 0$.

(iii) $\Rightarrow$ (i). Let N be locally μ^* -negligible. We know that local μ^* -negligibility implies μ^* -measurability (see Section 5). The fact that $\mu(K) = 0$, for all compact subsets $K \subset N$ is also trivial. $\square$

NOTATION. Let μ be a Radon measure on the locally compact space X , and let μ^* be the maximal outer extension of μ . We denote the σ -algebra $\mathcal{M}_{\mu^*}(X)$, of all μ^* -measurable subsets of X , simply by $\mathfrak{M}_\mu(X)$, and we define the measure $\tilde{\mu} = \mu^*|_{\mathfrak{M}_{\mu^*}(X)}$. Using the terminology introduced in Section 5, the pair $(\mathfrak{M}_\mu(X), \tilde{\mu})$ is the quasi-completion of $\text{Bor}(X)$ with respect to μ .

Our next goal is to examine the inclusion $Bor(X) \subset \mathfrak{M}_\mu(X)$ along the same lines used in the final part of Section 5. In preparation for the results that follow, it is helpful to introduce the following terminology.

DEFINITION. Let μ be a Radon measure on the locally compact space X . A non-empty compact subset $K \subset X$, is said to be μ -tight, if it has the property

- *there is no compact non-empty proper subset $L \subsetneq K$, with $\mu(K) = \mu(L)$.*

REMARK 7.3. Singleton sets are always μ -tight. If K is μ -tight, and $\mu(K) = 0$ then K must be a singleton.

For a non-empty compact set K with $\mu(K) > 0$, the μ -tightness is equivalent to the following condition¹¹:

$$(22) \quad \left. \begin{array}{l} D \subset X \text{ open} \\ D \cap K \neq \emptyset \end{array} \right\} \implies \mu(D \cap K) > 0.$$

Indeed, if K is μ -tight, and $D \subset X$ is an open set, such that $D \cap K \neq \emptyset$, then the compact set $L = K \setminus D$ is either empty, or a proper subset of K . In either case, we get $\mu(L) < \mu(K)$, and then the equality $D \cap K = K \setminus L$ gives $\mu(D \cap K) = \mu(K) - \mu(L) > 0$. Conversely, if K satisfies (22) and if L is a non-empty proper compact subset of K , then the set $D = X \setminus L$ is open, and satisfies $D \cap K \neq \emptyset$. By (22) this forces $\mu(D \cap K) > 0$, and since we have $L = K \setminus (D \cap K)$, we get $\mu(L) = \mu(K) - \mu(D \cap K) < \mu(K)$.

A μ -tight compact set K , with $\mu(K) > 0$, will be called *non-degenerate*.

LEMMA 7.2. *Let X be a locally compact space, let μ be a Radon measure on X . Every non-empty compact set $K \subset X$ has a μ -tight compact subset $K_0 \subset K$, with $\mu(K_0) = \mu(K)$.*

PROOF. If K is already tight, there is nothing to prove. Also, if $\mu(K) = 0$, then we can pick K_0 to be of the form $\{x\}$, with x any point in K .

For the remainder of the proof, we are going to assume that K is not μ -tight, and $\mu(K) > 0$. Consider the collection

$$\mathcal{L} = \{L \in \mathcal{C}_X : \emptyset \neq L \subsetneq K \text{ and } \mu(L) = \mu(K)\}.$$

Since K is not μ -tight, the collection $\mathcal{L}$ is non-empty. One key property of the collection $\mathcal{L}$ is the following.

Claim 1: If $L_1, \dots, L_n \in \mathcal{L}$, then $L_1 \cap \dots \cap L_n \in \mathcal{L}$.

Indeed, if we define the sets $A_j = K \setminus L_j$, $j = 1, \dots, n$, then $\mu(A_1) = \dots = \mu(A_n) = 0$, and then the equality

$$K \setminus [L_1 \cap \dots \cap L_n] = A_1 \cup \dots \cup A_n$$

will force $\mu(K \setminus [L_1 \cap \dots \cap L_n]) = 0$, thus giving $\mu(L_1 \cap \dots \cap L_n) = \mu(K) > 0$. (The last inequality forces of course $L_1 \cap \dots \cap L_n \neq \emptyset$.)

Using the *finite intersection property*, it follows that the intersection $K_0 = \bigcap_{L \in \mathcal{L}} L$ is non-empty.

Claim 2: $K_0 \in \mathcal{L}$.

Obviously K_0 is compact non-empty proper subset of K , so the only thing we need to prove is the equality $\mu(K_0) = \mu(K)$. Consider the Borel subset

$$B = K \setminus K_0 \subset K.$$

¹¹ Notice that using $D = X$, condition (22) actually forces $\mu(K) > 0$.

Since $B \subset K$, it follows that $\mu(B) < \infty$. By Corollary 7.3 we have

$$(23) \quad \mu(B) = \sup \{ \mu(P) : P \text{ compact, } P \subset B \}.$$

Notice however that if $P \subset B$ is compact, then we have, by the definition of B , the equality

$$\bigcap_{L \in \mathcal{L}} (L \cap P) = \left(\bigcap_{L \in \mathcal{L}} L \right) \cap P = (K \setminus B) \cap P = \emptyset,$$

so again by the *finite intersection property*, combined with Claim 1, it follows that there exists $L \in \mathcal{L}$, such that $P \cap L = \emptyset$. Then we have $\mu(K \setminus L) = 0$, so the inclusion $P \subset K \setminus L$ will force $\mu(P) = 0$. Using (23), this forces $\mu(B) = 0$.

We now show that K_0 is μ -tight. Indeed, if K_0 were not tight, we could find some non-empty compact proper subset $L \subsetneq K_0$, with $\mu(L) = \mu(K_0) = \mu(K)$. This will of course force L to belong to $\mathcal{L}$, and therefore it will force the inclusion $K_0 \subset L$, which is impossible. $\square$

LEMMA 7.3. *Let X be a locally compact space, let ν be a Radon measure on X , and let $\mathcal{G}$ be a pair-wise disjoint collection of non-degenerate μ -tight compact sets. For any set $A \subset X$, with $\mu^*(A) < \infty$, the collection*

$$S_{\mathcal{G}}(A) = \{ G \in \mathcal{G} : G \cap A \neq \emptyset \}$$

is at most countable.

PROOF. Since $\mu^*(A) < \infty$, by (13), there exists some open set $D \supset A$ with $\mu(D) < \infty$. It is obvious that $S_{\mathcal{G}}(A) \subset S_{\mathcal{G}}(D)$, so it suffices to prove that $S_{\mathcal{G}}(D)$ is at most countable.

On the one hand, we notice that, for every finite subset $\mathcal{F} \subset S_{\mathcal{G}}(D)$, one has

$$\sum_{G \in \mathcal{F}} \mu(G \cap D) = \mu\left(\bigcup_{G \in \mathcal{F}} [G \cap D]\right) \leq \mu(D) < \infty.$$

This means that the family $(\mu(G \cap D))_{G \in S_{\mathcal{G}}(D)}$ is *summable*, and we have

$$\sum_{G \in S_{\mathcal{G}}(D)} \mu(G \cap D) \leq \mu(D) < \infty.$$

On the other hand, by Remark 7.3, we know that all the terms $\mu(G \cap D)$, $G \in S_{\mathcal{G}}(D)$ are strictly positive. Using Proposition II.2.2, this forces $S_{\mathcal{G}}(D)$ to be countable. $\square$

The main application of the above result is the following.

THEOREM 7.4. *Let X be a locally compact space, and let μ be a Radon measure on X . Then there exists a partition $\mathcal{F}$ of X into μ -tight compact sets, with the property that the set*

$$N_{\mathcal{F}} = \bigcup_{\substack{F \in \mathcal{F} \\ \mu(F) = 0}} F$$

is locally μ^ -negligible.*

PROOF. Define the set

$$\Omega = \{ \mathcal{F} : \mathcal{F} \text{ pairwise disjoint collection of non-degenerate } \mu\text{-tight compact sets} \}.$$

We agree to consider the empty collection as an element of Ω , so that Ω is non-empty. Equip the set Ω with the order relation $\subset$ given by inclusion.

Claim 1: The ordered set $(\Omega, \subset)$ contains a maximal element.

This is a straightforward application of Zorn's Lemma. Start with some subset Λ of Ω , which is totally ordered with respect to $\subset$, and let us show that there is an upper bound for Λ (in Ω). If we write $\Lambda = \{\mathcal{G}_i : i \in I\}$, we define the collection $\mathcal{G} = \bigcup_{i \in I} \mathcal{G}_i$. It is clear that every element in $\mathcal{G}$ is a non-degenerate μ -tight compact set. If $K, L \in \mathcal{G}$ are different elements, then there exist $i, j \in I$ with $K \in \mathcal{G}_i$ and $L \in \mathcal{G}_j$. Since Λ is totally ordered, we either have $\mathcal{G}_i \subset \mathcal{G}_j$, or $\mathcal{G}_j \subset \mathcal{G}_i$. In either case, we conclude that there exists some $k \in I$, such that $K, L \in \mathcal{G}_k$, and then $K \cap L = \emptyset$. This shows that $\mathcal{G}$ is pairwise disjoint, hence $\mathcal{G}$ belongs to Ω . It is obvious that $\mathcal{G}$ is an upper bound for Λ .

Having proven Claim 1, we fix a maximal collection $\mathcal{G} \in \Omega$, and we define the set $T = \bigcup_{G \in \mathcal{G}} G$. It is quite possible that $\mathcal{G} = \emptyset$. In that case we define $T = \emptyset$.

Claim 2: For every compact subset $K \subset X \setminus T$, one has $\mu(K) = 0$.

We prove this by contradiction. Assume $\mu(K) > 0$. By Lemma 7.3 there exists a μ -tight compact subset $K_0 \subset K$, with $\mu(K_0) = \mu(K) > 0$ (in particular K_0 is non-degenerate). But then the collection $\mathcal{G} \cup \{K_0\}$ would obviously contradict the maximality of $\mathcal{G}$.

Claim 3: Whenever $D \subset X$ is an open set with $\mu(D) < \infty$, it follows that the set $D \setminus T$ is Borel, and $\mu(D \setminus T) = 0$.

By Lemma 7.3, the collection

$$S_{\mathcal{G}}(D) = \{G \in \mathcal{G} : G \cap D \neq \emptyset\}$$

is at most countable. Now we have

$$D \cap T = \bigcup_{G \in S_{\mathcal{G}}(D)} (D \cap G),$$

so $D \cap T$ is a countable union of Borel sets, hence $D \cap T$ itself is Borel, and so will be $D \setminus T = D \setminus (D \cap T)$. Since $\mu(D \setminus T) \leq \mu(D) < \infty$, by Corollary 7.3, we have

$$\mu(D \setminus T) = \sup \{\mu(K) : K \text{ compact, } K \subset D \setminus T\}.$$

By Claim 2 this, forces $\mu(D \setminus T) = 0$.

Going back to the proof of the theorem, we notice that, by Claim 2, none of the singletons $\{x\}$, $x \in X \setminus T$, has positive measure. We can then define collection

$$\mathcal{F} = \mathcal{G} \cup \{\{x\} : x \in X \setminus T\},$$

which is obviously a partition of X into μ -tight compact sets. For this partition, we obviously have the equality $N_{\mathcal{F}} = X \setminus T$. By Claim 3, we have

$$\mu^*(N_{\mathcal{F}} \cap D) = 0, \text{ for all open sets } D \subset X \text{ with } \mu(D) < \infty.$$

By Proposition 7.2, it follows that $N_{\mathcal{F}}$ is indeed locally μ^* -negligible. $\square$

DEFINITION. Let X be a locally compact space, and let μ be a Radon measure on X . A partition $\mathcal{F}$ of X into μ -tight compact sets, with the property stated in Theorem 7.3, will be called *non-degenerate*.

The existence of such partitions is significant, as indicated below.

THEOREM 7.5. *Let X be a locally compact space, let μ be a Radon measure on X , and let $\mathcal{F}$ be a non-degenerate partition of X into μ -tight compact sets. Then¹² $\mathcal{F}$ is a sufficient μ -finite Bor(X)-partition of X .*

¹² See Section 5 for the terminology.

PROOF. What we to prove are the following properties:

- (i) $\mathcal{F}$ is pairwise disjoint, and $\bigcup_{F \in \mathcal{F}} F = X$;
- (ii) $\mathcal{F} \subset \mathcal{B}$, and $\mu(F) < \infty$, for all $F \in \mathcal{F}$;
- (iii) for every set $B \in \text{Bor}(X)$, with $\mu(B) < \infty$, one has the equality¹³

$$(24) \quad \mu(B) = \sum_{F \in \mathcal{F}} \mu(B \cap F).$$

Conditions (i) and (ii) are obvious.

To prove condition (iii) we define the sub-collection $\mathcal{G} = \{F \in \mathcal{F} : \mu(F) > 0\}$, so that $\mathcal{G}$ consists on non-degenerate μ -tight compact sets, and the set

$$N_{\mathcal{F}} = X \setminus \left(\bigcup_{F \in \mathcal{G}} F \right)$$

is locally μ^* -negligeable. Assume now $B \in \text{Bor}(X)$ has $\mu(B) < \infty$. By Lemma 7.3, the collection

$$S_{\mathcal{G}}(B) = \{F \in \mathcal{G} : B \cap F \neq \emptyset\}$$

is at most countable. In particular, the set

$$B_0 = \bigcup_{F \in S_{\mathcal{G}}(B)} (B \cap F) = B \setminus N_{\mathcal{F}}$$

is Borel, and so will be $B \setminus B_0 = B \cap N_{\mathcal{F}}$. On the one hand, since $B \setminus B_0$ is a subset of $N_{\mathcal{F}}$, it follows that $B \setminus B_0$ is locally μ^* -negligeable. On the other hand, since $B \setminus B_0$ is a subset of B , it follows that $\mu(B \setminus B_0) < \infty$. This clearly forces $\mu(B \setminus B_0) = 0$, so we have the equality

$$(25) \quad \mu(B) = \mu(B_0) = \sum_{F \in S_{\mathcal{G}}(B)} \mu(B \cap F).$$

Notice that, if $F \in \mathcal{F} \setminus S_{\mathcal{G}}(B)$, then either $F \notin \mathcal{G}$, in which case we have $\mu(F) = 0$, or $F \in \mathcal{G} \setminus S_{\mathcal{G}}(B)$, in which case we have $\mu(B \cap F) = 0$. This shows that

$$\mu(B \cap F) = 0, \quad \forall F \in \mathcal{F} \setminus S_{\mathcal{G}}(B),$$

so the equality (25) immediately gives (24). $\square$

COROLLARY 7.4. *Under the hypothesis above, the collection $\mathcal{F}$ is a $\tilde{\mu}$ -finite decomposition for $\mathfrak{M}_{\mu}(X)$.*

PROOF. Immediate from Corollary 5.3. $\square$

In the remainder of this section we discuss two basic examples of methods for constructing (regular) contents.

To introduce the first construction, let us recall some notations and terminology introduced in II.5 For a locally compact space X , and $\mathbb{K}$ one of the fields $\mathbb{R}$ or $\mathbb{C}$, we denote by $C_c^{\mathbb{K}}(X)$ the space of all continuous functions $f : X \rightarrow \mathbb{K}$, with compact support. A $\mathbb{R}$ -linear map $\phi : C_c^{\mathbb{R}}(X) \rightarrow \mathbb{R}$ is said to be *positive*, if it has the property:

$$f \in C_c^{\mathbb{R}}(X), \quad f \geq 0 \Rightarrow \phi(f) \geq 0.$$

With these notations, we have the following result.

¹³ Here we use the summation convention from II.2

PROPOSITION 7.6. *Let X be a locally compact space, and let $\phi : C_c^{\mathbb{R}}(X) \rightarrow \mathbb{R}$ be a positive $\mathbb{R}$ -linear map. For every compact subset $K \subset X$, define the number*

$$\omega_{\phi}(K) = \inf \{ \phi(f) : f \in C_c^{\mathbb{R}}(X), f \geq \kappa_K \}.$$

Then the map $\mathcal{C}_X \ni K \mapsto \omega_{\phi}(K) \in [0, \infty)$ is a regular content on X .

PROOF. The inequality $f \geq \kappa_K$ forces $f \geq 0$, so we indeed have $\omega_{\phi}(K) \geq 0$, $\forall K \in \mathcal{C}_X$. We now check conditions (i)-(iv) in the definition of a content.

The constant function 0 satisfies $0 \geq \kappa_{\emptyset}$, which immediately gives the equality $\omega_{\phi}(\emptyset) = 0$, so condition (i) is satisfied.

By the definition of ω_{ϕ} , it is clear that one has the implication

$$K, L \in \mathcal{C}_X, K \subset L \implies \omega_{\phi}(K) \leq \omega_{\phi}(L),$$

thus giving condition (ii).

To check condition (iii), suppose $K, L \in \mathcal{C}_X$, and let us prove the inequality

$$(26) \quad \omega_{\phi}(K \cup L) \leq \omega_{\phi}(K) + \omega_{\phi}(L).$$

Start with some $\varepsilon > 0$, and choose functions $f, g \in C_c^{\mathbb{R}}(X)$, such that $f \geq \kappa_K$, $g \geq \kappa_L$, $\phi(f) \leq \omega_{\phi}(K) + \varepsilon$, and $\phi(g) \leq \omega_{\phi}(L)$. If we consider the function $h = f + g \in C_c^{\mathbb{R}}(X)$, then we clearly have $h \geq \kappa_{K \cup L}$, so we will have

$$\omega_{\phi}(K \cup L) \leq \phi(h) = \phi(f + g) = \phi(f) + \phi(g) \leq \omega_{\phi}(K) + \omega_{\phi}(L) + 2\varepsilon.$$

Since the inequality $\omega_{\phi}(K \cup L) \leq \omega_{\phi}(K) + \omega_{\phi}(L) + 2\varepsilon$ holds for arbitrary $\varepsilon > 0$, it will clearly force (26)

Finally, to check condition (iv) we need start with two disjoint sets $K, L \in \mathcal{C}_X$, and we prove the equality

$$(27) \quad \omega_{\phi}(K \cup L) = \omega_{\phi}(K) + \omega_{\phi}(L).$$

By (26) it only suffices to show the inequality

$$(28) \quad \omega_{\phi}(K \cup L) \geq \omega_{\phi}(K) + \omega_{\phi}(L).$$

Start with some arbitrary $\varepsilon > 0$, and choose a function $f \in C_c^{\mathbb{R}}(X)$, with $f \geq \kappa_{K \cup L}$ and $\phi(f) \leq \omega_{\phi}(K \cup L) + \varepsilon$. Use Uryshon Lemma for locally compact spaces (Theorem I.5.1) to find a continuous map $\theta : X \rightarrow [0, 1]$, such that $\theta|_K = 1$ and $\theta|_L = 0$. The functions $g = f\theta$ and $h = f(1 - \theta)$ are obviously continuous, and have compact supports. Moreover, one has the inequalities $g \geq \kappa_K$ and $h \geq \kappa_L$. Since $g + h = f$, we get

$$\omega_{\phi}(K \cup L) + \varepsilon \geq \phi(f) = \phi(g + h) = \phi(g) + \phi(h) \geq \omega_{\phi}(K) + \omega_{\phi}(L).$$

Since the inequality $\omega_{\phi}(K \cup L) + \varepsilon \geq \omega_{\phi}(K) + \omega_{\phi}(L)$ holds for all $\varepsilon > 0$, it will clearly force the inequality (28)

So far, we have shown that ω_{ϕ} is a content. We now prove that ω_{ϕ} is regular, which means that, for every $K \in \mathcal{C}_X$, one has the equality

$$\omega_{\phi}(K) = \inf \{ \omega_{\phi}(L) : L \in \mathcal{C}_X, K \subset \text{Int}(L) \}.$$

By property (ii) we always have the inequality

$$\omega_{\phi}(K) \leq \inf \{ \omega_{\phi}(L) : L \in \mathcal{C}_X, K \subset \text{Int}(L) \},$$

so all we need to prove is the inequality

$$(29) \quad \omega_{\phi}(K) \geq \inf \{ \omega_{\phi}(L) : L \in \mathcal{C}_X, K \subset \text{Int}(L) \}.$$

Start with some arbitrary $\varepsilon > 0$, and choose a function $f \in C_c^{\mathbb{R}}(X)$ with $f \geq \varkappa_K$, and $\phi(f) \leq \omega_\phi(K) + \varepsilon$. Consider the function $g = (1 + \varepsilon)f$, and the set

$$D = \{x \in X : g(x) > 1\}.$$

Obviously D is an open set, and since $f(x) \geq 1, \forall x \in K$, we get $g(x) \geq 1 + \varepsilon > 1, \forall x \in K$. In particular, this gives the inclusion $K \subset D$. Apply then Lemma I.5.1 to find some compact set $L \subset D$, with $K \subset \text{Int}(L)$. Since $g(x) > 1, \forall x \in L$, we clearly have

$$\omega_\phi(L) \leq \phi(g) = (1 + \varepsilon)\phi(f) \leq (1 + \varepsilon)(\omega_\phi(K) + \varepsilon).$$

This argument shows that, if we denote the right hand side of (29) by $\nu(K)$, then we have the inequality

$$\nu(K) \leq (1 + \varepsilon)(\omega_\phi(K) + \varepsilon).$$

Since this inequality holds for all $\varepsilon > 0$, it will force the inequality $\nu(K) \leq \omega_\phi(K)$, thus proving (29). $\square$

DEFINITION. Let X be a locally compact space, and let $\phi : C_c^{\mathbb{R}}(X) \rightarrow \mathbb{R}$ be a positive $\mathbb{R}$ -linear map. We apply Corollary 7.2 to the regular content ω_ϕ , and we will denote the Radon measure extension of ω_ϕ simply by μ_ϕ . The measure μ_ϕ on $\text{Bor}(X)$ is called the *Riesz measure associated with ϕ* .

An interesting property, which will later be generalized, is the following.

LEMMA 7.4 (Mean Value Property). *Let X be a locally compact space, let $\phi : C_c^{\mathbb{R}}(X) \rightarrow \mathbb{R}$ be a positive $\mathbb{R}$ -linear map, and let μ_ϕ be the Riesz measure associated with ϕ . For any function $f \in C_c^{\mathbb{R}}(X)$, and any compact subset $K \subset X$, with $K \supset \text{supp } f$, one has the inequality*

$$(30) \quad \left[\min_{x \in K} f(x) \right] \cdot \mu_\phi(K) \leq \phi(f) \leq \left[\max_{x \in K} f(x) \right] \cdot \mu_\phi(K).$$

PROOF. Since $\min_{x \in K} f(x) = -\max_{x \in K} (-f)(x)$, it suffices to prove only the inequality

$$(31) \quad \phi(f) \leq \left[\max_{x \in K} f(x) \right] \cdot \mu_\phi(K).$$

Fix $f \in C_c^{\mathbb{R}}(X)$, as well as the compact set $K \supset \text{supp } f$. Denote the number $\max_{x \in K} f(x)$ simply by M .

If $M < 0$ the inequality is pretty clear, because the function $g = \frac{f}{M}$ satisfies $g \geq \varkappa_K$, which gives $\phi(g) \geq \omega_\phi(K) = \mu_\phi(K)$, and then multiplying by M immediately gives (31).

The case $M = 0$ is also trivial, since this forces $f \leq 0$, so we get $\phi(f) \leq 0$.

Assume $M > 0$. Fix for the moment some $\varepsilon > 0$, and choose some function $h \in C_c^{\mathbb{R}}(X)$, with $h \geq \varkappa_K$, and $\phi(h) \leq \mu_\phi(K) + \varepsilon$.

Let us observe that $Mh - f \geq 0$. Indeed, if we start with some arbitrary point $x \in X$, then either $x \in K$, in which case we have $Mh(x) \geq M \geq f(x)$, or we have $x \in X \setminus K$, in which case $Mh(x) \geq 0 = f(x)$.

Using the positivity of ϕ we then get $\phi(Mh - f) \geq 0$, which by the choice of h gives

$$\phi(f) \leq \phi(Mh) = M\phi(h) \leq M(\mu_\phi(K) + \varepsilon).$$

Since the inequality $\phi(f) \leq M(\mu_\phi(K) + \varepsilon)$ holds for arbitrary $\varepsilon > 0$, it will clearly force $\phi(f) \leq M\mu_\phi(K)$. $\square$

The Riesz measure can be implicitly characterized by the following result.

PROPOSITION 7.7. *With the notations above, the Riesz measure μ_ϕ is the unique Radon measure which has the interpolation property:*

(I_ϕ) *whenever $F \subset X$ is compact, $D \subset X$ is open, and $f \in C_c^\mathbb{R}(X)$ satisfies $\kappa_F \leq f \leq \kappa_D$, it follows that one has the inequality*

$$\mu_\phi(F) \leq \phi(f) \leq \mu_\phi(D).$$

PROOF. Let us first show that μ_ϕ has property (I_ϕ). Start with F , D and f as in (I_ϕ). Since $\mu_\phi(F) = \omega_\phi(F)$, by the definition of ω_ϕ , we immediately get the inequality $\mu_\phi(F) \leq \phi(f)$.

To prove the inequality $\phi(f) \leq \mu_\phi(D)$, we need some preparations. For every integer $n \geq 1$ we define the sets

$$A_n = \{x \in X : f(x) > \frac{1}{n}\} \text{ and } B_n = \{x \in X : f(x) \geq \frac{1}{n}\}.$$

Define also the set $E = \{x \in X : f(x) > 0\}$, so that $\overline{E} = \text{supp } f$. (Here we use the obvious fact that $f \geq 0$.) The sets A_n , $n \geq 1$ are open. The sets B_n , $n \geq 1$ are closed subsets of $E \subset \overline{E}$, hence they are compact. Notice also that we have the inclusions

$$A_1 \subset B_1 \subset A_2 \subset B_2 \subset \cdots \subset E \subset D.$$

For every $n \geq 1$, we use Urysohn Lemma to find a continuous function $h_n : X \rightarrow [0, 1]$, with $h_n|_{B_n} = 1$ and $h_n|_{X \setminus A_{n+1}} = 0$. On the one hand, we notice that the function $f(1 - h_n)$ has the support contained in the compact set $\overline{E} \setminus A_n \subset X \setminus A_n$. Moreover, since we clearly have $f(x) \leq \frac{1}{n}$, $\forall x \in X \setminus A_n$, by Lemma 7.4 we get the inequality

$$\phi(f) = \phi(fh_n) + \phi(f(1 - h_n)) \leq \phi(fh_n) + \frac{\mu_\phi(\overline{E} \setminus A_n)}{n} \leq \phi(fh_n) + \frac{\mu_\phi(\overline{E})}{n}, \quad \forall n \geq 1,$$

which shows that

$$(32) \quad \phi(f) \leq \limsup_{n \rightarrow \infty} \phi(fh_n).$$

On the other hand, for each $n \geq 1$, the function fh_n has support contained in B_{n+1} , and $(fh_n)(x) \leq 1$, $\forall x \in B_{n+1}$, so again by Lemma 7.4 combined with the inclusion $B_{n+1} \subset D$, we get

$$\phi(fh_n) \leq \mu_\phi(B_{n+1}) \leq \mu_\phi(D).$$

Using (32) we immediately get $\phi(f) \leq \mu_\phi(D)$.

We now prove the uniqueness. Let μ be a Radon measure with property (I_ϕ).

Claim 1: For any compact set $K \subset X$ and any open set $D \subset X$, with $K \subset D$, one has the inequality

$$\mu_\phi(K) \leq \mu(D).$$

Choose a compact set $L \subset X$, with $K \subset \text{Int}(L) \subset L \subset D$, and use Urysohn Lemma to find a continuous function $f : X \rightarrow [0, 1]$ such that $f|_K = 1$ and $f|_{X \setminus \text{Int}(L)} = 0$. In particular, f has compact support, and satisfies $\kappa_K \leq f \leq \kappa_D$. Using (I_ϕ) for μ_ϕ and for μ , we then get $\mu_\phi(K) \leq \phi(f) \leq \mu(D)$, and we are done.

Claim 2: for every compact set $K \subset X$, one has the equality $\mu_\phi(K) = \mu(K)$.

On the one hand, by the definition of the Radon measure, we have

$$\mu(K) = \inf \{ \mu(D) : D \subset X \text{ open, with } D \supset K \}.$$

By Claim 1, this immediately gives the inequality $\mu_\phi(K) \leq \mu(K)$. On the other hand, if we choose, for every $\varepsilon > 0$, a function $f_\varepsilon \in C_c^\mathbb{R}(X)$ with $f \geq \kappa_K$ and $\phi(f) \leq \mu_\phi(K) + \varepsilon$, then the function $g_\varepsilon = \min\{f_\varepsilon, 1\}$ will also satisfy $g_\varepsilon \geq \kappa_K$, and $\phi(g_\varepsilon) \leq \phi(f_\varepsilon) \leq \mu_\phi(K) + \varepsilon$. Applying (I_ϕ) for μ , with $C = K$ and $X = D$ will then force $\mu(K) \leq \phi(g_\varepsilon) \leq \mu_\phi(K) + \varepsilon$. Since the inequality $\mu(K) \leq \mu_\phi(K) + \varepsilon$ holds for all $\varepsilon > 0$, it will force $\mu(K) \leq \mu_\phi(K)$.

Having proven Claim 2, we now see that, using condition (iii) in the definition of Radon measures, we get the equality $\mu(D) = \mu_\phi(D)$, for all open sets $D \subset X$. Using condition (ii) from the definition, it then follows that $\mu(B) = \mu_\phi(B)$, $\forall B \in \text{Bor}(X)$. $\square$

COMMENT. The *Riesz correspondence*

$$\left\{ \begin{array}{c} \text{positive } \mathbb{R}\text{-linear maps} \\ C_c^\mathbb{R}(X) \rightarrow \mathbb{R} \end{array} \right\} \ni \phi \longmapsto \mu_\phi \in \left\{ \begin{array}{c} \text{Radon measures} \\ \text{on } X \end{array} \right\}.$$

will be studied in Chapter IV, where we will eventually prove the fact that it is bijective. At this point we simply regard it as a method of constructing Radon measures.

PROPOSITION 7.8. *Let X be a locally compact space. Then the Riesz correspondence is “linear” in the following sense.*

- (i) *If $\phi : C_c^\mathbb{R}(X) \rightarrow \mathbb{R}$ is a positive $\mathbb{R}$ -linear map, and $t \in [0, \infty)$, then $t\phi$ is also a positive $\mathbb{R}$ -linear map, and one has the equality $\mu_{t\phi} = t\mu_\phi$.*
- (ii) *If $\phi_1, \phi_2 : C_c^\mathbb{R}(X) \rightarrow \mathbb{R}$ are positive $\mathbb{R}$ -linear maps, then $\phi_1 + \phi_2$ is also a positive $\mathbb{R}$ -linear map, and one has the equality $\mu_{\phi_1 + \phi_2} = \mu_{\phi_1} + \mu_{\phi_2}$.*

PROOF. (i). Assume ϕ is positive and $t \in [0, \infty)$. The fact that $t\phi$ is positive is trivial. We know, by Proposition 7.2, that $t\mu_\phi$ is a Radon measure. Then the equality $\mu_{t\phi} = t\mu_\phi$ follows from Proposition 7.5, combined with the obvious fact that $\mu_{t\phi}$ has the interpolation property $(I_{t\phi})$.

(ii). If ϕ_1 and ϕ_2 are positive, then so is $\phi_1 + \phi_2$. Define $\psi = \phi_1 + \phi_2$, and $\nu = \mu_{\phi_1} + \mu_{\phi_2}$. By Proposition 7.2, we again know that ν is a Radon measure. The equality $\mu_\psi = \nu$ follows from Proposition 7.5, combined with the obvious fact that ν has the interpolation property (I_ψ) . $\square$

The Riesz correspondence is also functorial, with respect to proper maps, in the following sense.

PROPOSITION 7.9. *Let X and Y be locally compact spaces, let $\Phi : X \rightarrow Y$ be a proper continuous map, and let $\phi : C_c^\mathbb{R}(X) \rightarrow \mathbb{R}$ be a positive linear map.*

- (i) *Whenever $f : Y \rightarrow \mathbb{R}$ is a continuous function with compact support, it follows that the composition $f \circ \Phi : X \rightarrow \mathbb{R}$ is also a continuous function with compact support.*
- (ii) *The map*

$$\psi : C_c^\mathbb{R}(Y) \ni f \longmapsto \phi(f \circ \Phi) \in \mathbb{R}$$

is $\mathbb{R}$ -linear and positive.

- (iii) If μ_ϕ is the Riesz measure on X defined by ϕ , and if μ_ψ is the Riesz measure on Y defined by ψ , then one has the equality

$$\mu_\psi(B) = \mu_\phi(\Phi^{-1}(B)), \quad \forall B \in \text{Bor}(Y).$$

PROOF. (i). This statement is trivial, since Φ is proper.

(ii). The linearity of ψ is a consequence of the linearity of the map

$$T : C_c^\mathbb{R}(Y) \ni f \longmapsto f \circ \Phi \in C_c^\mathbb{R}(X),$$

and of the obvious equality $\psi = \phi \circ T$.

(iii). Use Proposition 7.3, which states that the map $\nu : \text{Bor}(Y) \rightarrow [0, \infty]$, defined by

$$\nu(B) = \mu_\phi(\Phi^{-1}(B)), \quad \forall B \in \text{Bor}(Y),$$

is a Radon measure. In order to prove statement (iii), which reads $\mu_\psi = \nu$, we observe that, using Proposition 7.5, it suffices to prove that ν has the interpolation property (1_ψ) . Fix then a compact set K and an open set $D \subset Y$, as well as a function $f \in C_c^\mathbb{R}(Y)$, such that $\kappa_K \leq f \leq \kappa_D$, and let us prove the inequalities

$$(33) \quad \nu(K) \leq \psi(f) \leq \nu(D).$$

Define the compact set $L = \Phi^{-1}(K)$ (here we use the fact that Φ is proper), and define the open set $E = \Phi^{-1}(D) \subset X$, so that $\nu(K) = \mu_\phi(L)$ and $\nu(D) = \mu_\phi(E)$. If we define, using (i), the function $g = f \circ \Phi \in C_c^\mathbb{R}(X)$, then we have $\psi(f) = \phi(g)$, and the inequalities (33) are the same as the inequalities

$$\mu_\phi(L) \leq \phi(g) \leq \mu_\phi(E).$$

But these inequalities follow immediately from the interpolation property of μ_ϕ , combined with the obvious inequalities $\kappa_L \leq g \leq \kappa_E$. $\square$

REMARKS 7.4. Let X be a locally compact space, let $\phi : C_c^\mathbb{R}(X) \rightarrow \mathbb{R}$ be a positive $\mathbb{R}$ -linear map, and let μ_ϕ be the Riesz measure defined by ϕ .

A. One has the equality

$$(34) \quad \mu_\phi(X) = \sup \{ \phi(f) : f \in C_c^\mathbb{R}(X), 0 \leq f \leq 1 \}.$$

Indeed, if we denote the right hand side of (34) by M , then the inequality $\mu_\phi(X) \geq M$ is immediate from the interpolation property. In fact, if for each compact set $K \subset X$, we choose (use Urysohn Lemma) some continuous function $f_K : X \rightarrow [0, 1]$, with compact support, such that $f_K|_K = 1$, then by the interpolation property we get $M \geq \phi(f_K) \geq \mu_\phi(K)$, so we have

$$M \geq \sup \{ \mu_\phi(K) : K \in \mathcal{C}_X \} = \mu_\phi(X).$$

B. As a consequence of the equality (34), and of Remark II.5.4, we get the equivalence

$$\phi \text{ continuous} \Leftrightarrow \mu_\phi(X) < \infty.$$

Moreover, in this case one has the equality $\|\phi\| = \mu_\phi(X)$.

C. Assume X is non-compact, and ϕ is continuous. Then ϕ can be extended to a positive linear functional ϕ' on the completion $C_0^\mathbb{R}(X)$ of $C_c^\mathbb{R}(X)$. In this case the Riesz correspondence has a nice connection with the Alexandrov compactification $X^\alpha = X \sqcup \{\infty\}$ (see I.5 and II.5). Recall that $C_0^\mathbb{R}(X)$ is identified with the space of all continuous functions $f : X^\alpha \rightarrow \mathbb{R}$ with $f(\infty) = 0$. Moreover, ϕ' has a unique extension to a positive linear map $\psi : C^\mathbb{R}(X^\alpha) \rightarrow \mathbb{R}$, with $\|\phi\| = \|\phi'\| = \|\phi\|$.

We can then consider two Riesz measures μ_ϕ on X , and μ_ψ on X^α . One has the equality

$$(35) \quad \mu_\psi(B) = \mu_\phi(B \cap X), \quad \forall B \in \text{Bor}(X^\alpha).$$

First of all, remark that

$$(36) \quad \mu_\psi(K) = \mu_\phi(K), \quad \forall K \in \mathcal{C}_X.$$

This is a consequence of the fact that for every $g \in C^\mathbb{R}(X^\alpha)$ with $g \geq \varkappa_K$, there exists some $f \in C_c^\mathbb{R}(X)$, with $g \geq f \geq \varkappa_K$ (Simply take $f = gh$, for some continuous function $h : X \rightarrow [0, 1]$ with compact support, with $h|_K = 1$.) Using (36), we immediately get the equality

$$(37) \quad \mu_\psi(B \cap X) = \mu_\phi(B \cap X), \quad \forall B \in \text{Bor}(X^\alpha).$$

Using this with $B = X$, we get

$$\mu_\psi(X) = \mu_\phi(X) = \|\phi\| = \|\psi\| = \mu_\psi(X^\alpha),$$

which forces $\mu_\psi(\{\infty\}) = 0$, and then (35) is immediate from (37)

Exercise 6. Consider the case when $X = \mathbb{R}^n$. For every continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, with compact support, we define

$$\phi(f) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} f(x_1, x_2, \dots, x_n) dx_1 dx_2 \cdots dx_n,$$

where the numbers $a_1 < b_1, \dots, a_n < b_n$ are chosen (arbitrarily) such that

$$\text{supp } f \subset [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n].$$

(One can show that the multiple integral is independent of the choice of the a 's and the b 's.) It is obvious that this way we have constructed a positive $\mathbb{R}$ -linear map $\phi : C_c^\mathbb{R}(\mathbb{R}^n) \rightarrow \mathbb{R}$. The Riesz measure μ_ϕ , defined by ϕ , is precisely the Lebesgue measure λ_n .

HINT: Compute the values of μ_ϕ on compact boxes.

We conclude this section with an important result from harmonic analysis. The main object of study is explained in the following.

DEFINITION. A *topological group* is a group G , which comes also equipped with a topology, which is compatible with the group structure in the sense that the map

$$G \times G \ni (g, h) \longmapsto gh^{-1} \in G$$

is continuous. Remark that is equivalent to the fact that both maps $G \times G \ni (g, h) \longmapsto gh \in G$ and $G \ni g \longmapsto g^{-1} \in G$ are continuous. To avoid any complications, *all topological groups are assumed to be Hausdorff*.

EXAMPLES 7.2. A. Any group becomes a topological group, when equipped with the *discrete topology*. (This is the topology in which every subset is open.)

B. The group $(\mathbb{R}^n, +)$ is a topological group, when equipped with the norm topology.

C. The unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is a topological group, when equipped with the usual multiplication, and the topology induced from $\mathbb{C}$. More generally, for an integer $n \geq 1$, the n -dimensional torus $\mathbb{T}^n$, equipped with coordinate-wise multiplication, and the product topology, is a topological group.

D. Given an integer $n \geq 1$, the group $GL_n(\mathbb{R})$, of all invertible $n \times n$ matrices (with matrix multiplication as the group operation), is a topological group, when

equipped with the topology coming from the identification of $GL_n(\mathbb{R})$ as an open subset in $\mathbb{R}^{n^2}$.

NOTATIONS. Let G be a group. For a subset $A \subset G$ and an element $g \in G$, we define the left and right translations of A by g , as the sets

$$gA = \{gh : h \in A\} \text{ and } Ag = \{hg : h \in A\}.$$

For two subsets $A, B \subset G$, we define

$$A \cdot B = \{hk : h \in A, k \in B\}.$$

Finally, for a subset $A \subset G$, we define $A^{-1} = \{h^{-1} : h \in A\}$.

REMARK 7.5. There is some similarity between topological groups and metric spaces. The subsets that play role of open balls are the open neighborhoods of the identity. More explicitly, if G is a topological group, with identity element e , then one has the equalities

$$\begin{aligned} \{N \subset G : N \text{ open neighborhood of } g\} &= \\ &= \{gV \subset G : V \text{ open neighborhood of } e\} = \\ &= \{Wg \subset G : W \text{ open neighborhood of } e\}. \end{aligned}$$

For example, given a metric space (X, d) , a map $f : G \rightarrow X$ is continuous at some point $g \in G$, if and only if, for every $\varepsilon > 0$, there exists some neighborhood V_ε of e , such that

$$d(f(gh), f(g)) < \varepsilon, \quad \forall h \in V_\varepsilon.$$

The following two results will be used several times.

LEMMA 7.5. *Suppose G is a topological group, with identity element e . For any open neighborhood U of e , there exists an open neighborhoods V of e , such that $V = V^{-1}$ and $V \cdot V \subset U$.*

PROOF. Fix the open neighborhood U . Use the continuity of the map $G \times H \ni (g, h) \rightarrow gh \in G$, at (e, e) , to find an open neighborhood D of (e, e) in $G \times G$, such that

$$gh \in U, \quad \forall (g, h) \in D.$$

Since D is open in the product topology, there exist open neighborhoods U_1 and U_2 , of e , such that $U_1 \times U_2 \subset D$. Then we obviously have

$$U_1 \cdot U_2 \subset U.$$

Consider the open neighborhood $W = U_1 \cap U_2$. We still have $W \cdot W \subset U$. Finally, using the continuity of the map $G \ni g \mapsto g^{-1} \in G$, it follows that W^{-1} is also a neighborhood of e . Then we are done, if we take $V = W \cap W^{-1}$. $\square$

PROPOSITION 7.10. *Let G be a topological group, and let $K, L \subset G$ be two compact disjoint sets. Then there exists an open neighborhood V of the identity element e , such that $V = V^{-1}$ and $(K \cdot V) \cap (L \cdot V) = (V \cdot K) \cap (V \cdot L) = \emptyset$.*

PROOF. Consider the continuous map $\phi : G \times G \ni (g, h) \mapsto gh^{-1} \in G$, and the compact set $C = (K \times L) \cup (L \times K) \subset G \times G$. Since ϕ is continuous, it follows that $\phi(C)$ is a compact subset of G . The condition $K \cap L = \emptyset$ obviously gives the fact that $e \notin \phi(C)$. Since $\phi(C)$ is closed, there exists some open neighborhood U of e , such that $\phi(C) \cap U = \emptyset$. Use Lemma 7.5 to find some open neighborhood V of e , such that $V = V^{-1}$ and $V \cdot V \subset U$.

We now show that $(K \cdot V) \cap (L \cdot V) = \emptyset$. Suppose the contrary, i.e. there exist $g \in K$, $h \in L$, and $v, w \in V$, such that $gv = hw$. Then we get $h^{-1}g = wv^{-1} \in V \cdot V^{-1} = V \cdot V \subset U$, which is impossible, since $h^{-1}g$ also belongs to $\phi(C)$.

Finally, let us show that we also have $(V \cdot K) \cap (W \cdot L) = \emptyset$. Suppose the contrary, i.e. there exist $g \in K$, $h \in L$, and $v, w \in V$, such that $vg = wh$. Then we get $hg^{-1} = w^{-1}v \in V^{-1} \cdot V = V \cdot V \subset U$, which is impossible, since gh^{-1} also belongs to $\phi(C)$. $\square$

In what follows we are going to restrict our attention to those topological groups which are locally compact in their respective topology. The topological groups listed in Examples 7.2.A-D are all locally compact.

DEFINITION. Let G be a locally compact group. A Radon measure μ on G is called a *Haar measure on G* , if $\mu(G) > 0$, and μ has the *left invariance property*:

$$\mu(gA) = \mu(A), \quad \forall g \in G, \quad A \in \text{Bor}(G).$$

Remark that, for every $g \in G$ the map $\ell_g : G \ni h \mapsto gh \in G$ is a homeomorphism, so for a subset $A \subset G$, one has the equivalence $A \in \text{Bor}(G) \Leftrightarrow gA = \ell_g(A) \in \text{Bor}(G)$. Likewise, the map $r_g : G \ni h \mapsto hg \in G$ is a homeomorphism, so $A \in \text{Bor}(G) \Leftrightarrow Ag \in \text{Bor}(G)$.

REMARK 7.6. Let G be a locally compact group. For any element $g \in G$, and any function $F \in C_c^{\mathbb{R}}(G)$, we define the continuous functions $L_g F, R_g F : G \rightarrow \mathbb{R}$ by $L_g F = F \circ \ell_{g^{-1}}$ and $R_g F = F \circ r_g$. In other words,

$$(L_g F)(h) = F(g^{-1}h) \text{ and } (R_g F)(h) = F(hg), \quad \forall h \in G.$$

It is fairly obvious that $L_g F$ and $R_g F$ both have compact support. Moreover, for a fixed $g \in G$, the maps $L_g, R_g : C_c^{\mathbb{R}}(G) \rightarrow C_c^{\mathbb{R}}(G)$ are linear, and continuous in the norm defined in Exercise 5. One has the equalities

$$L_{gh} = L_g \circ L_h \text{ and } R_{gh} = R_g \circ R_h, \quad \forall g, h \in G,$$

as well as $L_e = R_e = \text{Id}$, where e denotes the identity element in G .

The following result gives a sufficient condition for a Riesz measure to be a Haar measure.

PROPOSITION 7.11. *Let G be a locally compact group, and let $\phi : C_c^{\mathbb{R}}(G) \rightarrow \mathbb{R}$ be a positive $\mathbb{R}$ -linear map, which is not identically zero, and has the left invariance property:*

$$\phi \circ L_g = \phi, \quad \forall g \in G.$$

Then the Riesz measure μ_ϕ is a Haar measure on G .

PROOF. The key property we need is contained in the following

Claim 1: For any $g \in G$, and any compact subset $K \subset G$, one has the equality $\mu_\phi(gK) = \mu_\phi(K)$.

Fix for the moment $g \in G$, as well as the compact set $K \subset G$. The set gK is compact, so we have

$$(38) \quad \mu_\phi(gK) = \inf \{ \phi(F) : F \in C_c^{\mathbb{R}}(G), F \geq \chi_{gK} \}.$$

Notice that if $F \in C_c^{\mathbb{R}}(G)$ satisfies $F \geq \chi_{gK}$, this means that $F(gh) \geq \chi_{gK}(gh)$, $\forall h \in G$. Notice that, for any $h \in G$, one has the equivalences

$$\chi_{gK}(gh) = 1 \Leftrightarrow gh \in gK \Leftrightarrow h \in K,$$

which means that

$$\kappa_{gK}(gh) = \kappa_K(h), \quad \forall h \in K.$$

The inequality $F \geq \kappa_{gK}$ then gives

$$F(gh) \geq \kappa_K(h), \quad \forall h \in G,$$

which reads

$$L_{g^{-1}}f \geq \kappa_K.$$

Using the invariance property, we get

$$\mu_\phi(K) \leq \phi(L_{g^{-1}}(F)) = (\phi \circ L_{g^{-1}})(F) = \phi(F).$$

In other words, we have

$$\phi(F) \geq \mu_\phi(K), \text{ for all } F \in C_c^\mathbb{R}(G) \text{ with } F \geq \kappa_{gK}.$$

Using (38) this immediately gives

$$\mu_\phi(K) \leq \mu_\phi(gK).$$

Applying the same inequality with g replaced by g^{-1} and K replaced by gK , yields

$$\mu_\phi(gK) \leq \mu_\phi(g^{-1}(gK)) = \mu_\phi(K),$$

so the Claim follows.

Claim 2: For any $g \in G$, and any open subset $D \subset G$, one has the equality

$$\mu_\phi(gD) = \mu_\phi(D).$$

For a compact subset $L \subset G$, one clearly has the equivalence $L \subset gD \Leftrightarrow g^{-1}L \subset D$. So, using Claim 1, for every compact subset $L \subset gD$, one has

$$\mu_\phi(L) = \mu_\phi(g^{-1}L) \leq \mu_\phi(D),$$

and using property (iii) for Radon measures, we immediately get the inequality

$$\mu_\phi(gD) = \sup \{ \mu_\phi(L) : L \text{ compact, } L \subset gD \} \leq \mu_\phi(D).$$

The inequality $\mu_\phi(D) \leq \mu_\phi(gD)$ is proven by replacing g with g^{-1} and D with gD , in the above inequality.

We now prove that μ_ϕ is a Haar measure. Start with some Borel set $A \subset G$. For every open set $D \supset gA$, one has the inclusion $g^{-1}D \supset A$, which using Claim 2, gives $\mu_\phi(D) = \mu_\phi(g^{-1}D) \geq \mu_\phi(A)$. Using property (ii) in the definition of Radon measures, we then have

$$\mu_\phi(gA) = \inf \{ \mu_\phi(D) : D \text{ open, } D \supset gA \} \geq \mu_\phi(A).$$

The inequality $\mu_\phi(A) \geq \mu_\phi(gA)$ is proven by replacing g with g^{-1} and A with gA , in the above inequality. $\square$

COMMENT. Later on, in Chapter IV, we are going to prove that the left invariance property of ϕ is also a necessary condition for μ_ϕ to be a Haar measure.

EXAMPLES 7.3. Let us examine the examples 7.2.A-D and let us construct Haar measures on these groups.

A. On a discrete group G , one has the *counting measure* $\mu(A) = \text{Card } A$, $\forall A \subset G$, which is obviously a Haar measure.

B. On $(\mathbb{R}^n, +)$, the Lebesgue measure is a Haar measure.

C. On the n -dimensional torus $\mathbb{T}^n$, we consider the Riesz measure μ_Λ , associated with the positive $\mathbb{R}$ -linear map $\Lambda : C^\mathbb{R}(\mathbb{T}^n) \rightarrow \mathbb{R}$, defined by

$$\Lambda(F) = \int_0^1 \cdots \int_0^1 F(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_n}) d\theta_1 \cdots d\theta_n.$$

It is not hard to see that $\Lambda \circ L_g = \Lambda$, $\forall g \in \mathbb{T}^n$. One easy way is to check directly the equality $(\Lambda \circ L_g)(P) = \Lambda(P)$, for functions of the form $P(z_1, \dots, z_n) = z_1^{m_1} \cdots z_n^{m_n}$, with $m_1, \dots, m_n \in \mathbb{Z}$, and then use continuity and the Stone-Weierstrass Theorem which gives the fact that the linear span of all these P 's is dense in $C^\mathbb{R}(\mathbb{T}^n)$. Using Proposition 7.6 it follows that μ_Λ is a Haar measure on $\mathbb{T}^n$.

D. The construction of a Haar measure on $GL_n(\mathbb{R})$ is outlined in the following.

Exercise 7.* Identify $GL_n(\mathbb{R})$ as an open subset in $\mathbb{R}^{n^2}$. For every continuous function $F : GL_n(\mathbb{R}) \rightarrow \mathbb{R}$, with compact support, $\check{F} : \mathbb{R}^{n^2} \rightarrow \mathbb{R}$ by

$$\check{F}(\mathbf{x}) = \begin{cases} F(\mathbf{x}) \cdot |\det \mathbf{x}|^{-n} & \text{if } \mathbf{x} \in GL_n(\mathbb{R}) \\ 0 & \text{if } \mathbf{x} \in \mathbb{R}^{n^2} \setminus GL_n(\mathbb{R}) \end{cases}$$

and we define

$$\psi(F) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n^2}}^{b_{n^2}} f(x_1, x_2, \dots, x_{n^2}) dx_1 dx_2 \cdots dx_{n^2},$$

where the numbers $a_1 < b_1, \dots, a_{n^2} < b_{n^2}$ are chosen (arbitrarily) such that

$$\text{supp } \check{F} \subset [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_{n^2}, b_{n^2}].$$

(One has the equality $\text{supp } \check{F} = \text{supp } F$, and the multiple integral is independent of the choice of the a 's and the b 's.) Prove that $\psi \circ L_s = \psi$, $\forall s \in GL_n(\mathbb{R})$. Conclude that the Riesz measure μ_ψ associated with ψ is a Haar measure on $GL_n(\mathbb{R})$.

HINTS: Fix $s \in GL_n(\mathbb{R})$. The map $\ell_{s^{-1}} : GL_n(\mathbb{R}) \rightarrow GL_n(\mathbb{R})$ has an obvious linear extension $\Phi_s : \mathbb{R}^{n^2} \rightarrow \mathbb{R}^{n^2}$, defined by

$$\Phi_s(\mathbf{x}) = s^{-1}\mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^{n^2},$$

where the vector space $\mathbb{R}^{n^2}$ is identified with $Mat_{n \times n}(\mathbb{R})$. Fix now $F \in C_c^\mathbb{R}(GL_n(\mathbb{R}))$ and consider the function $H = F \circ \ell_{s^{-1}}$, so that $(\psi \circ L_s)(F) = \psi(H)$. Prove the equality

$$\check{H}(\mathbf{x}) = \check{F}(\Phi_s(x)) \cdot |\det s|^{-n}, \quad \forall \mathbf{x} \in \mathbb{R}^{n^2}.$$

Prove that the Jacobian of Φ_s is given as

$$|\det[(D\Phi_s)(\mathbf{x})]| = |\det s|^{-n}, \quad \forall \mathbf{x} \in \mathbb{R}^{n^2}.$$

Use this equality, combined with the above formula for $\check{H}$, to get the equality $\psi(H) = \psi(F)$, as a result of the change of variable theorem. (Use the fact that in the definition of ψ , instead of integrating over rectangles one can integrate over arbitrary compact sets $\Omega \subset GL_n(\mathbb{R})$, with Jordan negligible boundary, and $\text{Int}(\Omega) \supset \text{supp } F$.)

COMMENTS. The Haar measures defined in Examples 7.3.A-D are peculiar in the sense that they also have the *right invariance property*:

$$\mu(Ag) = \mu(A), \quad \forall g \in G, \quad A \in \text{Bor}(G).$$

In general such a property does not hold. At this point, we can only speculate on this matter, by examining the following example.

Exercise 8.* Consider the group G of all affine orientation preserving affine transformations of $\mathbb{R}$, i.e. the collection

$$G = \{T_{ab} : a, b \in \mathbb{R}, a > 0\},$$

where $T_{ab} : \mathbb{R} \ni x \mapsto ax + b \in \mathbb{R}$. (Some people call this the “ $ax + b$ ” group.) It is not hard to see that compositions and inverses of such transformations are again of this form. In fact one can identify G as the subgroup of $GL_2(\mathbb{R})$ given by

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a, b \in \mathbb{R}, a > 0 \right\}.$$

The topology on G is the one induced from this inclusion. Equivalently, G can be identified with the right half-plane $(0, \infty) \times \mathbb{R}$. We use this identification to define a positive $\mathbb{R}$ -linear map $\Lambda : C_c^\mathbb{R}(G) \rightarrow \mathbb{R}$ as follows. For every $F \in C_c^\mathbb{R}(G)$, we choose $0 < c_1 < d_1$ and $c_2 < d_2$, such that $\text{supp } F \subset [c_1, d_1] \times [c_2, d_2]$, and we define

$$\Lambda(F) = \int_{c_1}^{d_1} \int_{c_2}^{d_2} \frac{F(a, b)}{a^2} da db.$$

The integral does not depend on the particular choice of the rectangle. Prove that $\Lambda \circ L_g = \Lambda$, $\forall g \in G$, so that the Riesz measure μ_Λ is a Haar measure. In general the equality $\Lambda \circ R_g = \Lambda$ fails. As indicated in the comment that followed Proposition 7.6, the fact that $\Lambda \circ R_g \neq \Lambda$ would prevent the Riesz measure μ_Λ from having the right invariance property.

HINTS: Use similar arguments to the ones in Exercise 8. If $g = T_{ab} \in G$, then the map $\ell_{g^{-1}} : G \rightarrow G$ extends to a linear map $\Phi_g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, defined by

$$\Phi_g(x, y) = (ax + by, y), \quad \forall (x, y) \in \mathbb{R}^2.$$

Argue as in Exercise 8, and use the change of variable theorem.

Exercise 9. As indicated above, in general, Haar measures need not have the right invariance property. Prove that when μ is a Haar measure on G , then the map $\nu : \text{Bor}(G) \rightarrow [0, \infty]$ defined by

$$\nu(B) = \mu(B^{-1}), \quad \forall B \in \text{Bor}(G),$$

is a Radon measure, which has the right invariance property.

HINT: The map $G \ni g \mapsto g^{-1} \in G$ is a homeomorphism.

The main result we are interested in is the *existence of a Haar measure*. The following result reduces the problem to the existence of a left invariant content.

LEMMA 7.6. *Let G be a locally compact group, and let ω be a content on G , with the left invariance property:*

$$\omega(gK) = \omega(K), \quad \forall g \in G, K \in \mathcal{C}_G.$$

If ω is not identically zero, then the outer measure ω^ , induced by ω , also has the left invariance property:*

$$\omega^*(gA) = \omega^*(A), \quad \forall g \in G, A \subset G.$$

The measure $\mu = \omega^|_{\text{Bor}(G)}$ is a Haar measure on G .*

PROOF. We trace the construction outlined in Theorem 7.1. Denote by $\mathcal{T}_G$ the collection of all open subsets of G , and define the map $\hat{\omega} : \mathcal{T}_G \rightarrow [0, \infty]$ by

$$\hat{\omega}(D) = \sup \{ \omega(K) : K \in \mathcal{C}_G, K \subset D \}, \quad \forall D \in \mathcal{T}_G.$$

The outer measure ω^* is then defined by

$$\omega^*(A) = \inf \{ \hat{\omega}(D) : D \in \mathcal{T}_G, D \supset A \}, \quad \forall A \subset G.$$

Claim: The map $\hat{\omega} : \mathcal{T}_G \rightarrow [0, \infty]$ has the left invariance property:

$$\hat{\omega}(gD) = \hat{\omega}(D), \quad \forall g \in G, D \in \mathcal{T}_G.$$

Start with some arbitrary compact subset $K \subset gD$. Then $g^{-1}K$ is a compact subset of D , so by the left invariance property of ω , we get

$$\omega(K) = \omega(g^{-1}K) \leq \hat{\omega}(D).$$

This means that we have $\omega(K) \leq \hat{\omega}(D)$, for all compact subsets $K \subset gD$, so by the definition of $\hat{\omega}$ we get

$$\hat{\omega}(gD) = \sup \{ \omega(K) : K \in \mathcal{C}_G, K \subset gD \} \leq \hat{\omega}(D).$$

The other inequality $\hat{\omega}(D) \leq \hat{\omega}(gD)$, follows from the one above if we replace g with g^{-1} and D with gD .

We are now in position to prove that ω^* has the left invariance property. Fix for the moment $A \subset G$ and $g \in G$. For every open set $D \supset gA$, one has $g^{-1}D \supset A$, so by the Claim we get

$$\hat{\omega}(D) = \hat{\omega}(g^{-1}D) \geq \omega^*(A).$$

Since we have $\hat{\omega}(D) \geq \omega^*(A)$, for all open sets $D \supset gA$, by the definition of ω^* , we get

$$\omega^*(gA) = \inf \{ \hat{\omega}(D) : D \in \mathcal{T}_G, D \supset gA \} \geq \omega^*(A).$$

The other inequality $\omega^*(A) \geq \omega^*(gA)$, follows from the one above if we replace g with g^{-1} and A with gA .

In order to prove that μ is a Haar measure, all we need to prove is the fact that $\mu(G) > 0$. Start with some compact subset $K \subset G$, with $\omega(K) > 0$. We have

$$\mu(G) \geq \mu(K) = \omega^*(K) = \hat{\omega}(K) \geq \omega(K) > 0,$$

and we are done. $\square$

Before we prove the existence of Haar measures, we need more preparations.

NOTATIONS. Let G be a group. For two non-empty subsets $A, B \subset G$, we write $A \prec B$, if there exist elements $g_1, \dots, g_n \in G$, such that $A \subset g_1B \cup \dots \cup g_nB$. In this case we define the number

$$[A : B] = \min \{ n \in \mathbb{N} : \text{there exist } g_1, \dots, g_n \in G \text{ with } A \subset g_1B \cup \dots \cup g_nB \}.$$

The following result will be useful.

LEMMA 7.7. *Let G be a group.*

- (i) *If $A, B \subset G$ are non-empty sets with $A \subset B$, then $A \prec B$, and $[A : B] = 1$.*
- (ii) *The relation $\prec$ is transitive, i.e. whenever $A, B, C \subset G$ are non-empty subsets satisfying $A \prec B$ and $B \prec C$, it follows that $A \prec C$. Moreover, in this case one has the inequality*

$$[A : C] \leq [A : B] \cdot [B : C].$$

- (iii) *The relation $\prec$ is compatible with left translations. This means that for any two elements $g, h \in G$, and any two non-empty subsets $A, B \subset G$, one has the equivalence $A \prec B \Leftrightarrow gA \prec hB$. Moreover, in this case one has*

$$[gA : hB] = [A : B].$$

- (iv) If $A, B, C \subset G$ are non-empty subsets such that $A \prec C$ and $B \prec C$, then $A \cup B \prec C$. Moreover, in this case one has the inequality

$$[A \cup B : C] \leq [A : C] + [B : C].$$

- (v) If $A, B, C \subset G$ are non-empty sets, such that $A \prec C$, $B \prec C$, and $(A \cdot C^{-1}) \cap (B \cdot C^{-1}) = \emptyset$, then one has the equality

$$[A \cup B : C] = [A : C] + [B : C].$$

PROOF. (i) This part is trivial.

(ii) Put $m = [A : B]$ and $n = [B : C]$. Choose $g_1, \dots, g_m, h_1, \dots, h_n \in G$, such that $A \subset g_1 B \cup \dots \cup g_m B$, and $B \subset h_1 C \cup \dots \cup h_n C$. We then obviously have the inclusion

$$A \subset \bigcup_{i=1}^m \bigcup_{j=1}^n (g_i h_j) C,$$

which proves that $A \prec C$, but also shows that $[A : C] \leq mn$.

(iii) This follows immediately from (ii) plus the obvious relations $A \prec gA \prec A$, $B \prec hB \prec B$, and the equalities

$$[A : gA] = [gA : A] = [B : hB] = [hB : B] = 1.$$

(iv) Let $m = [A : C]$ and $n = [B : C]$. Choose $g_1, \dots, g_m, g_{m+1}, \dots, g_{m+n} \in G$ such that $A \subset g_1 C \cup \dots \cup g_m C$ and $B \subset g_{m+1} C \cup \dots \cup g_{m+n} C$. This clearly shows that $A \cup B \prec C$ and $[A \cup B : C] \leq m + n$.

(v) Let $p = [A \cup B : C]$, and choose $g_1, \dots, g_p \in G$, such that $A \cup B \subset g_1 C \cup \dots \cup g_p C$. Define the sets

$$M = \{j \in \{1, \dots, p\} : A \cap g_j C \neq \emptyset\} \text{ and } N = \{k \in \{1, \dots, p\} : B \cap g_k C \neq \emptyset\}.$$

Notice that $M \cap N = \emptyset$. Indeed, if there exists $j \in M \cap N$, this means that on the one hand, we have $A \cap g_j C \neq \emptyset$, which gives $g_j \in A \cdot C^{-1}$, and on the other hand, we have $B \cap g_j C \neq \emptyset$ which gives $g_j \in B \cdot C^{-1}$. But this clearly contradicts the assumption that $(A \cdot C^{-1}) \cap (B \cdot C^{-1}) = \emptyset$.

By the definition of M and N , we clearly have the inclusions

$$A \subset \bigcup_{j \in M} g_j C \text{ and } B \subset \bigcup_{k \in N} g_k C.$$

These immediately give the inequalities $[A : C] \leq \text{card } M$ and $[B : C] \leq \text{card } N$.

Since M and N are disjoint, and $M \cup N \subset \{1, \dots, p\}$, these inequalities give

$$[A : C] + [B : C] \leq \text{card } M + \text{card } N = \text{card}(M \cup N) \leq p = [A \cup B : C].$$

Using part (iv), we see that in fact we have equality $[A : C] + [B : C] = [A \cup B : C]$. $\square$

REMARK 7.7. If G is a topological group with identity element e , and if V is a neighborhood of e , then $K \prec V$, for every compact subset of G . Indeed, if we choose some open set D with $e \in D \subset V$, then using the compactness of K , and the obvious inclusion $K \subset \bigcup_{g \in K} gD$, it follows that there exists $g_1, \dots, g_n \in K$, such that $K \subset g_1 D \cup \dots \cup g_n D \subset g_1 V \cup \dots \cup g_n V$.

With these preparations we are in position to prove the following fundamental result.

THEOREM 7.6. *Let G be a locally compact group, and let A be a compact neighborhood of the identity element. Then there exists a Haar measure μ on G , such that $\mu(A) = 1$.*

PROOF. Denote the identity element of G by e . Throughout the proof the compact neighborhood A of e will be fixed. For every non-empty compact set $K \subset G$, we define $m(K) = [K : A]$. We also put $m(\emptyset) = 0$.

Let us define $\mathcal{V}$ to be the collection of all neighborhoods of e . For every $V \in \mathcal{V}$, we denote by $\Omega(V)$ the set of all maps $\omega : \mathcal{C}_G \rightarrow [0, \infty)$ with the following properties

- (i) $0 \leq \omega(K) \leq m(K)$, $\forall K \in \mathcal{C}_G$;
- (ii) $\omega(A) = 1$;
- (iii) $K, L \in \mathcal{C}_G$, $K \subset L \Rightarrow \omega(K) \leq \omega(L)$;
- (iv) $\omega(K \cup L) \leq \omega(K) + \omega(L)$, $\forall K, L \in \mathcal{C}_G$;
- (v) $\omega(gK) = \omega(K)$, $\forall g \in G$, $K \in \mathcal{C}_G$.
- (vi) $K, L \in \mathcal{C}_G$, $(K \cdot V) \cap (L \cdot V) = \emptyset \Rightarrow \omega(K \cup L) = \omega(K) + \omega(L)$.

Claim 1: For every $V \in \mathcal{V}$, the set $\Omega(V)$ is non-empty.

Fix V . We shall prove this Claim by an explicit construction of an element $\omega \in \Omega(V)$. Define $\omega(\emptyset) = 0$, and define

$$\omega(K) = \frac{[K : V^{-1}]}{[A : V^{-1}]},$$

for all non-empty compact subsets $K \subset G$. The fact that ω has properties (i)-(vi) is immediate from Lemma 7.7.

Let us regard the sets $\Omega(V)$, $V \in \mathcal{V}$ as subsets of the product space

$$\mathbf{P} = \prod_{K \in \mathcal{C}_G} [0, m(K)].$$

Notice that, when we equip $\mathbf{P}$ with the product topology, it becomes a compact space, by Tihonov's Theorem.

Claim 2: For every $V \in \mathcal{V}$, the set $\Omega(V)$ is closed in $\mathbf{P}$.

Define, for any $K \in \mathcal{C}_G$, the map

$$\pi_K : \mathbf{P} \ni \omega \longmapsto \omega(K) \in \mathbb{R}.$$

By the definition of the topology of $\mathbf{P}$, all maps $\pi_K : \mathbf{P} \rightarrow \mathbb{R}$ are continuous. For any two sets $K, L \in \mathcal{C}_G$, consider the functions $F_{KL}, T_{KL} : \mathbf{P} \rightarrow \mathbb{R}$, defined by

$$F_{KL}(\omega) = \omega(K) - \omega(L) \text{ and } T_{KL}(\omega) = \omega(K \cup L) - \omega(K) - \omega(L), \quad \forall \omega \in \mathbf{P}.$$

Since we have $F_{KL} = \pi_K - \pi_L$ and $T_{KL} = \pi_{K \cup L} - \pi_K - \pi_L$, it follows that the maps $F_{KL}, T_{KL} : \mathbf{P} \rightarrow \mathbb{R}$, $K, L \in \mathcal{C}_G$, are all continuous. As a consequence of the continuity of these maps, it follows that, for any two sets $K, L \in \mathcal{C}_G$, the sets

$$\begin{aligned} \Gamma(K, L) &= \{\omega \in \mathbf{P} : \omega(K) \leq \omega(L)\} = F_{KL}^{-1}((-\infty, 0]), \\ \Theta^-(K, L) &= \{\omega \in \mathbf{P} : \omega(K \cup L) \leq \omega(K) + \omega(L)\} = T_{KL}^{-1}((-\infty, 0]), \\ \Theta^+(K, L) &= \{\omega \in \mathbf{P} : \omega(K \cup L) \geq \omega(K) + \omega(L)\} = T_{KL}^{-1}([0, \infty)) \end{aligned}$$

are closed subsets of $\mathbf{P}$. It then follows that the sets

$$\begin{aligned}\Omega^1 &= \{\omega \in \mathbf{P} : \omega(A) = 1\} = \pi_A^{-1}(\{1\}), \\ \Omega^2 &= \bigcap_{\substack{(K,L) \in \mathcal{C}_G \times \mathcal{C}_G \\ K \subset L}} \Gamma(K, L), \\ \Omega^3 &= \bigcap_{(K,L) \in \mathcal{C}_G \times \mathcal{C}_G} \Theta^-(K, L), \\ \Omega^4 &= \bigcap_{K \in \mathcal{C}_G} \bigcap_{g \in G} [\Gamma(K, gK) \cap \Gamma(gK, K)],\end{aligned}$$

are all closed, so the intersection

$$\Omega^5 = \Omega^1 \cap \Omega^2 \cap \Omega^3 \cap \Omega^4$$

is again closed. Notice that

$$\Omega^5 = \{\omega \in \mathbf{P} : \omega \text{ has properties (i)-(v)}\}.$$

Finally, if we define, for every $V \in \mathcal{V}$, the set

$$\Omega_V^6 = \bigcap_{\substack{(K,L) \in \mathcal{C}_G \times \mathcal{C}_G \\ (K \cdot V) \cap (L \cdot V) = \emptyset}} \Theta^+(K, L),$$

then Ω_V^6 is also closed, and so will then be the intersection $\Omega^5 \cap \Omega_V^6 = \Omega(V)$.

Claim 3: The intersection $\bigcap_{V \in \mathcal{V}} \Omega(V)$ is non-empty.

Remark that, if $V_1, V_2 \in \mathcal{V}$ are such that $V_1 \subset V_2$, then we have the inclusion $\Omega(V_1) \subset \Omega(V_2)$. Indeed, if ω belongs to $\Omega(V_1)$, then properties (i)-(v) are clear. To check property (vi) for V_2 we need to show that whenever $K, L \subset G$ are compact sets, with $(K \cdot V_2) \cap (L \cdot V_2) = \emptyset$, it follows that $\omega(K \cup L) = \omega(K) + \omega(L)$. This is however trivial, since the inclusion $V_1 \subset V_2$ forces $(K \cdot V_1) \cap (L \cdot V_1) = \emptyset$, and then the desired equality follows from the property (vi) for V_1 . We now see that, for any finite number of sets $V_1, \dots, V_n \in \mathcal{V}$, we have the inclusion

$$\Omega(V_1 \cap \dots \cap V_n) \subset \Omega(V_1) \cap \dots \cap \Omega(V_n),$$

which by Claim 1, proves that $\Omega(V_1) \cap \dots \cap \Omega(V_n) \neq \emptyset$. Using Claim 2, and the compactness of $\mathbf{P}$, the Claim immediately follows.

Pick now an element $\omega \in \bigcap_{V \in \mathcal{V}} \Omega(V)$.

Claim 4: The map $\omega : \mathcal{C}_G \rightarrow [0, \infty)$ is a content on G with the left invariance property

$$\omega(gK) = \omega(K), \quad \forall g \in G, \quad K \in \mathcal{C}_G.$$

Moreover, one has the equality $\omega(A) = 1$.

The fact that $\omega(A) = 1$ is clear, from condition (ii) in the definition of $\Omega(V)$. The left invariance property follows from condition (v). In order to prove that ω is a content, we need to prove

- (a) $\omega(\emptyset) = 0$;
- (b) $K, L \in \mathcal{C}_G, \quad K \subset L \Rightarrow \omega(K) \leq \omega(L)$;
- (c) $\omega(K \cup L) \leq \omega(K) + \omega(L), \quad \forall K, L \in \mathcal{C}_G$;
- (d) $K, L \in \mathcal{C}_G, \quad K \cap L = \emptyset \Rightarrow \omega(K \cup L) = \omega(K) + \omega(L)$.

Properties (a), (b), and (c) are clear, because every element in $\Omega(V)$, $V \in \mathcal{V}$ satisfies them. (Property (a) is a consequence of condition (i), property (b) is a consequence of (iii), and property (c) is a consequence of (iv).) To prove property (d), we start with two disjoint compact sets K and L , and we use Proposition 7.5 to find some $V \in \mathcal{V}$ such that $(K \cdot V) \cap (L \cap V) = \emptyset$. Then we use the fact that ω belongs to $\Omega(V)$, and by condition (vi) we indeed get $\omega(K \cup L) = \omega(K) + \omega(L)$.

Having proven Claim 4, we now define the measure $\mu_0 = \omega^*|_{\text{Bor}(G)}$. By Lemma 7.7, μ_0 is a Haar measure on G . Notice that $\mu_0(A) = \check{\omega}(A) \geq \omega(A) = 1$, so if we define $\mu : \text{Bor}(G) \rightarrow [0, \infty]$ by (use the convention $\infty/\mu_0(A) = \infty$)

$$\mu(B) = \frac{\mu_0(B)}{\mu_0(A)}, \quad \forall B \in \text{Bor}(G),$$

then μ is a Haar measure on G , and satisfies $\mu(A) = 1$. □

COMMENT. Eventually (see Chapter IV) we are going to improve on the above result by proving the uniqueness of μ .

In concrete examples, it is possible to prove uniqueness.

*Exercise 10**. Let $S = [0, 1]^n$ be the unit square in $\mathbb{R}^n$, and let μ be a Haar measure on $(\mathbb{R}^n, +)$, with $\mu(S) = 1$. Prove that μ coincides with the n -dimensional Lebesgue measure λ_n .

HINT: Consider first the half open box $S_0 = [0, 1)^n$, and its measure $\beta = \mu(S_0)$. Prove that for a half open box of the form

$$B = [a_1, b_1) \times \cdots \times [a_n, b_n)$$

with $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{Q}$, one has $\mu(B) = \beta \lambda_n(B)$. Conclude that if a subset $A \subset \mathbb{R}^n$ is contained in a hyperplane of the form

$$\Pi_k(a) = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_k = a\},$$

then $\mu(A) = 0$. Use this to get $\beta = 1$, so

$$\mu(B) = \lambda_n(B),$$

for every “rational” half-open box. Prove that this equality holds for *all* half-open boxes. Use Corollary 5.1 to conclude that $\mu = \lambda_n$.

The following two exercises show how a Haar measure can be used to get some topological information.

Exercise 11. Let G be a locally compact group, and let μ be a Haar measure on G . Prove that $\mu(D) > 0$, for every open subset $D \subset G$.

HINT: Use the inequality $\mu(K) \leq [K : D] \cdot \mu(D)$, for all compact $K \subset G$.

*Exercise 12**. Let G be a locally compact group, and let μ be a Haar measure on G . Prove that the following are equivalent:

- (i) G is compact;
- (ii) $\mu(G) < \infty$.

HINT: For the implication (ii) $\Rightarrow$ (i), start with some compact neighborhood V of the identity, and choose a maximal subset $A \subset G$, such that the sets gV , $g \in A$ are disjoint. Prove that A is finite. Conclude that $G = \bigcup_{g \in A} (gV \cdot V^{-1})$, so G is a finite union of compact sets.

LECTURES 30-31

8. Signed measures and complex measures

In this section we discuss a generalization of the notion of a measure, to the case where the values are allowed to be outside $[0, \infty]$. The first notion is described by the following.

DEFINITION. Suppose $\mathcal{A}$ is a σ -algebra on a non-empty set X . A function $\mu : \mathcal{A} \rightarrow [-\infty, \infty]$ is called a *signed measure on $\mathcal{A}$* , if it has the properties below.

- (i) Either one of the following is true
 - $\mu(A) < \infty, \forall A \in \mathcal{A}$;
 - $\mu(A) > -\infty, \forall A \in \mathcal{A}$.
- (ii) $\mu(\emptyset) = 0$.
- (iii) For any pairwise disjoint sequence $(A_n)_{n=1}^\infty \subset \mathcal{A}$, one has the equality

$$(1) \quad \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Here we adopt the convention that if one term in the right hand side of (1) is equal to $\pm\infty$, then the entire sum is equal to $\pm\infty$. It is important to use condition (i), which avoids situations when one term is ∞ and another term is $-\infty$.

EXAMPLES 8.1. Let us agree, in this section only, to use the term “honest” measure, for a measure in the usual sense.

A. Any “honest” measure is of course a signed measure.

B. If μ is a signed measure, then $-\mu$ is again a signed measure.

C. If μ_1 and μ_2 are “honest” measures, one of which is *finite*, then $\mu_1 - \mu_2$ is a signed measure. Eventually (see Theorem 8.2) we are going to show that any signed measure can be written in this form.

One key technical result about signed measures is the following.

THEOREM 8.1. *Let $\mathcal{A}$ be a σ -algebra on a non-empty set X , and let μ be a signed measure on X . Then there exist sets $L, U \in \mathcal{A}$, such that*

$$(2) \quad \mu(L) = \inf \{ \mu(A) : A \in \mathcal{A} \};$$

$$(3) \quad \mu(M) = \sup \{ \mu(A) : A \in \mathcal{A} \}.$$

PROOF. Since $-\mu$ is also a signed measure, it suffices to prove only the existence of M satisfying (3). Denote the right hand side of (3) by α , and choose a sequence $(\alpha_n)_{n \geq 1} \subset \mathbb{R}$, such that $\lim_{n \rightarrow \infty} \alpha_n = \alpha$, and $\alpha_n < \alpha, \forall n \geq 1$. The key construction we need is contained in the following.

Claim 1: There exists a family of sets $\{B_k^n : k, n \in \mathbb{N}, 1 \leq k \leq n\} \subset \mathcal{A}$, with the following properties:

(i) for every $n \geq 1$, one has the inclusions

$$\begin{array}{ccccccc} B_1^n & \subset & B_2^n & \subset & \dots & \subset & B_n^n \\ \cup & & \cup & & \dots & & \cup \\ B_1^{n+1} & \subset & B_2^{n+1} & \subset & \dots & \subset & B_n^{n+1} \subset B_{n+1}^{n+1} \end{array}$$

(ii) for every $k \geq 1$ one has the inequalities

$$\mu(B_k^n \setminus B_k^{n+1}) \leq 0, \quad \forall n \geq k.$$

(iii) $\mu(B_n^n) \geq \alpha_n, \forall n \geq 1$.

We construct this sequence inductively, one row at a time (the rows are indexed by the upper index n). Choose $B_1^1 \in \mathcal{A}$ to be any set with $\mu(B_1^1) \geq \alpha_1$. Suppose we have constructed the first N rows, i.e. we have defined the sets $B_k^n, 1 \leq k \leq n \leq m$, so that property (i) holds for all $n = 1, \dots, m-1$, property (ii) holds in the form

$$\alpha_k \leq \mu(B_k^k) \leq \mu(B_k^{k+1}) \leq \dots \leq \mu(B_k^m), \quad \forall k = 1, \dots, m,$$

and property (iii) holds for all $n = 1, \dots, m$. Let us explain now how the next row $B_1^{m+1} \subset B_2^{m+1} \subset \dots \subset B_{m+1}^{m+1}$ is constructed. Define the sets $E_1, E_2, \dots, E_m \in \mathcal{A}$ by

$$E_1 = B_1^m, \text{ and } E_k = B_k^m \setminus B_{k-1}^m, \quad \forall k = 2, \dots, m.$$

The sets $E_k, k = 1, \dots, m$ are pairwise disjoint, and we have

$$B_k^m = \bigcup_{j=1}^k E_j, \quad \forall k = 1, \dots, m.$$

Choose now an arbitrary set $D \in \mathcal{A}$, with $\mu(D) \geq \alpha_{m+1}$, and define, for each $j \in \{1, \dots, m\}$, the set

$$G_j = \begin{cases} E_j & \text{if } \mu(E_j \setminus D) > 0 \\ E_j \cap D & \text{if } \mu(E_j \setminus D) \leq 0 \end{cases}$$

Notice that we have $E_j \supset G_j$, and using the equality $\mu(E_j) = \mu(E_j \cap D) + \mu(E_j \setminus D)$, we also have

$$(4) \quad \mu(E_j \setminus G_j) \leq 0 \text{ and } \mu(G_j) \geq \mu(E_j \cap D), \quad \forall j = 1, \dots, m.$$

Define also the set $G_{m+1} = D \setminus B_m^m$. It is clear that the sets $G_1, G_2, \dots, G_{m+1}$ are pairwise disjoint. Construct now the $m+1$ row by taking

$$B_k^{m+1} = \bigcup_{j=1}^k G_j, \quad \forall k = 1, 2, \dots, m+1.$$

It is obvious that one has the inclusions

$$B_1^{m+1} \subset B_2^{m+1} \subset \dots \subset B_{m+1}^{m+1}.$$

Since $E_k \supset G_k, \forall k = 1, \dots, m$, it is also clear that we have the vertical inclusions $B_k^m \supset B_k^{m+1}, \forall k = 1, \dots, m$. Using (4), for each $k = 1, \dots, m$, we have

$$\mu(B_k^m \setminus B_k^{m+1}) = \mu\left(\bigcup_{j=1}^k [E_j \setminus G_j]\right) = \sum_{j=1}^k [\mu(E_j \setminus G_j)] \leq 0.$$

Finally, again by (4), we have

$$\begin{aligned}\mu(B_{m+1}^{m+1}) &= \mu\left(\bigcup_{k=1}^{m+1} G_k\right) = \sum_{k=1}^{m+1} \mu(G_k) \geq \mu(G_{m+1}) + \sum_{k=1}^m \mu(E_k \cap D) = \\ &= \mu(G_{m+1}) + \mu\left(\bigcup_{k=1}^m [E_k \cap D]\right) = \mu(D \setminus B_m^m) + \mu(B_m^m \cap D) = \mu(D) \geq \alpha_{m+1}.\end{aligned}$$

Claim 2: There exists a sequence $(A_k)_{k=1}^\infty \subset \mathcal{A}$, such that

- (i) $A_1 \subset A_2 \subset A_3 \subset \dots$;
- (ii) $\mu(A_k) \geq \alpha_k, \forall k \geq 1$.

We fix a family $\{B_k^n : k, n \in \mathbb{N}, 1 \leq k \leq n\}$ satisfying the properties in Claim 1. For every $k \geq 1$, we define $A_k = \bigcap_{n=k}^\infty B_k^n$. Notice that, using property (i) from Claim 1 (the vertical inclusions), we have

$$B_k^k = A_k \cup \left(\bigcup_{n=k}^\infty [B_k^n \setminus B_k^{n+1}]\right),$$

and the sets $A_k, B_k^n \setminus B_k^{n+1}, n \geq k$, are pairwise disjoint, so using property (ii) from Claim 1, we have

$$\mu(B_k^k) = \mu(A_k) + \sum_{n=k}^\infty \mu(B_k^n \setminus B_k^{n+1}) \leq \mu(A_k).$$

Using property (iii) from Claim 1, we then get $\mu(A_k) \geq \alpha_k$. The fact that we have the inclusions $A_1 \subset A_2 \subset \dots$ is clear, from property (i) in Claim 1 (the horizontal inclusions).

Fix now the sequence $(A_k)_{k=1}^\infty \subset \mathcal{A}$ as in Claim 2, and let us consider the set $M = \bigcup_{k=1}^\infty A_k$. If we define the sets

$$M_1 = A_1 \text{ and } M_k = A_k \setminus A_{k-1}, \forall k \geq 2,$$

then we have $M = \bigcup_{k=1}^\infty M_k$, and the sets $M_1, M_2, M_3, \dots$ are pairwise disjoint. In particular, this gives

$$\mu(M) = \sum_{k=1}^\infty \mu(M_k) = \lim_{k \rightarrow \infty} \left[\sum_{j=1}^k \mu(M_j) \right] = \lim_{k \rightarrow \infty} \mu\left(\bigcup_{j=1}^k M_j\right).$$

Since we obviously have $\bigcup_{j=1}^k M_j = A_k, \forall k \geq 1$, the above equality proves that

$$(5) \quad \mu(M) = \lim_{k \rightarrow \infty} \mu(A_k).$$

Since we have $\alpha_k \leq \mu(A_k) \leq \alpha, \forall k \geq 1$, as well as $\lim_{k \rightarrow \infty} \alpha_k = \alpha$, the equality (5) forces $\mu(M) = \alpha$. $\square$

REMARK 8.1. One interesting application of the above result is the fact that, whenever μ is a signed measure on $\mathcal{A}$, such that

$$(6) \quad -\infty < \mu(A) < \infty, \forall A \in \mathcal{A},$$

then

$$-\infty < \inf\{\mu(A) : A \in \mathcal{A}\} \leq \sup\{\mu(A) : A \in \mathcal{A}\} < \infty.$$

A signed measure with property (6) is called *finite*.

We are now in position to prove the statement made in Example 8.1.C.

THEOREM 8.2. Let X be a non-empty set, let $\mathcal{A}$ be a σ -algebra on X , and let μ be a signed measure on $\mathcal{A}$. Then there exist subsets $X^+, X^- \in \mathcal{A}$, with the following properties:

- (i) $X^+ \cap X^- = \emptyset$, and $X^+ \cup X^- = X$;
- (ii) the maps $\mu^\pm : \mathcal{A} \rightarrow [-\infty, \infty]$, defined by

$$\mu^\pm(A) = \pm\mu(A \cap X^\pm), \quad \forall A \in \mathcal{A},$$

are “honest” measures on $\mathcal{A}$;

- (iii) one of the measures $\mu^\pm$ is finite, and one has the equality $\mu = \mu^+ - \mu^-$.

PROOF. Without any loss of generality, we can assume that

$$\mu(A) < \infty, \quad \forall A \in \mathcal{A}.$$

(Otherwise, we replace μ with $-\mu$, and the conclusion does not essentially change.) Put $\alpha = \sup \{\mu(A) : A \in \mathcal{A}\}$. By Theorem 8.1, it follows that $0 \leq \alpha < \infty$, and there exists a set $X^+ \in \mathcal{A}$, such that $\mu(X^+) = \alpha$. Define $X^- = X \setminus X^+$.

Claim: The sets $X^\pm$ have the following properties:

- (a) $0 \leq \mu(A) \leq \alpha$, for all $A \in \mathcal{A}$, with $A \subset X^+$;
- (b) $0 \geq \mu(B)$, for all $B \in \mathcal{A}$, with $B \subset X^-$.

To prove (a) start with some arbitrary subset $A \subset X^+$. First of all, by the definition of α , it is clear that $\mu(A) \leq \alpha$. Second, using the equality $\mu(X^+) = \mu(A) + \mu(X^+ \setminus A)$, it is clear that $\mu(A), \mu(X^+ \setminus A) > -\infty$, so we have

$$\alpha \geq \mu(X^+ \setminus A) = \mu(X^+) - \mu(A) = \alpha - \mu(A),$$

which clearly forces $\mu(A) \geq 0$. To prove (b), we start with some set $B \in \mathcal{A}$ with $B \subset X^-$. Using the fact that $X^+ \cap B = \emptyset$, we have

$$\alpha \geq \mu(X^+ \cup B) = \mu(X^+) + \mu(B) = \alpha + \mu(B),$$

which clearly forces $\mu(B) \leq 0$.

Having proven the Claim, we define the maps $\mu^\pm : \mathcal{A} \rightarrow [-\infty, \infty]$ as in the statement of the Theorem. By the Claim, we get $\mu^\pm(A) \geq 0, \forall A \in \mathcal{A}$. It is also pretty clear that both μ^+ and μ^- are σ -additive, so they define “honest” measures. Also, by the Claim, we have $\mu^+(A) \leq \alpha, \forall A \in \mathcal{A}$, so μ^+ is a finite measure. Finally, if we start with some arbitrary $A \in \mathcal{A}$, and we write it as $A = (A \cap X^+) \cup (A \cap X^-)$, then using the fact that $(A \cap X^+) \cap (A \cap X^-) = \emptyset$, we get

$$\mu(A) = \mu(A \cap X^+) + \mu(A \cap X^-) = \mu^+(A) - \mu^-(A). \quad \square$$

It will be helpful not only here, but also in some future discussions, to isolate a certain feature identified by the above result.

DEFINITION. Given a σ -algebra $\mathcal{A}$ on a non-empty set X , and two “honest” measures μ and ν on $\mathcal{A}$, we say that μ and ν are *mutually singular*, if there exists sets $M, N \in \mathcal{A}$, with $M \cup N = X$ and $M \cap N = \emptyset$, such that $\mu(N) = \nu(M) = 0$. Notice that this implies the equalities

$$\mu(A) = \mu(A \cap M) \text{ and } \nu(A) = \nu(A \cap N), \quad \forall A \in \mathcal{A}.$$

If this situation occurs, we write $\mu \perp \nu$.

With this terminology, Theorem 8.2 states that any signed measure μ can be written as $\mu = \mu^+ - \mu^-$, with μ^+ and μ^- “honest” mutually singular measures, and one of them finite.

Although the sets $X^\pm$ may not be uniquely determined, the decomposition $\mu = \mu^+ - \mu^-$ is unique, as indicated by the following result.

THEOREM 8.3 (Minimality). *Let X be a non-empty set, let $\mathcal{A}$ be a σ -algebra on X , and let μ be a signed measure on $\mathcal{A}$. Suppose μ^+ and μ^- are mutually singular “honest” measures on $\mathcal{A}$, one of them being finite, such that $\mu = \mu^+ - \mu^-$. Suppose ν and η are two “honest” measures on $\mathcal{A}$, one of which being finite, such that $\mu = \nu - \eta$. Then one has the inequalities $\mu^+ \leq \nu$ and $\mu^- \leq \eta$.*

PROOF. Fix sets $X^+, X^- \in \mathcal{A}$, such that $X^+ \cup X^- = X$, $X^+ \cap X^- = \emptyset$, and $\mu^+(X^-) = \mu^-(X^+) = 0$.

Start with some arbitrary set $A \in \mathcal{A}$. On the one hand, since $A = (A \cap X^+) \cup (A \cap X^-)$, with $A = (A \cap X^+) \cap (A \cap X^-) = \emptyset$, we see that if λ is either one of the measures μ , μ^+ , or μ^- , we have the equality

$$(7) \quad \lambda(A) = \lambda(A \cap X^+) + \lambda(A \cap X^-), \quad \forall A \in \mathcal{A}.$$

On the other hand, since μ^+ is an “honest” measure, and $\mu^+(X^-) = 0$, the inclusion $A \cap X^- \subset X^-$ will force

$$\mu^+(A \cap X^-) = 0, \quad \forall A \in \mathcal{A}.$$

Likewise, we have the equality

$$\mu^-(A \cap X^+) = 0, \quad \forall A \in \mathcal{A}.$$

These equalities, combined with $\mu = \mu^+ - \mu^-$, and with (7), give the equalities

$$(8) \quad \mu^+(A) = \mu^+(A \cap X^+) = \mu^+(A \cap X^+) - \mu^-(A \cap X^+) = \mu(A \cap X^+),$$

$$(9) \quad \mu^-(A) = \mu^-(A \cap X^-) = -\mu^+(A \cap X^-) + \mu^-(A \cap X^-) = -\mu(A \cap X^-),$$

for all $A \in \mathcal{A}$. Fix now some set $A \in \mathcal{A}$. Since ν is an “honest” measure, and $\eta(A \cap X^+) \geq 0$, using (8) we get

$$\nu(A) \geq \nu(A \cap X^+) \geq \nu(A \cap X^+) - \eta(A \cap X^-) = \mu(A \cap X^+) = \mu^+(A).$$

Likewise, we have

$$\eta(A) \geq \eta(A \cap X^-) \geq \eta(A \cap X^-) - \nu(A \cap X^-) = -\mu(A \cap X^-) = \mu^-(A). \quad \square$$

COROLLARY 8.1. *Let $\mathcal{A}$ be a σ -algebra on X , let μ be a signed measure on $\mathcal{A}$, and let μ^+ , μ^- , ν^+ and ν^- be “honest” measures on $\mathcal{A}$ with*

- $\mu^+ \perp \mu^-$, and one of the measures μ^+ and μ^- is finite;
- $\nu^+ \perp \nu^-$, and one of the measures ν^+ and ν^- is finite;
- $\mu = \mu^+ - \mu^- = \nu^+ - \nu^-$.

Then one has the equalities $\mu^+ = \nu^+$ and $\mu^- = \nu^-$.

PROOF. Apply Theorem 8.3 “both ways” to get $\mu^+ \leq \nu^+$ and $\mu^- \leq \nu^-$, as well as $\nu^+ \leq \mu^+$ and $\nu^- \leq \mu^-$. $\square$

DEFINITION. Given a signed measure μ , the decomposition $\mu = \mu^+ - \mu^-$, whose existence is shown in Theorem 8.2, and whose uniqueness is shown above, is called the *Hahn-Jordan decomposition of μ* . A pair of sets (X^+, X^-) , with $X^\pm \in \mathcal{A}$, $X^+ \cup X^- = X$, $X^+ \cap X^- = \emptyset$, and $\mu^+(X^-) = \mu^-(X^+) = 0$, is called a *Hahn-Jordan set decomposition of X relative to μ* .

Exercise 1. Let μ be a signed measure, and let $\mu = \mu^+ - \mu^-$ be the Hahn-Jordan decomposition. Prove that the following are equivalent

- (i) μ is finite, i.e. $-\infty < \mu(A) < \infty$, $\forall A \in \mathcal{A}$;
- (ii) both “honest” measures μ^+ and μ^- are finite.

The following result characterizes mutual singularity in an approximate fashion.

LEMMA 8.1. *Let $\mathcal{A}$ be a σ -algebra on X , and let μ and ν be “honest” measures on $\mathcal{A}$. The following are equivalent*

- (i) $\mu \perp \nu$;
- (ii) *for every $\varepsilon > 0$, there exist sets $D, E \in \mathcal{A}$, such that $\mu(D) < \varepsilon$, $\nu(E) < \varepsilon$, and $D \cup E = X$.*

PROOF. The implication (i) $\Rightarrow$ (ii) is trivial.

To prove the implication (ii) $\Rightarrow$ (i) construct, for each $\varepsilon > 0$, two sequences $(D_n^\varepsilon)_{n=1}^\infty$ and $(E_n^\varepsilon)_{n=1}^\infty$ of sets in $\mathcal{A}$, such that $\mu(D_n^\varepsilon) < \varepsilon/2^n$, $\nu(E_n^\varepsilon) < \varepsilon/2^n$, and $D_n^\varepsilon \cup E_n^\varepsilon = X$. Put $A_\varepsilon = \bigcap_{n=1}^\infty D_n^\varepsilon$ and $B_\varepsilon = \bigcup_{n=1}^\infty E_n^\varepsilon$. Fix for the moment $\varepsilon > 0$. On the one hand, using the inclusion $A_\varepsilon \subset D_n^\varepsilon$, $\forall n \geq 1$, we get $\mu(A_\varepsilon) \leq \varepsilon/2^n$, $\forall n \geq 1$, which clearly forces

$$(10) \quad \mu(A_\varepsilon) = 0.$$

On the other hand, using σ -subadditivity, we have

$$(11) \quad \nu(B_\varepsilon) = \nu\left(\bigcup_{n=1}^\infty E_n^\varepsilon\right) \leq \sum_{n=1}^\infty \nu(E_n^\varepsilon) < \sum_{n=1}^\infty \frac{\varepsilon}{2^n} = \varepsilon.$$

Finally, since we have, $X \setminus D_n^\varepsilon \subset E_n^\varepsilon$, $\forall n \geq 1$, we get

$$X \setminus A_\varepsilon = \bigcup_{n=1}^\infty (X \setminus D_n^\varepsilon) \subset \bigcup_{n=1}^\infty E_n^\varepsilon = B_\varepsilon,$$

which gives

$$(12) \quad A_\varepsilon \supset X \setminus B_\varepsilon.$$

Define now the sets $N = \bigcup_{n=1}^\infty A_{1/n}$ and $M = X \setminus N$. On the one hand, using σ -subadditivity, combined with (10), we get $\mu(N) = 0$. On the other hand, using (12), we have

$$M = X \setminus N = X \setminus \left(\bigcup_{n=1}^\infty A_{1/n}\right) = \bigcap_{n=1}^\infty (X \setminus A_{1/n}) \subset \bigcap_{n=1}^\infty B_{1/n} \subset B_{1/k}, \quad \forall k \geq 1,$$

which forces $\nu(M) = 0$. □

Although the next technical result seems a bit out of context at this point, we prove it here, and record it for future use.

LEMMA 8.2. *Let $\mathcal{A}$ be a σ -algebra on some non-empty set X , and let μ, η be signed measures on $\mathcal{A}$. Assume there is an “honest” finite measure ν on $\mathcal{A}$, with $\mu + \nu = \eta$.*

- (i) *If $\mu = \mu^+ - \mu^-$ and $\eta = \eta^+ - \eta^-$ are the Hahn-Jordan decompositions of μ and η respectively, then one has the inequalities*

$$(13) \quad \mu^+ \leq \eta^+ \leq \mu^+ + \nu$$

$$(14) \quad \eta^- \leq \mu^- \leq \eta^- + \nu.$$

- (ii) If (X^+, X^-) is a Hahn-Jordan set decomposition of X relative to μ , and if (Y^+, Y^-) is a Hahn-Jordan set decomposition of X relative to η , then one has the relations $X^+ \subseteq_{\nu} Y^+$ and $Y^- \subseteq_{\nu} X^-$.

PROOF. On the one hand, the signed measure η has a decomposition

$$\eta = \mu + \nu = (\mu^+ + \nu) - \mu^-,$$

with $\mu^+ + \nu$ and μ^- “honest” measures (one of them finite). Using the minimality Theorem 8.3, we get the inequalities

$$(15) \quad \eta^+ \leq \mu^+ + \nu \text{ and } \eta^- \leq \mu^-.$$

On the other hand, we can also consider the signed measure $\mu = \eta - \nu$, which has a decomposition

$$\mu = \eta^+ - (\eta^- + \nu),$$

with η^+ and $\eta^- + \nu$ “honest” measures (one of them finite). Using again the minimality Theorem 8.3, we get the inequalities

$$(16) \quad \mu^+ \leq \eta^+ \text{ and } \mu^- \leq \eta^- + \nu.$$

Clearly the inequalities (15) and (16) cover the desired inequalities (13) and (14)

- (ii). Recall (see Section 4) that the relation $A \subseteq_{\nu} B$ means that $\nu(A \setminus B) = 0$.

In our case, we have to look at the set

$$N = X^+ \setminus Y^+ = Y^- \setminus X^-,$$

for which we have to show that $\nu(N) = 0$. On the one hand, since $N \subset Y^-$, we get $\eta^+(N) = 0$. Using (13) this forces $\mu^+(N) = 0$. On the other hand, since $N \subset X^+$, we get $\mu^-(N) = 0$, and using (14) we also get $\eta^-(N) = 0$. In other words, we get the equalities

$$\mu(N) = \mu^+(N) - \mu^-(N) = 0,$$

$$\eta(N) = \eta^+(N) - \eta^-(N) = 0,$$

and then the equality $\eta = \mu + \nu$ clearly forces $\nu(N) = 0$. $\square$

The Hahn-Jordan decomposition has the following interesting application to the properties of the natural order relation on “honest” measures. The result below gives the existence of a “infimum” and a “supremum” for a pair of finite “honest” measures.

PROPOSITION 8.1 (Lattice Property). *Let $\mathcal{A}$ be a σ -algebra on a non-empty set X , and let μ and ν be “honest” measures on $\mathcal{A}$, with one of them finite.*

- (i) *There exists a unique measure $\mu \vee \nu$ with:*
 - (a) $\mu \vee \nu \geq \mu$ and $\mu \vee \nu \geq \nu$;
 - (b) *whenever ω is an “honest” measure on $\mathcal{A}$, with $\mu \leq \omega$ and $\nu \leq \omega$, it follows that one has the inequality $\mu \vee \nu \leq \omega$.*
- (ii) *There exists a unique measure $\mu \wedge \nu$ with:*
 - (a) $\mu \geq \mu \wedge \nu$ and $\nu \geq \mu \wedge \nu$;
 - (b) *whenever λ is an “honest” measure on $\mathcal{A}$, with $\mu \geq \lambda$ and $\nu \geq \lambda$, it follows that one has the inequality $\mu \wedge \nu \geq \lambda$.*

PROOF. Since the statement of the Theorem is “symmetric,” without any loss of generality we can assume that μ is finite.

Consider the signed measure $\eta = \mu - \nu$, and its Hahn-Jordan decomposition $\eta = \eta^+ - \eta^-$. Let (X^+, X^-) be a Hahn-Jordan set decomposition of X relative to η . This means that, for every $A \in \mathcal{A}$, one has

$$(17) \quad 0 \leq \eta^+(A) = \eta(A \cap X^+) = \mu(A \cap X^+) - \nu(A \cap X^+);$$

$$(18) \quad 0 \leq \eta^-(A) = -\eta(A \cap X^-) = \nu(A \cap X^-) - \mu(A \cap X^-).$$

In particular we get

$$(19) \quad \mu(A \cap X^+) \geq \nu(A \cap X^+) \text{ and } \mu(A \cap X^-) \leq \nu(A \cap X^-), \quad \forall A \in \mathcal{A}.$$

(i). Define the measure $\mu \vee \nu = \mu + \eta^-$. Using (18) we have

$$(20) \quad (\mu \vee \nu)(A) = \mu(A \cap X^+) + \nu(A \cap X^-), \quad \forall A \in \mathcal{A}.$$

Notice that, using (19), it follows that, for every $A \in \mathcal{A}$, one has the inequalities

$$(\mu \vee \nu)(A \cap X^+) = \mu(A \cap X^+) \geq \nu(A \cap X^+),$$

$$(\mu \vee \nu)(A \cap X^-) = \nu(A \cap X^-) \geq \mu(A \cap X^-),$$

In particular, this gives

$$(\mu \vee \nu)(A) = (\mu \vee \nu)(A \cap X^+) + (\mu \vee \nu)(A \cap X^-) \geq \mu(A \cap X^+) + \mu(A \cap X^-) = \mu(A),$$

$$(\mu \vee \nu)(A) = (\mu \vee \nu)(A \cap X^+) + (\mu \vee \nu)(A \cap X^-) \geq \nu(A \cap X^+) + \nu(A \cap X^-) = \mu(A),$$

for every $A \in \mathcal{A}$, so $\mu \vee \nu$ indeed has property (a).

To prove property (b), start with some “honest” measure ω on $\mathcal{A}$, with $\mu, \nu \leq \omega$, and let us show that $\mu \vee \nu \leq \omega$. This is quite clear, since for any $A \in \mathcal{A}$, using (20) we have

$$\omega(A) = \omega(A \cap X^+) + \omega(A \cap X^-) \geq \mu(A \cap X^+) + \nu(A \cap X^-) = (\mu \vee \nu)(A).$$

The uniqueness of $\mu \vee \nu$ is now clear from (a) and (b).

(ii). Remark that, using the Minimality Theorem 8.3, for the measure $\eta = \mu - \nu$, it follows that $\eta^+ \leq \mu$. In particular, η^+ is a finite “honest” measure, and so is the difference $\mu - \eta^+$. Put $\mu \wedge \nu = \mu - \eta^+$. Using (17) we have

$$(21) \quad (\mu \wedge \nu)(A) = \mu(A \cap X^-) + \nu(A \cap X^+), \quad \forall A \in \mathcal{A}.$$

Notice that, using (19), it follows that, for every $A \in \mathcal{A}$, one has the inequalities

$$(\mu \wedge \nu)(A \cap X^+) = \nu(A \cap X^+) \leq \mu(A \cap X^+),$$

$$(\mu \wedge \nu)(A \cap X^-) = \mu(A \cap X^-) \geq \nu(A \cap X^-),$$

In particular, this gives

$$(\mu \wedge \nu)(A) = (\mu \wedge \nu)(A \cap X^+) + (\mu \wedge \nu)(A \cap X^-) \leq \mu(A \cap X^+) + \mu(A \cap X^-) = \mu(A),$$

$$(\mu \wedge \nu)(A) = (\mu \wedge \nu)(A \cap X^+) + (\mu \wedge \nu)(A \cap X^-) \leq \nu(A \cap X^+) + \nu(A \cap X^-) = \mu(A),$$

for every $A \in \mathcal{A}$, so $\mu \wedge \nu$ indeed has property (a).

To prove property (b), start with some “honest” measure λ on $\mathcal{A}$, with $\mu, \nu \leq \omega$, and let us show that $\mu \wedge \nu \geq \lambda$. This is quite clear, since for any $A \in \mathcal{A}$, using (21) we have

$$\lambda(A) = \lambda(A \cap X^+) + \lambda(A \cap X^-) \leq \nu(A \cap X^+) + \mu(A \cap X^-) = (\mu \wedge \nu)(A).$$

The uniqueness of $\mu \wedge \nu$ is now clear from (a) and (b). $\square$

We conclude with a series of results that make a connection with the theory of Radon measures discussed in Section 7.

DEFINITION. Suppose X is a locally compact space, and μ is a signed measure on $\text{Bor}(X)$. We call μ a *signed Radon measure on X* , if there exist “honest” Radon measures ν and η on X , one of which is finite, such that $\mu = \nu - \eta$.

Exercise 2*. Let X be a locally compact space, and let μ be a signed measure on $\text{Bor}(X)$. Prove that the following are equivalent:

- (i) μ is a signed Radon measure on X ;
- (ii) if $\mu = \mu^+ - \mu^-$ denotes the Hahn-Jordan decomposition of μ , then both μ^+ and μ^- are Radon measures on X .

HINT: To prove the implication (i) $\Rightarrow$ (ii) use the fact that $\mu^+ \leq \nu$ and $\mu^- \leq \eta$. Moreover, show that, for any $B \in \text{Bor}(X)$, one has the implications $\mu^+(B) < \infty \Rightarrow \nu(B) < \infty$ and $\mu^-(B) < \infty \Rightarrow \eta(B) < \infty$. Then use Exercise 5 from Section 7.

REMARK 8.2. Suppose X is a locally compact space. In Section 7 we discussed the *Riesz correspondence*, which associates to each linear positive map $\phi : C_c^\mathbb{R}(X) \rightarrow \mathbb{R}$, a Radon measure μ_ϕ on X . As already suggested, this correspondence is in fact a bijection, although the proof of this fact will come later in Chapter IV. At this point we would like to analyze the Riesz correspondence in a simpler situation, namely the case when X is *compact*. In this case it is interesting to point out that Riesz correspondence can be extended beyond the positive case. The key fact (see Corollary II.5.3) is that every linear *continuous* map $\phi : C^\mathbb{R}(X) \rightarrow \mathbb{R}$ can be written as a difference $\phi = \phi_1 - \phi_2$, with $\phi_1, \phi_2 : C^\mathbb{R}(X) \rightarrow \mathbb{R}$ *positive* linear maps. (In fact ϕ_1 and ϕ_2 can be chosen such that $\|\phi\| = \|\phi_1\| + \|\phi_2\|$. This fact will be heavily exploited a little later.) We would like then to define a finite *signed* Radon measure μ_ϕ by the formula $\mu_\phi = \mu_{\phi_1} - \mu_{\phi_2}$. There is a minor problem here: *What if we find another pair of continuous positive linear maps $\psi_1, \psi_2 : C^\mathbb{R}(X) \rightarrow \mathbb{R}$, such that $\phi = \psi_1 - \psi_2$? Is it true that $\mu_{\psi_1} - \mu_{\psi_2} = \mu_{\phi_1} - \mu_{\phi_2}$?* The answer is affirmative, and this is an easy consequence of Proposition 7.6, which gives the equalities

$$\mu_{\phi_1} + \mu_{\psi_2} = \mu_{\phi_1 + \psi_2} = \mu_{\psi_1 + \phi_2} = \mu_{\psi_1} + \mu_{\phi_2}.$$

NOTATIONS. Suppose X is a compact Hausdorff space. We define

$$\begin{aligned} \mathcal{M}^\mathbb{R}(X) &= \{ \phi : C^\mathbb{R}(X) \rightarrow \mathbb{R} : \phi \text{ } \mathbb{R}\text{-linear continuous} \}, \\ \mathfrak{R}^\mathbb{R}(X) &= \{ \mu \text{ signed Radon measure on } X \}. \end{aligned}$$

The correspondence

$$(22) \quad \mathcal{M}^\mathbb{R}(X) \ni \phi \longmapsto \mu_\phi \in \mathfrak{R}^\mathbb{R}(X)$$

defined above, will still be referred to as the *extended Riesz correspondence*.

REMARK 8.3. If X is a compact Hausdorff space, then the extended Riesz correspondence (22) is a linear map. This is a consequence of Proposition 7.6.

Given $\phi \in \mathcal{M}^\mathbb{R}(X)$, the existence of a decomposition of ϕ , of the particular type described in Corollary II.5.3, is extremely significant, as suggested by the following result.

THEOREM 8.4. Let X be a compact Hausdorff space, let $\phi_1, \phi_2 : C^\mathbb{R}(X) \rightarrow \mathbb{R}$ be positive linear maps, and let μ_{ϕ_1} and μ_{ϕ_2} be the corresponding Riesz measures. Consider the linear continuous map $\phi = \phi_1 - \phi_2$, and the finite signed measure

$$(23) \quad \mu_\phi = \mu_{\phi_1} - \mu_{\phi_2}.$$

If $\|\phi\| = \|\phi_1\| + \|\phi_2\|$, then $\mu_{\phi_1} \perp \mu_{\phi_2}$, so (23) represents the Hahn-Jordan decomposition of μ_ϕ .

PROOF. We are going to show that the decomposition (23) satisfies condition (ii) in Lemma 8.1. The key step in proving this fact is contained in the following.

Claim: For every $\varepsilon > 0$, there exist functions $f_1, f_2 \in C^\mathbb{R}(X)$, with $f_1, f_2 \geq 0$, $f_1 + f_2 \geq 1$, and such that $\phi_1(f_2) < \varepsilon$ and $\phi_2(f_1) < \varepsilon$.

To prove this we fix $\varepsilon > 0$, and we use the definition of the norm, to find some function $g \in C^\mathbb{R}(X)$, with $\|g\| \leq 1$, and $|\phi(g)| \geq \|\phi\| - \varepsilon$. Replacing g with $-g$, if necessary, we can assume that

$$(24) \quad \phi(g) \geq \|\phi\| - \varepsilon.$$

Consider the functions $g^+ = \max\{g, 0\}$ and $g^- = \max\{-g, 0\}$, so that $g = g^+ - g^-$, and we clearly have $0 \leq g^\pm \leq 1$. On the one hand, since $\|\phi_k\| = \phi_k(1)$ (see Proposition II.5.4), we have $\phi_k(g^\pm) \leq \|\phi_k\|$, $k = 1, 2$. On the other hand, by (24), and the positivity of ϕ_1 and ϕ_2 , we know that

$$\begin{aligned} \|\phi\| - \varepsilon &\leq \phi(g) = \phi_1(g) - \phi_2(g) = \phi_1(g^+) + \phi_2(g^-) - \phi_1(g^-) - \phi_2(g^+) \leq \\ &\leq \phi_1(g^+) + \phi_2(g^-) \leq \|\phi_1\| + \|\phi_2\| = \|\phi\|, \end{aligned}$$

so we get

$$\begin{aligned} \varepsilon &\geq \|\phi\| - \phi_1(g^+) - \phi_2(g^-) = \|\phi_1\| + \|\phi_2\| - \phi_1(g^+) - \phi_2(g^-) = \\ &= \phi_1(1) + \phi_2(1) - \phi_1(g^+) - \phi_2(g^-) = \phi_1(1 - g^+) + \phi_2(1 - g^-). \end{aligned}$$

If we define $f_1 = 1 - g^-$ and $f_2 = 1 - g^+$, then it is clear that $f_1, f_2 \geq 0$. Using the fact that $g^+ + g^- = |g| \leq 1$, we get $f_1 + f_2 = 2 - |g| \geq 1$. Finally, the above estimate gives $\phi_1(f_2) + \phi_2(f_1) \leq \varepsilon$, and so the Claim immediately follows.

Having proven the Claim, we are now in position to prove that the two measures μ_{ϕ_1} and μ_{ϕ_2} satisfy condition (ii) in Lemma 8.1. Start with some arbitrary $\varepsilon > 0$, and use the Claim to find two functions $f_1, f_2 \in C^\mathbb{R}(X)$ with $f_1, f_2 \geq 0$, $f_1 + f_2 \geq 1$, such that $\phi_1(f_2) \leq \varepsilon/2$ and $\phi_2(f_1) \leq \varepsilon/2$. Consider the compact subsets

$$K_1 = \{x \in X : f_1(x) \geq \frac{1}{2}\} \text{ and } K_2 = \{x \in X : f_2(x) \geq \frac{1}{2}\}.$$

Since $f_1 + f_2 \geq 1$, it follows immediately that we have $K_1 \cup K_2 = X$. By construction, we have $2f_1 \geq \chi_{K_1}$ and $2f_2 \geq \chi_{K_2}$, so using the interpolation property (see Proposition 7.5), we get

$$\begin{aligned} \mu_{\phi_1}(K_2) &\leq \phi_1(2f_2) = 2\phi_1(f_2) \leq \varepsilon; \\ \mu_{\phi_2}(K_1) &\leq \phi_2(2f_1) = 2\phi_2(f_1) \leq \varepsilon. \end{aligned} \quad \square$$

The above result has several interesting consequences.

COROLLARY 8.2. *Suppose X is a compact Hausdorff space. Then the extended Riesz correspondence (22) is injective.*

PROOF. Since the correspondence (22) is linear, it suffices to prove the implication $\mu_\phi = 0 \Rightarrow \phi = 0$. Start with some linear continuous map $\phi : C^\mathbb{R}(X) \rightarrow \mathbb{R}$, such that $\mu_\phi = 0$. Use Corollary II.5.3 to find two positive linear maps $\phi_1, \phi_2 : C^\mathbb{R}(X) \rightarrow \mathbb{R}$, such that $\phi = \phi_1 - \phi_2$, and $\|\phi\| = \|\phi_1\| + \|\phi_2\|$. By Theorem 8.4 the difference $\mu_{\phi_1} - \mu_{\phi_2} = \mu_\phi = 0$ is the Hahn-Jordan decomposition of the zero measure. By the uniqueness (see Corollary 8.1) it follows that $\mu_{\phi_1} = \mu_{\phi_2} = 0$. By

the interpolation property, we know that $\|\phi_k\| = \phi_k(1) = \mu_{\phi_k}(X) = 0$, $k \geq 1$, so we get $\phi_1 = \phi_2 = 0$, thus forcing $\phi = 0$. $\square$

The injectivity of the extended Riesz correspondence has as a consequence the uniqueness of the decomposition of linear continuous as differences of positive ones, of the type described in Corollary II.5.3.

COROLLARY 8.3. *Let X be a compact Hausdorff space, and let $\phi : C^{\mathbb{R}}(X) \rightarrow \mathbb{R}$ be a linear continuous map. Assume one has positive linear maps $\phi_1, \phi_2, \psi_1, \psi_2 : C^{\mathbb{R}}(X) \rightarrow \mathbb{R}$, such that*

- $\phi = \phi_1 - \phi_2 = \psi_1 - \psi_2$;
- $\|\phi_1\| + \|\phi_2\| = \|\psi_1\| + \|\psi_2\| = \|\phi\|$.

Then one has the equalities $\phi_1 = \psi_1$ and $\phi_2 = \psi_2$.

PROOF. Consider the signed measure μ_{ϕ} . By Theorem 8.4, the decompositions

$$\mu_{\phi} = \mu_{\phi_1} - \mu_{\phi_2} = \mu_{\psi_1} - \mu_{\psi_2}$$

both represent the Hahn-Jordan decomposition of μ_{ϕ} . By the uniqueness (Corollary 8.1) we have $\mu_{\phi_1} = \mu_{\psi_1}$ and $\mu_{\phi_2} = \mu_{\psi_2}$. By Corollary 8.2 this forces $\phi_1 = \psi_1$ and $\phi_2 = \psi_2$. $\square$

COMMENT. If X is a compact Hausdorff space, and $\phi : C^{\mathbb{R}}(X) \rightarrow \mathbb{R}$ is a linear continuous map, then by the above result, combined with Corollary II.5.3, we know that there exist *unique* positive linear maps $\phi^{\pm} : C^{\mathbb{R}}(X) \rightarrow \mathbb{R}$ such that $\|\phi\| = \|\phi^+\| + \|\phi^-\|$, and

$$(25) \quad \phi = \phi^+ - \phi^-.$$

The decomposition (25) will be referred to as the *Hahn-Jordan decomposition* of ϕ . This notation and terminology are used for the following reason. If we take μ_{ϕ} the measure given by the extended Riesz correspondence, then

$$\mu_{\phi} = \mu_{\phi^+} - \mu_{\phi^-}$$

is precisely the Hahn-Jordan decomposition of μ_{ϕ} .

REMARKS 8.4. There is a version of the extended Riesz correspondence which works for general locally compact spaces. Start with a locally compact space X , and define the spaces

$$\mathcal{M}_0^{\mathbb{R}}(X) = \{\phi : C_0^{\mathbb{R}}(X) \rightarrow \mathbb{R} : \phi \text{ linear continuous}\},$$

$$\mathfrak{R}_0^{\mathbb{R}}(X) = \{\mu \text{ finite signed Radon measure on } X\}.$$

Since $C_0^{\mathbb{R}}(X)$ is the completion of $C_c^{\mathbb{R}}(X)$, the correspondence

$$\mathcal{M}_0^{\mathbb{R}}(X) \ni \phi \longmapsto \phi|_{C_c^{\mathbb{R}}(X)}$$

establishes an isometric linear isomorphism between $\mathcal{M}_0^{\mathbb{R}}(X)$ and the space of all continuous linear maps $C_c^{\mathbb{R}}(X) \rightarrow \mathbb{R}$. For every positive $\phi \in \mathcal{M}_0^{\mathbb{R}}(X)$, we denote by μ_{ϕ} the Riesz measure associated with the restriction $\phi^c = \phi|_{C_c^{\mathbb{R}}(X)}$. Since $\|\phi^c\| = \|\phi\|$, we have the equality $\mu_{\phi}(X) = \|\phi\|$.

We know (see Proposition II.5.10) that for every linear continuous map $\phi : C_0^{\mathbb{R}}(X) \rightarrow \mathbb{R}$, there exist linear *positive* continuous maps $\phi_1, \phi_2 : C_0^{\mathbb{R}}(X) \rightarrow \mathbb{R}$, with $\phi = \phi_1 - \phi_2$. (In fact ϕ_1 and ϕ_2 can be chosen such that $\|\phi_1\| + \|\phi_2\| = \|\phi\|$.) We use this fact to define the finite signed Radon measure $\mu_{\phi} = \mu_{\phi_1} - \mu_{\phi_2}$. Exactly as

in Remark 8.2, this definition is independent of the particular choice of ϕ_1 and ϕ_2 . This way we have constructed a map

$$(26) \quad \mathcal{M}_0^{\mathbb{R}}(X) \ni \phi \longmapsto \mu_\phi \in \mathfrak{R}_0^{\mathbb{R}}(X)$$

which we will call the *extended finite Riesz correspondence*. Of course, if X is already compact, we have $C_0^{\mathbb{R}}(X) = C^{\mathbb{R}}(X)$, $\mathcal{M}_0^{\mathbb{R}}(X) = \mathcal{M}^{\mathbb{R}}(X)$, and $\mathfrak{R}_0^{\mathbb{R}}(X) = \mathfrak{R}^{\mathbb{R}}(X)$, so (26) is the extended Riesz correspondence previously defined.

The following result generalizes the statements of Remark 8.3, Theorem 8.4, and Corollaries 8.2 and 8.3.

THEOREM 8.5. *Let X be a locally compact space.*

- A. *The extended finite Riesz correspondence (26) is an injective linear map.*
- B. *For every $\phi \in \mathcal{M}_0^{\mathbb{R}}(X)$, there exist unique positive maps $\phi^+, \phi^- \in \mathcal{M}_0^{\mathbb{R}}(X)$, such that $\phi = \phi^+ - \phi^-$, and $\|\phi\| = \|\phi^+\| + \|\phi^-\|$. Moreover, in this case*

$$\mu_\phi = \mu_{\phi^+} - \mu_{\phi^-}$$

is precisely the Hahn-Jordan decomposition of μ_ϕ .

PROOF. First of all, the correspondence (26) is clearly linear, again as a consequence of Proposition 7.6.

Second, we remark that the existence part in B is already known, from Proposition II.5.10. We are going to use the following version of Theorem 8.4.

Claim: Suppose $\phi \in \mathcal{M}_0^{\mathbb{R}}(X)$ is written as a difference $\phi = \phi_1 - \phi_2$, with $\phi_1, \phi_2 \in \mathcal{M}_0^{\mathbb{R}}(X)$ positive, and $\|\phi\| = \|\phi_1\| + \|\phi_2\|$. Then

$$\mu_\phi = \mu_{\phi_1} - \mu_{\phi_2}$$

is the Hahn-Jordan decomposition of μ_ϕ .

One way to prove this is by employing the Alexandrov compactification $X^\alpha = X \sqcup \{\infty\}$. We use the identification

$$C_0^{\mathbb{R}}(X) = \{f \in C^{\mathbb{R}}(X^\alpha) : f(\infty) = 0\}.$$

We know that there exist positive linear maps $\psi_1, \psi_2 : C^{\mathbb{R}}(X) \rightarrow \mathbb{R}$, such that $\psi_k|_{C_0^{\mathbb{R}}(X)} = \phi_k$, and $\|\psi_k\| = \|\phi_k\|$, $k = 1, 2$. If we define $\psi : C^{\mathbb{R}}(X^\alpha) \rightarrow \mathbb{R}$ by $\psi = \psi_1 - \psi_2$, it is not hard to see that $\|\psi\| = \|\psi_1\| + \|\psi_2\|$, so if we consider the Radon measures μ_ψ, μ_{ψ_1} and μ_{ψ_2} on the compact space X^α , then using Theorem 8.4, we get the fact that

$$\mu_\psi = \mu_{\psi_1} - \mu_{\psi_2}$$

is precisely the Hahn-Jordan decomposition of μ_ψ . This means that there are sets $B_1, B_2 \in \text{Bor}(X^\alpha)$, with $B_1 \cup B_2 = X^\alpha$, $B_1 \cap B_2 = \emptyset$, and $\mu_{\psi_1}(B_2) = \mu_{\psi_2}(B_1) = 0$. We know (see Remarks 7.4) that

$$\mu_{\psi_k}(B) = \mu_{\phi_k}(B \cap X), \quad \forall B \in \text{Bor}(X^\alpha), \quad k = 1, 2,$$

so if we define $A_k = B_k \cap X$, we immediately get $A_1 \cup A_2 = X$, $A_1 \cap A_2 = \emptyset$, and $\mu_{\phi_1}(A_2) = \mu_{\phi_2}(A_1) = 0$, thus proving that $\mu_{\phi_1} \perp \mu_{\phi_2}$.

Having proven the above Claim, the proof follows line by line the proofs of Corollaries 8.3 and 8.4. $\square$

The notion of a finite signed measure can be generalized to the complex case.

DEFINITION. Suppose $\mathcal{A}$ is a σ -algebra on a non-empty set X . A function $\mu : \mathcal{A} \rightarrow \mathbb{C}$ is called a *complex measure on $\mathcal{A}$* , if it is σ -additive in the sense that

(ADD $_{\sigma}$) for any pairwise disjoint sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{A}$, one has the equality

$$(27) \quad \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Remark that the condition $\mu(\emptyset) = 0$ is automatic in this case. Note also that a map $\mu : \mathcal{A} \rightarrow \mathbb{C}$ is a complex measure, if and only if the maps $\operatorname{Re} \mu$ and $\operatorname{Im} \mu$ are finite signed measures.

The following result describes an important construction.

THEOREM 8.6. *Let $\mathcal{A}$ be a σ -algebra, and let μ be either a signed measure, or a complex measure on $\mathcal{A}$. For every $A \in \mathcal{A}$, we define*

$$(28) \quad \nu(A) = \sup \left\{ \sum_{k=1}^{\infty} |\mu(A_k)| : (A_k)_{k=1}^{\infty} \subset \mathcal{A}, \text{ pairwise disjoint, } \bigcup_{k=1}^{\infty} A_k = A \right\}.$$

The map $\nu : \mathcal{A} \rightarrow [0, \infty]$ is an "honest" measure on $\mathcal{A}$.

PROOF. The first step in the proof is contained in the following.

Claim 1: For any pairwise disjoint sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{A}$, one has the inequality

$$(29) \quad \nu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \nu(A_n).$$

Denote the right hand side of (29) by S , and denote the union $\bigcup_{n=1}^{\infty} A_n$ simply by A . Start now with some pairwise disjoint sequence $(D_k)_{k=1}^{\infty} \subset \mathcal{A}$, with $\bigcup_{k=1}^{\infty} D_k = A$. For every $k \geq 1$, we have $D_k = \bigcup_{n=1}^{\infty} (D_k \cap A_n)$, with $(D_k \cap A_n)_{n=1}^{\infty} \subset \mathcal{A}$ pairwise disjoint, so we have

$$|\mu(D_k)| = \left| \sum_{n=1}^{\infty} \mu(D_k \cap A_n) \right| \leq \sum_{n=1}^{\infty} |\mu(D_k \cap A_n)|, \quad \forall k \geq 1.$$

Summing up then yields

$$(30) \quad \sum_{k=1}^{\infty} |\mu(D_k)| \leq \sum_{k=1}^{\infty} \left[\sum_{n=1}^{\infty} |\mu(D_k \cap A_n)| \right] = \sum_{n=1}^{\infty} \left[\sum_{k=1}^{\infty} |\mu(D_k \cap A_n)| \right].$$

Since for each $n \geq 1$, the sequence $(D_k \cap A_n)_{k=1}^{\infty} \subset \mathcal{A}$ is pairwise disjoint, and satisfies $\bigcup_{k=1}^{\infty} (D_k \cap A_n) = A_n$, by the definition of ν , we get

$$\sum_{k=1}^{\infty} |\mu(D_k \cap A_n)| \leq \nu(A_n), \quad \forall n \geq 1.$$

Using these estimates in (30), we then get

$$\sum_{k=1}^{\infty} |\mu(D_k)| \leq \sum_{n=1}^{\infty} \nu(A_n).$$

Since the inequality $\sum_{k=1}^{\infty} |\mu(D_k)| \leq S$ holds for all pairwise disjoint sequences $(D_k)_{k=1}^{\infty} \subset \mathcal{A}$, with $\bigcup_{k=1}^{\infty} D_k = A$, by the definition of ν we get $\nu(A) \leq S$, and the Claim is proven.

Claim 2: For any finite pairwise disjoint collection $(A_n)_{n=1}^N \subset \mathcal{A}$, one has the inequality

$$\nu(A_1 \cup \cdots \cup A_N) \geq \nu(A_1) + \cdots + \nu(A_N).$$

We use induction on N , and we see immediately that it suffices only to prove the case $N = 2$. Fix for the moment a pairwise disjoint sequence $(D_k)_{k=1}^\infty \subset \mathcal{A}$, with $\bigcup_{k=1}^\infty D_k = A_1$, and denote the sum $\sum_{k=1}^\infty |\mu(D_k)|$ by R . Suppose we have a pairwise disjoint sequence $(E_j)_{j=1}^\infty \subset \mathcal{A}$, with $\bigcup_{j=1}^\infty E_j = A_2$. If we combine it with the D_k 's, i.e. we define

$$F_p = \begin{cases} D_{p/2} & \text{if } p \text{ is even} \\ E_{(p+1)/2} & \text{if } p \text{ is odd} \end{cases}$$

then we get a new pairwise disjoint sequence $(F_p)_{p=1}^\infty \subset \mathcal{A}$, with $\bigcup_{p=1}^\infty F_p = A_1 \cup A_2$. By the definition of ν we will then get

$$\nu(A_1 \cup A_2) \geq \sum_{p=1}^\infty |\mu(F_p)| = \sum_{k=1}^\infty |\mu(D_k)| + \sum_{j=1}^\infty |\mu(E_j)| = R + \sum_{j=1}^\infty |\mu(E_j)|.$$

Taking supremum over all pairwise disjoint sequences $(E_j)_{j=1}^\infty \subset \mathcal{A}$, with $\bigcup_{j=1}^\infty E_j = A_2$, the above inequality yields $\mu(A_1 \cup A_2) \geq R + \nu(A_2)$, so now we have

$$\nu(A_1 \cup A_2) \geq \nu(A_2) + \sum_{k=1}^\infty |\mu(D_k)|.$$

Taking supremum over all pairwise disjoint sequences $(D_k)_{k=1}^\infty \subset \mathcal{A}$, with $\bigcup_{k=1}^\infty D_k = A_1$, the above inequality finally gives $\nu(A_1 \cup A_2) \geq \nu(A_2) + \nu(A_1)$, and the Claim is proven.

We are now in position to prove that ν is a measure on $\mathcal{A}$. The equality $\nu(\emptyset) = 0$ is trivial. To prove σ -additivity, we start with some pairwise disjoint sequence $(A_n)_{n=1}^\infty \subset \mathcal{A}$, and we must prove the equality

$$\nu\left(\bigcup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \nu(A_n).$$

On the one hand, using Claim 1, we know that we have the inequality $\nu\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \nu(A_n)$. On the other hand, if we denote the union $\bigcup_{n=1}^\infty A_n$ simply by A , then using Claim 2, we see that

$$\nu(A) \geq \nu(A \setminus [A_1 \cup \dots \cup A_N]) + \nu(A_1) + \dots + \nu(A_N) \geq \nu(A_1) + \dots + \nu(A_N), \quad \forall N \geq 1,$$

which immediately gives the other inequality $\nu(A) \geq \sum_{n=1}^\infty \nu(A_n)$. $\square$

DEFINITION. With the notations above, and under the hypothesis of Theorem 8.6, the “honest” measure ν , defined by (28), is called the *variation measure of μ* , and will be denoted by $|\mu|$. By construction, we have the inequality

$$|\mu(A)| \leq |\mu|(A), \quad \forall A \in \mathcal{A}.$$

REMARK 8.5. Let μ be either a signed measure, or a complex measure on the σ -algebra $\mathcal{A}$. Exactly as with numbers (or functions), the measure $|\mu|$ has a minimality property, which can be stated as follows. *Whenever ν is an “honest” measure on $\mathcal{A}$ with*

$$|\mu(A)| \leq \nu(A), \quad \forall A \in \mathcal{A},$$

it follows that we have

$$|\mu|(A) \leq \nu(A), \quad \forall A \in \mathcal{A}.$$

This is quite clear, because for any pairwise disjoint sequence $(A_n)_{n=1}^\infty \subset \mathcal{A}$, with $\bigcup_{n=1}^\infty A_n = A$, one has the inequality

$$\sum_{n=1}^\infty |\mu(A_n)| \leq \sum_{n=1}^\infty \nu(A_n) = \nu(A),$$

and then the desired inequality follows by taking the supremum in the left hand side.

In the case of signed measures, the variation measure is also given by the following.

PROPOSITION 8.2. *Let μ be a signed measure on the σ -algebra $\mathcal{A}$. Then one has the equality*

$$|\mu| = \mu^+ + \mu^-,$$

where $\mu = \mu^+ - \mu^-$ is the Hahn-Jordan decomposition of μ .

PROOF. Denote the measure $\mu^+ + \mu^-$ simply by ν . Remark that we obviously have

$$-\nu(A) = -\mu^+(A) - \mu^-(A) \leq \mu^+(A) - \mu^-(A) \leq \mu^+(A) + \mu^-(A) = \nu(A), \quad \forall A \in \mathcal{A},$$

which gives

$$|\mu(A)| \leq \nu(A), \quad \forall A \in \mathcal{A}.$$

By Remark 8.5, this forces the inequality $|\mu| \leq \nu$.

To prove the other inequality, we start by fixing sets $X^+, X^- \in \mathcal{A}$ as in Theorem 8.2. We decompose each set $A \in \mathcal{A}$ as $A = A^+ \cup A^-$, where $A^\pm = A \cap X^\pm$, so that we have

$$\nu(A) = \nu(A^+) + \nu(A^-) = \mu^+(A^+) + \mu^+(A^-) + \mu^-(A^+) + \mu^-(A^-) = \mu^+(A^+) + \mu^-(A^-).$$

Notice now that $\mu(A^+) = \mu^+(A^+) \geq 0$, and $-\mu(A^-) = \mu^-(A^-) \geq 0$, which means that we have the equalities $\mu^+(A^+) = |\mu(A^+)|$ and $\mu^-(A^-) = |\mu(A^-)|$, so the above equality reads

$$\nu(A) = |\mu(A^+)| + |\mu(A^-)|,$$

and by the definition of $|\mu|$ we then immediately get $\nu(A) \leq |\mu|(A)$. $\square$

An interesting consequence is the following.

COROLLARY 8.4. *Let μ be either a finite signed measure, or a complex measure on the σ -algebra $\mathcal{A}$. Then the variation measure $|\mu|$ is finite.*

PROOF. The signed measure case is clear from the above result.

In the complex case, we write $\mu = \nu + i\eta$, with ν and η finite signed measures on $\mathcal{A}$. We apply the signed case, to get the fact that both $|\nu|$ and $|\eta|$ are finite. Notice that we have

$$|\mu(A)| = |\nu(A) + i\eta(A)| \leq |\nu(A)| + |\eta(A)| \leq |\nu|(A) + |\eta|(A), \quad \forall A \in \mathcal{A},$$

so by Remark 8.5 we get $|\mu| \leq |\nu| + |\eta|$, and then the finiteness of $|\mu|$ is a consequence of the finiteness of $|\nu|$ and $|\eta|$. $\square$

Exercise 3. Let $\mathcal{A}$ be a σ -algebra, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. For the purpose of this exercise, let us agree to use the term $\mathbb{K}$ -measure for designating either a finite signed measure (when $\mathbb{K} = \mathbb{R}$), or a complex measure (when $\mathbb{K} = \mathbb{C}$). Prove the following.

- (i) The collection of all $\mathbb{K}$ -measures on $\mathcal{A}$ is a vector space.

(ii) For any two $\mathbb{K}$ -measures μ and ν , one has the inequality

$$|\mu + \nu| \leq |\mu| + |\nu|.$$

(iii) For any $\mathbb{K}$ -measure μ and any $\alpha \in \mathbb{K}$, one has the equality

$$|\alpha\mu| = |\alpha| \cdot |\mu|.$$

Proposition 8.2 has another interesting consequence, which is relevant for the study of the extended finite Riesz correspondence.

COROLLARY 8.5. *Let X be a locally compact space. Then the extended finite Riesz correspondence (26) has the property*

$$(31) \quad |\mu_\phi|(X) = \|\phi\|, \quad \forall \phi \in \mathcal{M}_0^{\mathbb{R}}(X).$$

PROOF. From Proposition 8.1 and Theorem 8.5, we know that $|\mu_\phi| = \mu_{\phi^+} + \mu_{\phi^-}$. Using Remark 8.4, and Theorem 8.5 again, we have

$$|\mu_\phi|(X) = \mu_{\phi^+}(X) + \mu_{\phi^-}(X) = \|\phi^+\| + \|\phi^-\| = \|\phi\|. \quad \square$$

COMMENTS. Given a locally compact space X , we can define a *complex Radon measure on X* as being a complex measure on X , whose real and imaginary part are both (finite) signed Radon measures. The extended finite Riesz correspondence can be then defined also over the complex numbers, as a map

$$\mathcal{M}_0(X) \ni \phi \longmapsto \mu_\phi \in \mathfrak{R}_0(X),$$

where

$$\mathcal{M}_0(X) = \{\phi : C_0(X) \rightarrow \mathbb{C} : \phi \text{ linear continuous}\},$$

$$\mathfrak{R}_0(X) = \{\mu \text{ complex Radon measure on } X\}.$$

This correspondence is again linear. One will still have the equality (31), but the proof of this fact will appear later in Chapter IV.

Chapter IV

Integration Theory

LECTURES 32-33

1. Construction of the integral

In this section we construct the abstract integral. As a matter of terminology, we define a *measure space* as being a triple $(X, \mathcal{A}, \mu)$, where X is some (non-empty) set, $\mathcal{A}$ is a σ -algebra on X , and μ is a measure on $\mathcal{A}$. The measure space $(X, \mathcal{A}, \mu)$ is said to be *finite*, if $\mu(X) < \infty$.

DEFINITION. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. A $\mathbb{K}$ -valued elementary μ -integrable function on $(X, \mathcal{A}, \mu)$ is an function $f : X \rightarrow \mathbb{K}$, with the following properties

- the range $f(X)$ of f is a finite set;
- $f^{-1}(\{\alpha\}) \in \mathcal{A}$, and $\mu(f^{-1}(\{\alpha\})) < \infty$, for all $\alpha \in f(X) \setminus \{0\}$.

We denote by $\mathfrak{L}_{\mathbb{K}, \text{elem}}^1(X, \mathcal{A}, \mu)$ the collection of all such functions.

REMARKS 1.1. Let $(X, \mathcal{A}, \mu)$ be a measure space.

A. Every $\mathbb{K}$ -valued elementary μ -integrable function f on $(X, \mathcal{A}, \mu)$ is measurable, as a map $f : (X, \mathcal{A}) \rightarrow (\mathbb{K}, \text{Bor}(\mathbb{K}))$. In fact, any such f can be written as

$$f = \alpha_1 \chi_{A_1} + \cdots + \alpha_n \chi_{A_n},$$

with $\alpha_k \in \mathbb{K}$, $A_k \in \mathcal{A}$ and $\mu(A_k) < \infty$, $\forall k = 1, \dots, n$. Using the notations from III.1, we have the inclusion

$$\mathfrak{L}_{\mathbb{K}, \text{elem}}^1(X, \mathcal{A}, \mu) \subset \mathcal{A}\text{-Elem}_{\mathbb{K}}(X).$$

B. If we consider the collection $\mathcal{R} = \{A \in \mathcal{A} : \mu(A) < \infty\}$, then $\mathcal{R}$ is a ring, and, we have the equality

$$\mathfrak{L}_{\mathbb{K}, \text{elem}}^1(X, \mathcal{A}, \mu) = \mathcal{R}\text{-Elem}_{\mathbb{K}}(X).$$

In particular, it follows that $\mathfrak{L}_{\mathbb{K}, \text{elem}}^1(X, \mathcal{A}, \mu)$ is a $\mathbb{K}$ -vector space.

The following result is the first step in the construction of the integral.

THEOREM 1.1. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. Then there exists a unique $\mathbb{K}$ -linear map $I_{\text{elem}}^\mu : \mathfrak{L}_{\mathbb{K}, \text{elem}}^1(X, \mathcal{A}, \mu) \rightarrow \mathbb{K}$, such that

$$(1) \quad I_{\text{elem}}^\mu(\chi_A) = \mu(A),$$

for all $A \in \mathcal{A}$, with $\mu(A) < \infty$.

PROOF. For every $f \in \mathfrak{L}_{\mathbb{K}, \text{elem}}^1(X, \mathcal{A}, \mu)$, we define

$$I_{\text{elem}}^\mu(f) = \sum_{\alpha \in f(X) \setminus \{0\}} \alpha \cdot \mu(f^{-1}(\{\alpha\})),$$

with the convention that, when $f(X) = \{0\}$ (which is the same as $f = 0$), we define $I_{elem}^\mu(f) = 0$. It is obvious that I_{elem}^μ satisfies the equality (1) for all $A \in \mathcal{A}$ with $\mu(A) < \infty$.

One key feature we are going to use is the following.

Claim 1: Whenever we have a finite pairwise disjoint sequence $(A_k)_{k=1}^n \subset \mathcal{A}$, with $\mu(A_k) < \infty$, $\forall k = 1, \dots, n$, one has the equality

$$I_{elem}^\mu(\alpha_1 \chi_{A_1} + \dots + \alpha_n \chi_{A_n}) = \alpha_1 \mu(A_1) + \dots + \alpha_n \mu(A_n), \quad \forall \alpha_1, \dots, \alpha_n \in \mathbb{K}.$$

It is obvious that we can assume $\alpha_j \neq 0$, $\forall j = 1, \dots, n$. To prove the above equality, we consider the elementary μ -integrable function $f = \alpha_1 \chi_{A_1} + \dots + \alpha_n \chi_{A_n}$, and we observe that $f(X) \setminus \{0\} = \{\alpha_1\} \cup \dots \cup \{\alpha_n\}$. It may be the case that some of the α 's are equal. We list $f(X) \setminus \{0\} = \{\beta_1, \dots, \beta_p\}$, with $\beta_j \neq \beta_k$, for all $j, k \in \{1, \dots, p\}$ with $j \neq k$. For each $k \in \{1, \dots, p\}$, we define the set

$$J_k = \{j \in \{1, \dots, n\} : \alpha_j = \beta_k\}.$$

It is obvious that the sets $(J_k)_{k=1}^p$ are pairwise disjoint, and we have $J_1 \cup \dots \cup J_p = \{1, \dots, n\}$. Moreover, for each $k \in \{1, \dots, p\}$, one has the equality

$$f^{-1}(\{\beta_k\}) = \bigcup_{j \in J_k} A_j,$$

so we get

$$\beta_k \mu(f^{-1}(\{\beta_k\})) = \beta_k \sum_{j \in J_k} \mu(A_j) = \sum_{j \in J_k} \alpha_j \mu(A_j), \quad \forall k \in \{1, \dots, p\}.$$

By the definition of I_{elem}^μ we then get

$$I_{elem}^\mu(f) = \sum_{k=1}^p \beta_k \mu(f^{-1}(\{\beta_k\})) = \sum_{k=1}^p \left[\sum_{j \in J_k} \alpha_j \mu(A_j) \right] = \sum_{j=1}^n \alpha_j \mu(A_j).$$

Claim 2: For every $f \in \mathfrak{L}_{\mathbb{K}, elem}^1(X, \mathcal{A}, \mu)$, and every $A \in \mathcal{A}$ with $\mu(A) < \infty$, one has the equality

$$(2) \quad I_{elem}^\mu(f + \alpha \chi_A) = I_{elem}^\mu(f) + \alpha \mu(A), \quad \forall \alpha \in \mathbb{K}.$$

Write $f = \alpha_1 \chi_{A_1} + \dots + \alpha_n \chi_{A_n}$, with $(A_j)_{j=1}^n \subset \mathcal{A}$ pairwise disjoint, and $\mu(A_j) < \infty$, $\forall j = 1, \dots, n$. In order to prove (2), we are going to write the function $f + \alpha \chi_A$ in a similar way, and we are going to apply Claim 1. Consider the sets $B_1, B_2, \dots, B_{2n}, B_{2n+1} \in \mathcal{A}$ defined by $B_{2n+1} = A \setminus (A_1 \cup \dots \cup A_n)$, and $B_{2k-1} = A_k \cap A$, $B_{2k} = A_k \setminus A$, $\forall k = 1, \dots, n$. It is obvious that the sets $(B_p)_{p=1}^{2n+1}$ are pairwise disjoint. Moreover, one has the equalities

$$(3) \quad B_{2k-1} \cup B_{2k} = A_k, \quad \forall k \in \{1, \dots, n\},$$

as well as the equality

$$(4) \quad A = \bigcup_{k=1}^{n+1} B_{2k-1}.$$

Using these equalities, now we have $f + \alpha \chi_A = \sum_{p=1}^{2n+1} \beta_p \chi_{B_p}$, where $\beta_{2n+1} = \alpha$, and $\beta_{2k} = \alpha_k$ and $\beta_{2k-1} = \alpha_k + \alpha$, $\forall k \in \{1, \dots, n\}$. Using these equalities,

combined with Claim 1, and (3) and (4), we now get

$$\begin{aligned}
 I_{elem}^\mu(f + \alpha \chi_A) &= \sum_{p=1}^{2n+1} \beta_p \mu(B_p) = \\
 &= \alpha \mu(B_{2n+1}) + \sum_{k=1}^n [(\alpha_k + \alpha) \mu(B_{2k-1}) + \alpha_k \mu(B_{2k})] = \\
 &= \left[\alpha \sum_{k=1}^{n+1} \mu(B_{2k-1}) \right] + \left[\sum_{k=1}^n \alpha_k [\mu(B_{2k-1}) + \mu(B_{2k})] \right] = \\
 &= \alpha \mu\left(\bigcup_{k=1}^{n+1} B_{2k-1}\right) + \sum_{k=1}^n \alpha_k \mu(B_{2k-1} \cup B_{2k}) = \\
 &= \alpha \mu(A) + \sum_{k=1}^n \alpha_k \mu(A_k) = \alpha \mu(A) + I_{elem}^\mu(f),
 \end{aligned}$$

and the Claim is proven.

We now prove that I_{elem}^μ is linear. The equality

$$I_{elem}^\mu(f + g) = I_{elem}^\mu(f) + I_{elem}^\mu(g), \quad \forall f, g \in \mathfrak{L}_{\mathbb{K}, elem}^1(X, \mathcal{A}, \mu)$$

follows from Claim 2, using an obvious inductive argument. The equality

$$I_{elem}^\mu(\alpha f) = \alpha I_{elem}^\mu(f), \quad \forall \alpha \in \mathbb{K}, f \in \mathfrak{L}_{\mathbb{K}, elem}^1(X, \mathcal{A}, \mu).$$

is also pretty obvious, from the definition.

The uniqueness is also clear. □

DEFINITION. With the notations above, the linear map

$$I_{elem}^\mu : \mathfrak{L}_{\mathbb{K}, elem}^1(X, \mathcal{A}, \mu) \rightarrow \mathbb{K}$$

is called the *elementary μ -integral*.

In what follows we are going to encounter also situations when certain relations among measurable functions hold “almost everywhere.” We are going to use the following.

CONVENTION. Let T be one of the spaces $[-\infty, \infty]$ or $\mathbb{C}$, and let R be some relation on T (in our case R will be either “=,” or “ $\geq$,” or “ $\leq$,” on $[-\infty, \infty]$). Given a measurable space $(X, \mathcal{A}, \mu)$, and two measurable functions $f_1, f_2 : X \rightarrow T$,

$$f_1 R f_2, \quad \mu\text{-a.e.}$$

if the set

$$A = \{x \in X : f_1(x) R f_2(x)\}$$

belongs to $\mathcal{A}$, and it has μ -null complement in X , i.e. $\mu(X \setminus A) = 0$. (If R is one of the relations listed above, the set A automatically belongs to $\mathcal{A}$.) The abbreviation “ μ -a.e.” stands for “ μ -almost everywhere.”

REMARK 1.2. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f \in \mathcal{A}\text{-Elem}_{\mathbb{K}}(X)$ be such that

$$f = 0, \quad \mu\text{-a.e.}$$

Then $f \in \mathfrak{L}_{\mathbb{K}, elem}^1(X, \mathcal{A}, \mu)$, and $I_{elem}^\mu(f) = 0$. Indeed, if we define the set

$$N = \{x \in X : f(x) \neq 0\},$$

then $N \in \mathcal{A}$ and $\mu(N) = 0$. Since $f^{-1}(\{\alpha\}) \subset N$, $\forall \alpha \in f(X) \setminus \{0\}$, it follows that $\mu(f^{-1}(\{\alpha\})) = 0$, $\forall \alpha \in f(X) \setminus \{0\}$, and then by the definition of the elementary μ -integral, we get $I_{elem}^\mu(f) = 0$.

One useful property of elementary integrable functions is the following.

PROPOSITION 1.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f, g \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, and let $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$ be such that*

$$f \leq h \leq g, \quad \mu\text{-a.e.}$$

Then $h \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, and

$$(5) \quad I_{elem}^\mu(f) \leq I_{elem}^\mu(h) \leq I_{elem}^\mu(g).$$

PROOF. Consider the sets

$$A = \{x \in X : f(x) > h(x)\} \text{ and } B = \{x \in X : h(x) > g(x)\},$$

which both belong to $\mathcal{A}$, and have $\mu(A) = \mu(B) = 0$. The set $M = A \cup B$ also belongs to $\mathcal{A}$ and has $\mu(M) = 0$. Define the functions $f_0 = f(1 - \chi_M)$, $g_0 = g(1 - \chi_M)$, and $h_0 = h(1 - \chi_M)$. It is clear that f_0 , g_0 , and h_0 are all in $\mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$. Moreover, we have the equalities $f_0 = f$, μ -a.e., $g_0 = g$, μ -a.e., and $h_0 = h$, μ -a.e., so by Remark ??, combined with Theorem 1.1, the functions $f_0 = f + (f_0 - f)$ and $g_0 = (g_0 - g) + g$ both belong to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, and we have the equalities

$$(6) \quad I_{elem}^\mu(f_0) = I_{elem}^\mu(f) \text{ and } I_{elem}^\mu(g_0) = I_{elem}^\mu(g).$$

Notice now that we have the (absolute) inequality

$$f_0 \leq h_0 \leq g_0.$$

Let us show that h_0 is elementary integrable. Start with some $\alpha \in h_0(X) \setminus \{0\}$. If $\alpha > 0$, then, using the inequality $h_0 \leq g_0$, we get

$$h_0^{-1}(\{\alpha\}) \subset g_0^{-1}((0, \infty)) \subset \bigcup_{\lambda \in g_0(X) \setminus \{0\}} g_0^{-1}(\{\lambda\}),$$

which proves that $\mu(h_0^{-1}(\{\alpha\})) < \infty$. Likewise, if $\alpha < 0$, then, using the inequality $h_0 \geq f_0$, we get

$$h_0^{-1}(\{\alpha\}) \subset f_0^{-1}((-\infty, 0)) \subset \bigcup_{\lambda \in f_0(X) \setminus \{0\}} f_0^{-1}(\{\lambda\}),$$

which proves again that $\mu(h_0^{-1}(\{\alpha\})) < \infty$.

Having shown that h_0 is elementary integrable, we now compare the numbers $I_{elem}^\mu(f)$, $I_{elem}^\mu(h_0)$, and $I_{elem}^\mu(g)$. Define the functions $f_1 = h_0 - f_0$, and $g_1 = g_0 - h_0$. By Theorem 1.1, we know that $f_1, g_1 \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$. Since $f_1, g_1 \geq 0$, we have $f_1(X), g_1(X) \subset [0, \infty)$, so it follows immediately that $I_{elem}^\mu(f_1) \geq 0$ and $I_{elem}^\mu(g_1) \geq 0$. Now, again using Theorem 1.1, and (6), we get

$$I_{elem}^\mu(h_0) = I_{elem}^\mu(f_0 + f_1) = I_{elem}^\mu(f_0) + I_{elem}^\mu(f_1) \geq I_{elem}^\mu(f_0) = I_{elem}^\mu(f);$$

$$I_{elem}^\mu(h_0) = I_{elem}^\mu(g_0 - g_1) = I_{elem}^\mu(g_0) - I_{elem}^\mu(g_1) \leq I_{elem}^\mu(g_0) = I_{elem}^\mu(g).$$

Since $h = h_0$, μ -a.e., by the above Remark it follows that $h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, and $I_{elem}^\mu(h) = I_{elem}^\mu(h_0)$, so the desired inequality (5) follows immediately. $\square$

We now define another type of integral.

DEFINITION. Let $(X, \mathcal{A}, \mu)$ be a measure space. A measurable function $f : X \rightarrow [0, \infty]$ is said to be μ -integrable, if

- (a) every $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with $0 \leq h \leq f$, is elementary μ -integrable;
- (b) $\sup \{I_{elem}^{\mu}(h) : h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X), 0 \leq h \leq f\} < \infty$.

If this is the case, the above supremum is denoted by $I_+^{\mu}(f)$. The space of all such functions is denoted by $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. The map

$$I_+^{\mu} : \mathfrak{L}_+^1(X, \mathcal{A}, \mu) \rightarrow [0, \infty)$$

is called the *positive μ -integral*.

The first (legitimate) question is whether there is an overlap between the two definitions. This is answered by the following.

PROPOSITION 1.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$ be a function with $f \geq 0$. The following are equivalent*

- (i) $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$;
- (ii) $f \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$.

Moreover, if f is as above, then $I_{elem}^{\mu}(f) = I_+^{\mu}(f)$.

PROOF. The implication (i) $\Rightarrow$ (ii) is trivial.

To prove the implication (ii) $\Rightarrow$ (i) we start with an arbitrary elementary $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with $0 \leq h \leq f$. Using Proposition 1.1, we clearly get

- (a) $h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$;
- (b) $I_{elem}^{\mu}(h) \leq I_{elem}^{\mu}(f)$.

Using these two facts, it follows that $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, as well as the equality

$$\sup \{I_{elem}^{\mu}(h) : h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X), h \leq f\} = I_{elem}^{\mu}(f),$$

which gives $I_+^{\mu}(f) = I_{elem}^{\mu}(f)$. $\square$

We now examine properties of the positive integral, which are similar to those of the elementary integral. The following is an analogue of Proposition 1.1.

PROPOSITION 1.3. *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and let $g : X \rightarrow [0, \infty]$ be a measurable function, such that $g \leq f$, μ -a.e., then $g \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and $I_+^{\mu}(g) \leq I_+^{\mu}(f)$.*

PROOF. Start with some elementary function $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with $0 \leq h \leq g$. Consider the sets

$$M = \{x \in X : h(x) > f(x)\} \text{ and } N = \{x \in X : g(x) > f(x)\},$$

which obviously belong to $\mathcal{A}$. Since $N \subset M$, and $\mu(N) = 0$, we have $\mu(M) = 0$. If we define the elementary function $h_0 = h(1 - \chi_M)$, then we have $h = h_0$, μ -a.e., and $0 \leq h_0 \leq f$, so it follows that $h_0 \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, and $I_{elem}^{\mu}(h_0) \leq I_{elem}^{\mu}(f)$. Since $h = h_0$, μ -a.e., by Proposition 1.1., it follows that $h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, and $I_{elem}^{\mu}(h) = I_{elem}^{\mu}(h_0) \leq I_+^{\mu}(f)$. By definition, this gives $g \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$ and $I_+^{\mu}(g) \leq I_+^{\mu}(f)$. $\square$

REMARK 1.3. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. Although f is allowed to take the value ∞ , it turns out that this is inessential. More precisely one has

$$\mu(f^{-1}(\{\infty\})) = 0.$$

This is in fact a consequence of the equality

$$(7) \quad \lim_{t \rightarrow \infty} \mu(f^{-1}([t, \infty])) = 0.$$

Indeed, if we define, for each $t \in (0, \infty)$, the set $A_t = f^{-1}([t, \infty]) \in \mathcal{A}$, then we have $0 \leq t\chi_{A_t} \leq f$. This forces the functions $t\chi_{A_t}$, $t \in (0, \infty)$ to be elementary integrable, and

$$\mu(A_t) \leq \frac{I_+^\mu(f)}{t}, \quad \forall t \in (0, \infty).$$

This forces $\lim_{t \rightarrow \infty} \mu(A_t) = 0$.

The next result explains the fact that positive integrability is a “decomposable” property.

PROPOSITION 1.4. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Suppose $(A_k)_{k=1}^n \subset \mathcal{A}$ is a pairwise disjoint finite sequence, with $A_1 \cup \cdots \cup A_n = X$. For a measurable function $f : X \rightarrow [0, \infty]$, the following are equivalent.*

- (i) $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$;
- (ii) $f\chi_{A_k} \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, $\forall k = 1, \dots, n$.

Moreover, if f satisfies these equivalent conditions, one has

$$I_+^\mu(f) = \sum_{k=1}^n I_+^\mu(f\chi_{A_k}).$$

PROOF. The implication (i) $\Rightarrow$ (ii) is trivial, since we have $0 \leq f\chi_{A_k} \leq f$, so we can apply Proposition 1.3.

To prove the implication (ii) $\Rightarrow$ (i), start by assuming that f satisfies condition (ii). We first observe that every elementary function $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with $0 \leq h \leq f$, has the properties:

- (a) $h \in \mathfrak{L}_{\mathbb{R}, \text{elem}}^1(X, \mathcal{A}, \mu)$;
- (b) $I_{\text{elem}}^\mu(h) \leq \sum_{k=1}^n I_+^\mu(f\chi_{A_k})$.

This is immediate from the fact that we have the equality $h = \sum_{k=1}^n h\chi_{A_k}$, and all function $h\chi_{A_k}$ are elementary, and satisfy $0 \leq h\chi_{A_k} \leq f\chi_{A_k}$, and then everything follows from Theorem 1.1 and the definition of the positive integral which gives $I_{\text{elem}}^\mu(h\chi_{A_k}) \leq I_+^\mu(f\chi_{A_k})$.

Of course, the properties (a) and (b) above prove that $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, as well as the inequality

$$I_+^\mu(f) \leq \sum_{k=1}^n I_+^\mu(f\chi_{A_k}).$$

To prove that we have in fact equality, we start with some $\varepsilon > 0$, and we choose, for each $k \in \{1, \dots, n\}$, a function $h_k \in \mathfrak{L}_{\mathbb{R}, \text{elem}}^1(X, \mathcal{A}, \mu)$, such that $0 \leq h_k \leq f\chi_{A_k}$, and $I_{\text{elem}}^\mu(h_k) \geq I_+^\mu(f\chi_{A_k}) - \frac{\varepsilon}{n}$. By Theorem 1.1, the function $h = h_1 + \cdots + h_n$ belongs to $\mathfrak{L}_{\mathbb{R}, \text{elem}}^1(X, \mathcal{A}, \mu)$, and has

$$(8) \quad I_{\text{elem}}^\mu(h) = \sum_{k=1}^n I_{\text{elem}}^\mu(h_k) \geq \left(\sum_{k=1}^n I_+^\mu(f\chi_{A_k}) \right) - \varepsilon.$$

We obviously have

$$h = \sum_{k=1}^n h_k \leq \sum_{k=1}^n f\chi_{A_k} = f,$$

so we get $I_{elem}^\mu(h) \leq I^\mu(f)$, thus the inequality (8) gives

$$I^\mu(f) \geq \left(\sum_{k=1}^n I_+^\mu(f \chi_{A_k}) \right) - \varepsilon.$$

Since this inequality holds for all $\varepsilon > 0$, we get $I^\mu(f) \geq \sum_{k=1}^n I_+^\mu(f \chi_{A_k})$, and we are done. $\square$

REMARK 1.4. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $S \in \mathcal{A}$. We can

$$\mathcal{A}|_S = \{A \cap S : A \in \mathcal{A}\} = \{A \in \mathcal{A} : A \subset S\},$$

so that $\mathcal{A}|_S \subset \mathcal{A}$ is a σ -algebra on S . The restriction of μ to $\mathcal{A}|_S$ will be denoted by $\mu|_S$. With these notations, $(S, \mathcal{A}|_S, \mu|_S)$ is a measure space. It is not hard to see that for a measurable function $f : X \rightarrow [0, \infty]$, the conditions

- $f \chi_S \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$,
- $f|_S \in \mathfrak{L}_+^1(S, \mathcal{A}|_S, \mu|_S)$

are equivalent. Moreover, in this case one has the equality

$$I_+^\mu(f \chi_S) = I_+^{\mu|_S}(f|_S).$$

This is a consequence of the fact that these two conditions are equivalent if f is elementary, combined with the fact that the restriction map $h \mapsto h|_S$ establishes a bijection between the sets

$$\begin{aligned} & \{h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X) : 0 \leq h \leq f \chi_S\}, \\ & \{k \in \mathcal{A}|_S\text{-Elem}_{\mathbb{R}}(S) : 0 \leq k \leq f|_S\}. \end{aligned}$$

The next result gives an alternative definition of the positive integral, for functions that are dominated by elementary integrable ones.

PROPOSITION 1.5. *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f : X \rightarrow [0, \infty]$ be a measurable function. Assume there exists $h_0 \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, with $h_0 \geq f$. Then $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and one has the equality*

$$(9) \quad I_+^\mu(f) = \inf \{ I_{elem}^\mu(h) : h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu), h \geq f \}.$$

PROOF. Since $h_0 \geq 0$, by Proposition 1.2, we know that $h_0 \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. The fact that $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$ then follows from Proposition 1.3, combined with the inequality $h_0 \geq f$. More generally, again by Propositions 1.2 and 1.3, we know that for any $h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, with $h \geq f$, we have $h \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, as well as the inequality

$$I_+^\mu(f) \leq I_+^\mu(h) = I_{elem}^\mu(h).$$

So, if we denote the right hand side of (9) by $J(f)$, we have $I_+^\mu(f) \leq J(f) \leq I_{elem}^\mu(h_0)$.

We now prove the other inequality $I_+^\mu(f) \geq J(f)$. If $h_0 = 0$, there is nothing to prove. Assume h_0 is not identically zero. Without any loss of generality, we can assume that $h_0 = \beta \chi_B$, for some $\beta \in (0, \infty)$ and $B \in \mathcal{A}$ with $\mu(B) < \infty$. (If we define $B = h_0^{-1}((0, \infty)) = \bigcup_{\alpha \in h_0(X) \setminus \{0\}} h_0^{-1}(\{\alpha\})$, and if we set $\beta = \max h_0(X)$, then we clearly have $\mu(B) < \infty$, and $h_0 \leq \beta \chi_B$.)

For every integer $n \geq 1$, we define the sets $A_1^n, \dots, A_n^n \in \mathcal{A}$ by

$$A_k^n = f^{-1}\left(\left(\frac{(k-1)\beta}{n}, \frac{k\beta}{n}\right]\right), \quad \forall k = 1, \dots, n,$$

and we define the elementary functions

$$g_n = \sum_{k=1}^n \frac{(k-1)\beta}{n} \chi_{A_k^n} \text{ and } h_n = \sum_{k=1}^n \frac{k\beta}{n} \chi_{A_k^n}.$$

The main features of these constructions are collected in the following.

Claim: For every $n \geq 1$, the functions g_n and h_n are elementary integrable, and satisfy the inequalities $0 \leq g_n \leq f \leq h_n \leq h_0$, as well as

$$I_{elem}^\mu(h_n) \leq I_{elem}^\mu(g_n) + \frac{\beta\mu(B)}{n}.$$

To prove this fact, we fix $n \geq 1$, and we first remark that the sets $(A_k^n)_{k=1}^n$ are pairwise disjoint. Since $0 \leq f \leq h_0 = \beta\chi_B$, we have

$$A_1^n \cup \dots \cup A_n^n = f^{-1}((0, \beta]) \subset B.$$

In particular, if we define $A_n = A_1^n \cup \dots \cup A_n^n \subset B$, we have

$$h_n = \sum_{k=1}^n \frac{k\beta}{n} \chi_{A_k^n} \leq \beta \sum_{k=1}^n \chi_{A_k^n} = \beta\chi_{A_n} \leq \beta\chi_B.$$

Let us prove the inequalities $g_n \leq f \leq h_n$. Start with some arbitrary point $x \in X$, and let us show that $g_n(x) \leq f(x) \leq h_n(x)$. If $f(x) = 0$, there is nothing to prove, because this forces $\chi_{A_k^n}(x) = 0$, $\forall k = 1, \dots, n$. Assume now $f(x) > 0$. Since $f \leq \beta\chi_B$, we now that $f(x) \in (0, \beta]$, so there exists a unique $k \in \{1, \dots, n\}$, such that $\frac{(k-1)\beta}{n} < f(x) \leq \frac{k\beta}{n}$, i.e. $x \in A_k^n$. We then obviously have

$$g_n(x) = \frac{(k-1)\beta}{n} \chi_{A_k^n}(x) = \frac{(k-1)\beta}{n} < f(x) \leq \frac{k\beta}{n} = \frac{k\beta}{n} \chi_{A_k^n}(x) = h_n(x),$$

and we are done. Finally, let us observe that since $g_n \leq h_n \leq h_0$, it follows that g_n and h_n are in $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, so g_n and h_n are elementary integrable. Notice that

$$h_n - g_n = \frac{\beta}{n} \sum_{k=1}^n \chi_{A_k^n} = \frac{\beta}{n} \chi_{A_n} \leq \frac{\beta}{n} \chi_B,$$

so we have $I_{elem}^\mu(h_n - g_n) \leq I_{elem}^\mu(\frac{\beta}{n} \chi_B) = \frac{\beta\mu(B)}{n}$, so using Theorem 1.1, we get

$$I_{elem}^\mu(h_n) = I_{elem}^\mu(g_n) + I_{elem}^\mu(h_n - g_n) \leq I^\mu(g_n) + \frac{\beta\mu(B)}{n}.$$

Having proven the Claim, we immediately see that by the definition of the positive integral, we have

$$J(f) \leq I_{elem}^\mu(h_n) \leq I^\mu(g_n) + \frac{\beta\mu(B)}{n} \leq I_+^\mu(f) + \frac{\beta\mu(B)}{n}.$$

Since the inequality $J(f) \leq I_+^\mu(f) + \frac{\beta\mu(B)}{n}$ holds for all $n \geq 1$, it will clearly force $J(f) \leq I_+^\mu(f)$. $\square$

Our next goal is to prove an analogue of Theorem 1.1, for the positive integral (Theorem 1.2 below). We discuss first a weaker version.

LEMMA 1.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

- (i) *If $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$ and $g \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$ are such that $g + f \geq 0$, then $g + f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and $I_+^\mu(g + f) = I_{elem}^\mu(g) + I_+^\mu(f)$.*
- (ii) *If $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$ and $g \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$ are such that $g - f \geq 0$, then $g - f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and $I_+^\mu(g - f) = I_{elem}^\mu(g) - I_+^\mu(f)$.*

PROOF. (i). We start with a weaker version.

Claim: If $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$ and $g \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, are such that $g + f \geq 0$, then $g + f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and $I_+^\mu(g + f) \leq I_{elem}^\mu(g) + I_+^\mu(f)$.

What we need to prove is the fact that, for every $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with $0 \leq h \leq g + f$, we have:

- (a) $h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$;
- (b) $I_{elem}^\mu(h) \leq I_{elem}^\mu(g) + I_+^\mu(f)$.

Consider the elementary function $h_1 = \max\{h - g, 0\}$. It is obvious that $0 \leq h_1 \leq f$, so by Proposition 1.3, it follows that $h_1 \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and $I_+^\mu(h_1) \leq I_+^\mu(f)$. By Proposition 1.2, this gives $h_1 \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, and

$$(10) \quad I_{elem}^\mu(h_1) = I_+^\mu(h_1) \leq I_+^\mu(f).$$

Using the obvious inequality $-g \leq h - g \leq h_1$, again by Proposition 1.2, it follows that $h - g \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, and

$$(11) \quad I_{elem}^\mu(h - g) \leq I_{elem}^\mu(h_1).$$

Of course, by Theorem 1.1, this gives the fact that $h = (h - g) + g$ is elementary μ -integrable, as well as the equality

$$I_{elem}^\mu(h) = I_{elem}^\mu(h - g) + I_{elem}^\mu(g).$$

Combining this with (11) and (10) immediately gives

$$I_{elem}^\mu(h) \leq I_{elem}^\mu(h_1) + I_{elem}^\mu(g) \leq I_+^\mu(f) + I_{elem}^\mu(g),$$

and the Claim is proven.

Having proven the above Claim, we now proceed with the proof of (i). If $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$ and $g \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$ are such that $g + f \geq 0$, then by the Claim, we already know that $g + f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and $I^\mu(g + f) \leq I_{elem}^\mu(g) + I_+^\mu(f)$. We apply now again the Claim to the functions $f_1 = g + f$ and $g_1 = -g$, to get

$$I_+^\mu(f) = I_+^\mu(g_1 + f_1) \leq I_{elem}^\mu(g_1) + I_+^\mu(f_1) = -I_+^\mu(g) + I_+^\mu(g + f),$$

which gives the other inequality $I_{elem}^\mu(g) + I_+^\mu(f) \leq I_+^\mu(g + f)$.

(ii). Start with $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$ and $g \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, with $g - f \geq 0$. First of all, since $0 \leq g - f \leq g$, by Proposition 1.5, it follows that $g - f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and

$$(12) \quad I_+^\mu(g - f) = \inf \{ I_{elem}^\mu(k) : k \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu), k \geq g - f \}.$$

Second, remark that, whenever $k \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$ is such that $g - f \leq k$, it follows that $k + f \geq g$, so using part (i) combined with Proposition 1.3, we see that $k + f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and

$$I_{elem}^\mu(g) = I_+^\mu(g) \leq I_+^\mu(k + f) = I_{elem}^\mu(k) + I_+^\mu(f).$$

This means that we have

$$I_{elem}^\mu(k) \geq I_{elem}^\mu(g) - I_+^\mu(f),$$

for all $k \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, with $k \geq g - f$, and then by (12), we immediately get

$$I_+^\mu(g - f) \geq I_{elem}^\mu(g) - I_+^\mu(f).$$

To prove the other inequality, we use the definition of the positive integral, which gives

$$(13) \quad I_+^\mu(g - f) = \sup \{ I_{elem}^\mu(h) : h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu), 0 \leq h \leq g - f \}.$$

Remark that, whenever $h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$ is such that $0 \leq h \leq g - f$, it follows that $0 \leq h + f \leq g$, so using part (i) combined with Proposition 1.3, we see that $h + f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and

$$I_{elem}^\mu(g) = I_+^\mu(g) \geq I_+^\mu(h + f) = I_{elem}^\mu(h) + I_+^\mu(f).$$

This means that we have

$$I_{elem}^\mu(h) \leq I_{elem}^\mu(g) - I_+^\mu(f),$$

for all $h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, with $0 \leq h \leq g - f$, and then by (13), we immediately get $I_+^\mu(g - f) \leq I_{elem}^\mu(g) - I_+^\mu(f)$. $\square$

We are now in position to prove the following result (compare with Theorem 1.1).

THEOREM 1.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

- (i) *If $f_1, f_2 \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, then $f_1 + f_2 \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and one has the equality $I_+^\mu(f_1 + f_2) = I_+^\mu(f_1) + I_+^\mu(f_2)$.*
- (ii) *If $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and $\alpha \in [0, \infty)$, then¹⁴ $\alpha f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and one has the equality $I_+^\mu(\alpha f) = \alpha I_+^\mu(f)$.*

PROOF. (i). Fix $f_1, f_2 \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$.

Claim 1: Whenever $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$ satisfies $0 \leq h \leq f_1 + f_2$, it follows that

- (a) $h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$,
- (b) $I_{elem}^\mu(h) \leq I_+^\mu(f_1) + I_+^\mu(f_2)$.

Fix an elementary function $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with $0 \leq h \leq f_1 + f_2$, and let us first show that h is elementary integrable. Fix some $\alpha \in h(X) \setminus \{0\}$, and let us prove that $\mu(h^{-1}(\{\alpha\})) < \infty$. If we define the sets $A_j = f_j^{-1}([\alpha/2, \infty]) \in \mathcal{A}$, $j = 1, 2$, then the elementary functions $h_j = \frac{\alpha}{2} \chi_{A_j}$ satisfy $0 \leq h_j \leq f_j$, $j = 1, 2$. In particular, it follows that $h_1, h_2 \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, which forces $\mu(A_1) < \infty$ and $\mu(A_2) < \infty$. Notice however that, for every $x \in h^{-1}(\{\alpha\})$, we have $f_1(x) + f_2(x) \geq h(x) = \alpha$, which forces either $f_1(x) \geq \frac{\alpha}{2}$ or $f_2(x) \geq \frac{\alpha}{2}$. This argument shows that we have the inclusion $h^{-1}(\{\alpha\}) \subset A_1 \cup A_2$, so it follows that we indeed have $\mu(h^{-1}(\{\alpha\})) < \infty$.

Having shown property (a), let us prove property (b). Define the sets

$$B = \{x \in X : h(x) \geq f_1(x)\} \text{ and } D = X \setminus B.$$

It is obvious that $B, D \in \mathcal{A}$ are pairwise disjoint, and $B \cup D = X$. Define the elementary functions $h' = h \chi_B$, and $h'' = h - h' = h \chi_D$. On the one hand, we have

$$f_1 \chi_B \leq h' \leq f_1 \chi_B + f_2 \chi_B,$$

which gives

$$0 \leq h' - f_1 \chi_B \leq f_2 \chi_B.$$

¹⁴ Here we use the convention that when $\alpha = 0$, we take $\alpha f = 0$.

By Lemma 1.1.(ii), combined with Proposition 1.4, it follows that $h' - f_1\chi_B \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$ and $I_{elem}^\mu(h') - I_+^\mu(f_1\chi_B) = I_+^\mu(h' - f_1\chi_B) \leq I_+^\mu(f_2\chi_B)$, so we get

$$(14) \quad I_{elem}^\mu(h') \leq I_+^\mu(f_1\chi_B) + I_+^\mu(f_2\chi_B).$$

On the other hand, we have

$$h'' = h\chi_D \leq f_1\chi_D,$$

which gives

$$(15) \quad I_{elem}^\mu(h'') \leq I_+^\mu(f_1\chi_D) \leq I_+^\mu(f_1\chi_D) + I_+^\mu(f_2\chi_D).$$

Since $h = h' + h''$, with h' and h'' elementary integrable, using Theorem 1.1 combined with Proposition 1.4, by adding the inequalities (14) and (15) we get

$$\begin{aligned} I_{elem}^\mu(h) &= I_{elem}^\mu(h') + I_{elem}^\mu(h'') \leq \\ &\leq I_+^\mu(f_1\chi_B) + I_+^\mu(f_2\chi_B) + I_+^\mu(f_1\chi_D) + I_+^\mu(f_2\chi_D) = I_+^\mu(f_1) + I_+^\mu(f_2), \end{aligned}$$

and the Claim is proven.

Claim 1 obviously implies the fact that $f_1 + f_2 \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, as well as the inequality

$$I_+^\mu(f_1 + f_2) \leq I_+^\mu(f_1) + I_+^\mu(f_2).$$

To prove the other inequality, we use the following.

Claim 2: For every $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with $0 \leq h \leq f_1$, one has the inequality

$$I_{elem}^\mu(h) \leq I_+^\mu(f_1 + f_2) - I_+^\mu(f_2).$$

Indeed, if h is as above, then h is in $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, hence elementary integrable, and we obviously have $0 \leq h + f_2 \leq f_1 + f_2$. Then by Lemma 1.1.(i), combined with Proposition 1.3, we get

$$I_{elem}^\mu(h) + I_+^\mu(f_2) = I_+^\mu(h + f_2) \leq I_+^\mu(f_1 + f_2),$$

and the Claim follows.

Using Claim 2, and the definition of the positive integral, we get

$$I_+^\mu(f_1) = \sup \{ I_{elem}^\mu(h) : h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X), 0 \leq h \leq f_1 \} \leq I_+^\mu(f_1 + f_2) - I_+^\mu(f_2),$$

which then gives

$$I_+^\mu(f_1) + I_+^\mu(f_2) \leq I_+^\mu(f_1 + f_2).$$

(ii). This part is obvious. $\square$

DEFINITIONS. Let $(X, \mathcal{A}, \mu)$ be a measure space. Denote the extended real line $[-\infty, \infty]$ by $\mathbb{R}$. A measurable function $f : X \rightarrow \mathbb{R}$ is said to be μ -integrable, if there exist functions $f_1, f_2 \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, such that

$$(16) \quad f(x) = f_1(x) - f_2(x), \quad \forall x \in X \setminus [f_1^{-1}(\{\infty\}) \cup f_2^{-1}(\{\infty\})].$$

By Remark 1.3, we know that the sets $f_k^{-1}(\{\infty\})$, $k = 1, 2$, have measure zero. The equality (16) gives then the fact $f = f_1 - f_2$, μ -a.e. We define

$$\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu) = \{ f : X \rightarrow \mathbb{R} : f \text{ } \mu\text{-integrable} \}.$$

We also define the space of “honest” real-valued μ -integrable functions, as

$$\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu) = \{ f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu) : f - \infty < f(x) < \infty, \forall x \in X \}.$$

Finally, we define the space of complex-valued μ -integrable functions as

$$\mathfrak{L}_{\mathbb{C}}^1(X, \mathcal{A}, \mu) = \{ f : X \rightarrow \mathbb{C} : \operatorname{Re} f, \operatorname{Im} f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu) \}.$$

The next result collects the basic properties of $\mathfrak{L}_{\mathbb{R}}^1$. Among other things, it states that it is an “almost” vector space.

THEOREM 1.3. *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

- (i) *For a measurable function $f : X \rightarrow \bar{\mathbb{R}}$, the following are equivalent:*
 (a) $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$;
 (b) $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$.
 (ii) *If $f, g \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, and if $h : X \rightarrow \bar{\mathbb{R}}$ is a measurable function, such that*

$$h(x) = f(x) + g(x), \quad \forall x \in X \setminus [f^{-1}(\{-\infty, \infty\}) \cup g^{-1}(\{-\infty, \infty\})],$$

then $h \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$.

- (iii) *If $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, and $\alpha \in \mathbb{R}$, and if $g : X \rightarrow \bar{\mathbb{R}}$ is a measurable function, such that*

$$g(x) = \alpha f(x), \quad \forall x \in X \setminus f^{-1}(\{-\infty, \infty\}),$$

then $g \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$.

- (iv) *One has the inclusion*

$$\mathfrak{L}_{\mathbb{R}, \text{elem}}^1(X, \mathcal{A}, \mu) \cup \mathfrak{L}_+^1(X, \mathcal{A}, \mu) \subset \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu).$$

PROOF. (i). Consider the functions measurable functions $f^{\pm} : X \rightarrow [0, \infty]$ defined as

$$f^+ = \max\{f, 0\} \text{ and } f^- = \max\{-f, 0\}.$$

To prove the implication (a) $\Rightarrow$ (b), assume $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, which means there exist $f_1, f_2 \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, such that

$$f(x) = f_1(x) - f_2(x), \quad \forall x \in X \setminus [f_1^{-1}(\{\infty\}) \cup f_2^{-1}(\{\infty\})].$$

Notice that we have the inequalities

$$(17) \quad f^+ \leq f_1, \quad \mu\text{-a.e.},$$

$$(18) \quad f^- \leq f_2, \quad \mu\text{-a.e.}.$$

Indeed, if we put $N = f_1^{-1}(\{\infty\}) \cup f_2^{-1}(\{\infty\})$, then $\mu(N) = 0$, and if we start with some $x \in X \setminus N$, we either have $f_1(x) \geq f_2(x) \geq 0$, in which case we get

$$f^+(x) = f(x) = f_1(x) - f_2(x) \leq f_1(x),$$

$$f^-(x) = 0 \leq f_2(x),$$

or we have $f_1(x) \leq f_2(x)$, in which case we get

$$f^+(x) = 0 \leq f_1(x),$$

$$f^-(x) = -f(x) = f_2(x) - f_1(x) \leq f_2(x).$$

In other words, we have

$$f^+(x) \leq f_1(x) \text{ and } f^-(x) \leq f_2(x), \quad \forall x \in X \setminus N,$$

so we indeed get (17) and (18). Using these inequalities, and Proposition 1.3, it follows that $f^{\pm} \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, so by Theorem 1.2, it follows that $f^+ + f^- = |f|$ also belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$.

To prove the implication $(b) \Rightarrow (a)$, start by assuming that $|f| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. Then, since we obviously have the inequalities $0 \leq f^\pm \leq |f|$, again by Proposition 1.3, it follows that $f^\pm \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. Since we obviously have

$$f(x) = f^+(x) - f^-(x), \quad \forall x \in X \setminus f^{-1}(\{-\infty, \infty\}),$$

it follows that f indeed belongs to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$.

(ii). Assume f, g , and h are as in (ii). By (i), both functions $|f|$ and $|g|$ are in $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. By Theorem 1.2, it follows that the function $k = |f| + |g|$ also belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. Notice that we have the equality

$$f^{-1}(\{-\infty, \infty\}) \cup g^{-1}(\{-\infty, \infty\}) = k^{-1}(\{\infty\}),$$

so the hypothesis on h reads

$$h(x) = f(x) + g(x), \quad \forall x \in X \setminus k^{-1}(\{\infty\}),$$

which then gives

$$|h(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)|, \quad \forall x \in X \setminus k^{-1}(\{\infty\}).$$

Of course, since $\mu(k^{-1}(\{\infty\})) = 0$, this gives

$$|h| \leq k, \quad \mu\text{-a.e.},$$

and using (i) it follows that h indeed belongs to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$.

(iii). Assume f, α , and g are as in (iii). Exactly as above, we have $|g| = |\alpha| \cdot |f|$, μ -a.e., and then by Theorem 1.2 it follows that $|g| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$.

(iv). The inclusion $\mathfrak{L}_+^1(X, \mathcal{A}, \mu) \subset \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$ is trivial. To prove the inclusion $\mathfrak{L}_{\mathbb{R}, \text{elem}}^1(X, \mathcal{A}, \mu) \subset \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, we use parts (ii) and (iii) to reduce this to the fact that $\varkappa_A \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, for all $A \in \mathcal{A}$, with $\mu(A) < \infty$. But this fact is now obvious, because any such function belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu) \subset \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$. $\square$

COROLLARY 1.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$.*

- (i) *For a $\mathbb{K}$ -valued measurable function $f : X \rightarrow \mathbb{K}$, the following are equivalent:*
 - (a) $f \in \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$;
 - (b) $|f| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$.
- (ii) *When equipped with the pointwise addition and scalar multiplication, the space $\mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ becomes a $\mathbb{K}$ -vector space.*

PROOF. (i). The case $\mathbb{K} = \mathbb{R}$ is immediate from Theorem 1.3

In the case when $\mathbb{K} = \mathbb{C}$, we use the obvious inequalities

$$(19) \quad \max \{ |\operatorname{Re} f|, |\operatorname{Im} f| \} \leq |f| \leq |\operatorname{Re} f| + |\operatorname{Im} f|.$$

If $f \in \mathfrak{L}_{\mathbb{C}}^1(X, \mathcal{A}, \mu)$, then both $\operatorname{Re} f$ and $\operatorname{Im} f$ belong to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, so by Theorem 1.3, both $|\operatorname{Re} f|$ and $|\operatorname{Im} f|$ belong to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. By Theorem 1.2, the function $g = |\operatorname{Re} f| + |\operatorname{Im} f|$ belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and then using the second inequality in (19), it follows that $|f|$ belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$.

Conversely, if $|f|$ belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, then using the first inequality in (19), it follows that both $|\operatorname{Re} f|$ and $|\operatorname{Im} f|$ belong to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, so by Theorem 1.3, both $\operatorname{Re} f$ and $\operatorname{Im} f$ belong to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, i.e. f belongs to $\mathfrak{L}_{\mathbb{C}}^1(X, \mathcal{A}, \mu)$.

(ii). This part is pretty clear. If $f, g \in \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, then by (i) both $|f|$ and $|g|$ belong to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and by Theorem 1.2, the function $|f| + |g|$ will

also belong to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. Since $|f + g| \leq |f| + |g|$, it follows that $|f + g|$ itself belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, so using (i) again, it follows that $f + g$ indeed belongs to $\mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$. If $f \in \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ and $\alpha \in \mathbb{K}$, then $|f|$ belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, so $|\alpha f| = |\alpha| \cdot |f|$ again belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, which by (i) gives the fact that αf belongs to $\mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$. $\square$

REMARK 1.5. Let $(X, \mathcal{A}, \mu)$ be a measure space. Then one has the equalities

$$(20) \quad \mathfrak{L}_+^1(X, \mathcal{A}, \mu) = \{f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu) : f(X) \subset [0, \infty]\};$$

$$(21) \quad \mathfrak{L}_{\mathbb{K}, elem}^1(X, \mathcal{A}, \mu) = \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu) \cap \mathcal{A}\text{-Elem}_{\mathbb{K}}(X).$$

Indeed, by Theorem 1.3 that we have the inclusion

$$\mathfrak{L}_+^1(X, \mathcal{A}, \mu) \subset \{f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu) : f(X) \subset [0, \infty]\}.$$

The inclusion in the other direction follows again from Theorem 1.3, since any function that belongs to the right hand side of (20) satisfies $f = |f|$. The inclusion

$$\mathfrak{L}_{\mathbb{K}, elem}^1(X, \mathcal{A}, \mu) \subset \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu) \cap \mathcal{A}\text{-Elem}_{\mathbb{K}}(X)$$

is again contained in Theorem 1.3. To prove the inclusion in the other direction, it suffices to consider the case $\mathbb{K} = \mathbb{R}$. Start with $h \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu) \cap \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, which gives $|h| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. The function $|h|$ is obviously in $\mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, so we get $|h| \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$. Since $\mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$ is a vector space, it will also contain the function $-|h|$. The fact that h itself belongs to $\mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$ then follows from Proposition 1.1, combined with the obvious inequalities

$$-|h| \leq h \leq |h|.$$

The following result deals with the construction of the integral.

THEOREM 1.4. *Let $(X, \mathcal{A}, \mu)$ be a measure space. There exists a unique map $I_{\mathbb{R}}^{\mu}(X, \mathcal{A}, \mu) \rightarrow \mathbb{R}$, with the following properties:*

(i) *Whenever $f, g, h \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$ are such that*

$$h(x) = f(x) + g(x), \quad \forall x \in X \setminus [f^{-1}(\{-\infty, \infty\}) \cup g^{-1}(\{-\infty, \infty\})],$$

$$\text{it follows that } I_{\mathbb{R}}^{\mu}(h) = I_{\mathbb{R}}^{\mu}(f) + I_{\mathbb{R}}^{\mu}(g).$$

(ii) *Whenever $f, g \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$ and $\alpha \in \mathbb{R}$ are such that*

$$g(x) = \alpha f(x), \quad \forall x \in X \setminus f^{-1}(\{-\infty, \infty\}),$$

$$\text{it follows that } I_{\mathbb{R}}^{\mu}(g) = \alpha I_{\mathbb{R}}^{\mu}(f).$$

(iii) $I_{\mathbb{R}}^{\mu}(f) = I_+^{\mu}(f)$, $\forall f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$.

PROOF. Let us first show the existence. Start with some $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, and define the functions $f^{\pm} : X \rightarrow [0, \infty]$ by $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$ so that $f = f^+ - f^-$, and $f^+, f^- \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. We then define

$$I_{\mathbb{R}}^{\mu}(f) = I_+^{\mu}(f^+) - I_+^{\mu}(f^-).$$

It is obvious that $I_{\mathbb{R}}^{\mu}$ satisfies condition (iii).

The key fact that we need is contained in the following.

Claim: Whenever $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, and $f_1, f_2 \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$ are such that

$$f(x) = f^+(x) - f^-(x), \quad \forall x \in X. \setminus [^{-1}(\{\infty\}) \cup f_2^{-1}(\{\infty\})],$$

it follows that we have the equality

$$I_{\mathbb{R}}^{\mu}(f) = I_+^{\mu}(f_1) - I_+^{\mu}(f_2).$$

Indeed, since we have $f = f^+ - f^-$, it follows immediately that we have the equality

$$f_2(x) + f^+(x) = f_1(x) + f^-(x), \quad \forall x \in X. \setminus [f_1^{-1}(\{\infty\}) \cup f_2^{-1}(\{\infty\})],$$

which gives

$$f_2 + f^+ = f_1 + f^-, \quad \mu\text{-a.e.}$$

By Theorem 1.2, this immediately gives

$$I_+^{\mu}(f_2) + I_+^{\mu}(f^+) = I_+^{\mu}(f_1) + I_+^{\mu}(f^-),$$

which then gives

$$I_+^{\mu}(f_1) - I_+^{\mu}(f_2) = I_+^{\mu}(f^+) - I_+^{\mu}(f^-) = I_{\mathbb{R}}^{\mu}(f).$$

Having prove the above Claim, let us show now that $I_{\mathbb{R}}^{\mu}$ has properties (i) and (ii). Assume f, g and h are as in (i). Notice that if we define $h_1 = f^+ + g^+$ and $h_2 = f^- + g^-$, then we clearly have $0 \leq h_1 \leq |f| + |g|$ and $0 \leq h_2 \leq |f| + |g|$, so h_1 and h_2 both belong to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. By Theorem 1.2, we then have

$$(22) \quad I_+^{\mu}(h_1) = I_+^{\mu}(f^+) + I_+^{\mu}(g^+) \text{ and } I_+^{\mu}(h_2) = I_+^{\mu}(f^-) + I_+^{\mu}(g^-).$$

Notice also that, because of the equalities

$$h_1^{-1}(\{\infty\}) = f^{-1}(\{\infty\}) \cup g^{-1}(\{\infty\}) \text{ and } h_2^{-1}(\{\infty\}) = f^{-1}(\{-\infty\}) \cup g^{-1}(\{-\infty\}),$$

we have

$$h = h_1(x) - h_2(x), \quad \forall x \in X. \setminus [h_1^{-1}(\{\infty\}) \cup h_2^{-1}(\{\infty\})],$$

so by the above Claim, combined with (22), we get

$$I_{\mathbb{R}}^{\mu}(h) = I_+^{\mu}(h_1) - I_+^{\mu}(h_2) = I_+^{\mu}(f^+) + I_+^{\mu}(g^+) - I_+^{\mu}(f^-) - I_+^{\mu}(g^-) = I_{\mathbb{R}}^{\mu}(f) + I_{\mathbb{R}}^{\mu}(g).$$

Property (ii) is pretty obvious.

The uniqueness is also obvious. If we start with a map $J : \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu) \rightarrow \mathbb{R}$ with properties (i)-(iii), then for every $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, we must have

$$J(f) = J(f^+) - J(f^-) = I_+^{\mu}(f^+) - I_+^{\mu}(f^-).$$

(For the second equality we use condition (iii), combined with the fact that both f^+ and f^- belong to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$.) $\square$

COROLLARY 1.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. There exists a unique linear map $I_{\mathbb{K}}^{\mu}(X, \mathcal{A}, \mu) \rightarrow \mathbb{K}$, such that*

$$I_{\mathbb{K}}^{\mu}(f) = I_+^{\mu}(f), \quad \forall f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu) \cap \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu).$$

PROOF. Let us start with the case $\mathbb{K} = \mathbb{R}$. In this case, we have the inclusion

$$\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu) \subset \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu),$$

so we can define $I_{\mathbb{R}}^{\mu}$ as the restriction of $I_{\mathbb{R}}^{\mu}$ to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$. The uniqueness is again clear, because of the equalities

$$I_{\mathbb{R}}^{\mu}(f) = I_{\mathbb{R}}^{\mu}(f^+) - I_{\mathbb{R}}^{\mu}(f^-) = I_+^{\mu}(f^+) - I_+^{\mu}(f^-).$$

In the case $\mathbb{K} = \mathbb{C}$, we define

$$I_{\mathbb{C}}^{\mu}(f) = I_{\mathbb{R}}^{\mu}(\operatorname{Re} f) + iI_{\mathbb{R}}^{\mu}(\operatorname{Im} f).$$

The linearity is obvious. The uniqueness is also clear, because the restriction of $I_{\mathbb{C}}^{\mu}$ to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$ must agree with $I_{\mathbb{R}}^{\mu}$. $\square$

DEFINITION. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathbf{K}$ be one of the symbols $\mathbb{R}$, $\mathbb{R}$, or $\mathbb{C}$. For any $f \in \mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu)$, the number $I_{\mathbf{K}}^{\mu}(f)$ (which is real, if $\mathbf{K} = \mathbb{R}$ or $\mathbb{R}$, and is complex if $\mathbf{K} = \mathbb{C}$) will be denoted by

$$\int_X f d\mu,$$

and is called the μ -integral of f . This notation is unambiguous, because if $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, then we have $I_{\mathbb{R}}^{\mu}(f) = I_{\mathbb{C}}^{\mu}(f) = I_{\mathbb{R}}^{\mu}(f)$.

REMARK 1.6. If $(X, \mathcal{A}, \mu)$ is a measure space, then for every $A \in \mathcal{A}$, with $\mu(A) < \infty$, using the above Corollary, we get

$$\int_X \chi_A d\mu = I_{+}^{\mu}(\chi_A) = \mu(A).$$

By linearity, if $\mathbb{K} = \mathbb{R}, \mathbb{C}$, one has then the equality

$$\int_X h d\mu = I_{elem}^{\mu}(h), \quad \forall h \in \mathfrak{L}_{\mathbb{K}, elem}^1(X, \mathcal{A}, \mu).$$

To make the exposition a bit easier, it will adopt the following.

CONVENTION. If $(X, \mathcal{A}, \mu)$ is a measure space, and if $f : X \rightarrow [0, \infty]$ is a measurable function, which *does not belong to* $\mathfrak{L}_{+}^1(X, \mathcal{A}, \mu)$, then we *define*

$$\int_X f d\mu = \infty.$$

REMARKS 1.7. Let $(X, \mathcal{A}, \mu)$ be a measure space.

A. Using the above convention, when $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$ is a function with $h(X) \subset [0, \infty)$, the condition $\int_X h d\mu = \infty$ is equivalent to the existence of some $\alpha \in h(X) \setminus \{0\}$, with $\mu(h^{-1}(\{\alpha\})) = \infty$.

B. Using the above convention, for every measurable function $f : X \rightarrow [0, \infty]$, one has the equality

$$\int_X f d\mu = \sup \left\{ \int_X h d\mu : h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X), \quad 0 \leq h \leq f \right\}.$$

C. If $f, g : X \rightarrow [0, \infty]$ are measurable, then one has the equalities

$$\begin{aligned} \int_X (f + g) d\mu &= \int_X f d\mu + \int_X g d\mu, \\ \int_X (\alpha f) d\mu &= \alpha \int_X f d\mu, \quad \forall \alpha \in [0, \infty), \end{aligned}$$

even in the case when some term is infinite. (We use the convention $\infty + t = \infty$, $\forall t \in [0, \infty]$, as well as $\alpha \cdot \infty = \infty$, $\forall \alpha \in (0, \infty)$, and $0 \cdot \infty = 0$.)

D. If $f, g : X \rightarrow [0, \infty]$ are measurable, and $f \leq g$, μ -a.e., then (using B) one has the inequality

$$\int_X f d\mu \leq \int_X g d\mu,$$

even if one side (or both) is infinite.

E. Let $\mathbf{K}$ be one of the symbols $\bar{\mathbb{R}}$, $\mathbb{R}$, or $\mathbb{C}$, and let $f : X \rightarrow \mathbf{K}$ be a measurable function. Then the function $|f| : X \rightarrow [0, \infty]$ is measurable. Using the above convention, the condition that f belongs to $\mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu)$ is equivalent to the inequality $\int_X |f| d\mu < \infty$.

In the remainder of this section we discuss several properties of integration that are analogous to those of the positive/elementary integration.

We begin with a useful estimate

PROPOSITION 1.6. *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathbf{K}$ be one of the symbols $\bar{\mathbb{R}}$, $\mathbb{R}$, or $\mathbb{C}$. For every function $f \in \mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu)$, one has the inequality*

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu.$$

PROOF. Let us first examine the case when $\mathbf{K} = \bar{\mathbb{R}}, \mathbb{R}$. In this case we define $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$, so we have $f = f^+ - f^-$, as well as $|f| = f^+ + f^-$. Using the inequalities $I_+^\mu(f^\pm) \geq 0$, we have

$$\begin{aligned} \int_X f d\mu &= I_+^\mu(f^+) - I_+^\mu(f^-) \leq I_+^\mu(f^+) + I_+^\mu(f^-) = \int_X |f| d\mu; \\ -\int_X f d\mu &= -I_+^\mu(f^+) + I_+^\mu(f^-) \leq I_+^\mu(f^+) + I_+^\mu(f^-) = \int_X |f| d\mu. \end{aligned}$$

In other words, we have

$$\pm \int_X f d\mu \leq \int_X |f| d\mu,$$

and the desired inequality immediately follows.

Let us consider now the case $\mathbf{K} = \mathbb{C}$. Consider the number $\lambda = \int_X f d\mu$, and let us choose some complex number $\alpha \in \mathbb{C}$, with $|\alpha| = 1$, and $\alpha\lambda = |\lambda|$. (If $\lambda \neq 0$, we take $\alpha = \lambda^{-1}|\lambda|$; otherwise we take $\alpha = 1$.) Consider the measurable function $g = \alpha f$. Notice now that

$$\left(\int_X \operatorname{Re} g d\mu \right) + i \left(\int_X \operatorname{Im} g d\mu \right) = \int_X g d\mu = \alpha \int_X f d\mu = \alpha\lambda = |\lambda| \geq 0,$$

so in particular we get

$$|\lambda| = \int_X \operatorname{Re} g d\mu.$$

If we apply the real case, we then get

$$(23) \quad |\lambda| \leq \int_X |\operatorname{Re} g| d\mu.$$

Notice now that, we have the inequality $|\operatorname{Re} g| \leq |g| = |f|$, which gives

$$I_+^\mu(|\operatorname{Re} g|) \leq I_+^\mu(|f|) = \int_X |f| d\mu,$$

so the inequality (23) immediately gives

$$\left| \int_X f d\mu \right| = |\lambda| \leq \int_X |f| d\mu. \quad \square$$

COROLLARY 1.3. *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathbf{K}$ be one of the symbols $\bar{\mathbb{R}}, \mathbb{R}$, or $\mathbb{C}$. If a measurable function $f : X \rightarrow \mathbf{K}$ satisfies $f = 0$, μ -a.e., then $f \in \mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu)$, and $\int_X f d\mu = 0$.*

PROOF. Consider the measurable function $|f| : X \rightarrow [0, \infty]$, which satisfies $|f| = 0$, μ -a.e. By Proposition 1.3, it follows that $|f| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, hence $f \in \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, and $\int_X |f| d\mu = 0$. Of course, the last equality forces $\int_X f d\mu = 0$. $\square$

COROLLARY 1.4. *Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. If $(X, \mathcal{A}, \mu)$ is a finite measure space, then every bounded measurable function $f : X \rightarrow \mathbb{K}$ belongs to $\mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, and satisfies*

$$\left| \int_X f d\mu \right| \leq \mu(X) \cdot \sup_{x \in X} |f(x)|.$$

PROOF. If we put $\beta = \sup_{x \in X} |f(x)|$, then we clearly have $|f| \leq \beta \chi_X$, which shows that $|f| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and also gives $\int_X |f| d\mu \leq \int_X \beta \chi_X d\mu = \mu(X) \cdot \beta$. Then everything follows from Proposition 1.6. $\square$

COMMENT. The introduction of the space $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, of extended real-valued μ -integrable functions, is useful mostly for technical reasons. In effect, everything can be reduced to the case when only “honest” real-valued functions are involved. The following result clarifies this matter.

LEMMA 1.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f : X \rightarrow \overline{\mathbb{R}}$ be a measurable function. The following are equivalent*

- (i) $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$;
- (ii) *there exists $g \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, such that $g = f$, μ -a.e.*

Moreover, if f satisfies these equivalent conditions, then any function g , satisfying (ii), also has the property

$$\int_X f d\mu = \int_X g d\mu.$$

PROOF. Consider the set $F = \{x \in X : -\infty < f(x) < \infty\}$, which belongs to $\mathcal{A}$. We obviously have the equality $X \setminus F = |f|^{-1}(\{\infty\})$.

(i) $\Rightarrow$ (ii). Assume $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, which means that $|f| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. In particular, we get $\mu(X \setminus F) = 0$. Define the measurable function $g = f \chi_F$. On the one hand, it is clear, by construction, that we have $-\infty < g(x) < \infty$, $\forall x \in X$. On the other hand, it is clear that $g|_F = f|_F$, so using $\mu(X \setminus F) = 0$, we get the fact that $f = g$, μ -a.e. Finally, the inequality $0 \leq |g| \leq |f|$, combined with Proposition 1.3, gives $|g| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, so g indeed belongs to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$.

(ii) $\Rightarrow$ (i). Suppose there exists $g \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, with $f = g$, μ -a.e., and let us prove that

- (a) $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$;
- (b) $\int_X f d\mu = \int_X g d\mu$.

The first assertion is clear, because by Proposition 1.3, the equality $|f| = |g|$, μ -a.e., combined with $|g| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, forces $|f| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, i.e. $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$. To prove (b), we consider the difference $h = f - g$, which is a measurable function $h : X \rightarrow \overline{\mathbb{R}}$, and satisfies $h = 0$, μ -a.e. By Corollary 1.3, we know that $h \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, and $\int_X h d\mu = 0$. By Theorem 1.3, we get

$$\int_X f d\mu = \int_X g d\mu + \int_X h d\mu = \int_X g d\mu. \quad \square$$

The following result is an analogue of Proposition 1.1 (see also Proposition 1.3).

PROPOSITION 1.7. *Let $(X, \mathcal{A}, \mu)$, and let $f_1, f_2 \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$. Suppose $f : X \rightarrow \mathbb{R}$ is a measurable function, such that $f_1 \leq f \leq f_2$, μ -a.e. Then $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, and one has the inequality*

$$\int_X f_1 d\mu \leq \int_X f d\mu \leq \int_X f_2 d\mu.$$

PROOF. First of all, since f_1 and f_2 belong to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, it follows that $|f_1|$ and $|f_2|$, hence also $|f_1| + |f_2|$, belong to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$. Second, since we have $f_2 \leq |f_2| \leq |f_1| + |f_2|$ and $f_1 \geq -|f_1| \geq -|f_1| - |f_2|$ (everywhere!), the inequalities $f_1 \leq f \leq f_2$, μ -a.e., give

$$-|f_1| - |f_2| \leq f \leq |f_1| + |f_2|, \quad \mu\text{-a.e.},$$

which reads

$$|f| \leq |f_1| + |f_2|, \quad \mu\text{-a.e.}$$

Since $|f_1| + |f_2| \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, by Proposition 1.3., we get $|f| \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, so f indeed belongs to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$.

To prove the inequality for integrals, we use Lemma 1.2, to find functions $g_1, g_2, g \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, such that $f_1 = g_1$, μ -a.e., $f_2 = g_2$, μ -a.e., and $f = g$, μ -a.e. Lemma 1.2 also gives the equalities $\int_X f_1 d\mu = \int_X g_1 d\mu$, $\int_X f_2 d\mu = \int_X g_2 d\mu$, and $\int_X f d\mu = \int_X g d\mu$, so what we need to prove are the inequalities

$$(24) \quad \int_X g_1 d\mu \leq \int_X g d\mu \leq \int_X g_2 d\mu.$$

Of course, we have

$$g_1 \leq g \leq g_2, \quad \mu\text{-a.e.}$$

To prove the first inequality in (24), we consider the function $h = g - g_1 \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, and we prove that $\int_X h d\mu \geq 0$. But this is quite clear, because we have $h \geq 0$, μ -a.e., which means that $h = |h|$, μ -a.e., so by Lemma 1.2, we get

$$\int_X h d\mu = \int_X |h| d\mu = I_+^\mu(|h|) \geq 0.$$

The second inequality in (24) is proved the exact same way. $\square$

The next result is an analogue of Proposition 1.4.

PROPOSITION 1.8. *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathbf{K}$ be one of the symbols $\mathbb{R}$, $\mathbb{R}$, or $\mathbb{C}$. Suppose $(A_k)_{k=1}^n \subset \mathcal{A}$ is a pairwise disjoint finite sequence, with $A_1 \cup \cdots \cup A_n = X$. For a measurable function $f : X \rightarrow \mathbf{K}$, the following are equivalent.*

- (i) $f \in \mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu)$;
- (ii) $f \chi_{A_k} \in \mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu)$, $\forall k = 1, \dots, n$.

Moreover, if f satisfies these equivalent conditions, one has

$$(25) \quad \int_X f d\mu = \sum_{k=1}^n \int_X f \chi_{A_k} d\mu.$$

PROOF. It is fairly obvious that $|f \chi_{A_k}| = |f| \chi_{A_k}$. Then the equivalence (i) $\Leftrightarrow$ (ii) follows from Proposition 1.4 applied to the function $|f| : X \rightarrow [0, \infty]$. In the cases when $\mathbf{K} = \mathbb{R}, \mathbb{C}$, the equality (25) follows immediately from linearity, and the obvious equality $f = \sum_{k=1}^n f \chi_{A_k}$. In the case when $\mathbf{K} = \mathbb{R}$, we take $g \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, such that $f = g$, μ -a.e. Then we obviously have $f \chi_{A_k} = g \chi_{A_k}$,

μ -a.e., for all $k = 1, \dots, n$, and the equality (25) follows from the corresponding equality that holds for g . $\square$

REMARK 1.8. The equality (25) also holds for arbitrary measurable functions $f : X \rightarrow [0, \infty]$, if we use the convention that preceded Remarks 1.7. This is an immediate consequence of Proposition 1.4, because the left hand side is infinite, if and only if one of the terms in the right hand side is infinite.

The following is an obvious extension of Remark 1.4.

REMARK 1.9. Let $\mathbf{K}$ be one of the symbols $\bar{\mathbb{R}}$, $\mathbb{R}$, or $\mathbb{C}$, let $(X, \mathcal{A}, \mu)$ be a measure space. For a set $S \in \mathcal{A}$, and a measurable function $f : X \rightarrow \mathbf{K}$, one has the equivalence

$$f \chi_S \in \mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu) \iff f|_S \in \mathfrak{L}_{\mathbf{K}}^1(S, \mathcal{A}|_S, \mu|_S).$$

If this is the case, one has the equality

$$(26) \quad \int_X f \chi_S d\mu = \int_S f|_S d\mu|_S.$$

The above equality also holds for arbitrary measurable functions $f : X \rightarrow [0, \infty]$, again using the convention that preceded Remarks 1.7.

NOTATION. The above remark states that, whenever the quantities in (26) are defined, they are equal. (This only requires the fact that $f|_S$ is measurable, and either $f|_S \in \mathfrak{L}_{\mathbf{K}}^1(S, \mathcal{A}|_S, \mu|_S)$, or $f(S) \subset [0, \infty]$.) In this case, the equal quantities in (26) will be simply denoted by $\int_S f d\mu$.

Exercise 1. Let I be some non-empty set. Consider the σ -algebra $\mathcal{P}(I)$, of all subsets of I , equipped with the counting measure

$$\mu(A) = \begin{cases} \text{Card } A & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite} \end{cases}$$

Prove that $\mathfrak{L}_{\bar{\mathbb{R}}}^1(I, \mathcal{P}(I), \mu) = \mathfrak{L}_{\bar{\mathbb{R}}}^1(I, \mathcal{P}(I), \mu)$. Prove that, if $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, then

$$\mathfrak{L}_{\mathbb{K}}^1(I, \mathcal{P}(I), \mu) = \ell_{\mathbb{K}}^1(I),$$

the Banach space discussed in II.2 and II.3.

Exercise 2. There is an instance when the entire theory developed here is essentially vacuous. Let X be a non-empty set, and let $\mathcal{A}$ be a σ -algebra on X . For a measure μ on $\mathcal{A}$, prove that the following are equivalent

- (i) $\mathfrak{L}_+^1(X, \mathcal{A}, \mu) = \{f : X \rightarrow [0, \infty] : f \text{ measurable, and } f = 0, \mu\text{-a.e.}\};$
- (ii) for every $A \in \mathcal{A}$, one has $\mu(A) \in \{0, \infty\}$.

A measure space $(X, \mathcal{A}, \mu)$, with property (ii), is said to be *degenerate*.

Exercise 3 $^\diamond$. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f : X \rightarrow [0, \infty]$ be a measurable function, with $\int_X f d\mu = 0$. Prove that $f = 0, \mu$ -a.e.

HINT: Define the measurable sets $A_n = \{x \in X : f(x) \geq \frac{1}{n}\}$, and analyze the relationship between f and χ_{A_n} .

LECTURE 34

2. Convergence theorems

In this section we analyze the dynamics of integrability in the case when sequences of measurable functions are considered. Roughly speaking, a “convergence theorem” states that integrability is preserved under taking limits. In other words, if one has a sequence $(f_n)_{n=1}^{\infty}$ of integrable functions, and if f is some kind of a limit of the f_n ’s, then we would like to conclude that f itself is integrable, as well as the equality $\int f = \lim_{n \rightarrow \infty} \int f_n$.

Such results are often employed in two instances:

- A. *When we want to prove that some function f is integrable.* In this case we would look for a sequence $(f_n)_{n=1}^{\infty}$ of integrable approximants for f .
- B. *When we want to construct an integrable function.* In this case, we will produce first the approximants, and then we will examine the existence of the limit.

The first convergence result, which is somehow primitive, but very useful, is the following.

LEMMA 2.1. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space, let $a \in (0, \infty)$ and let $f_n : X \rightarrow [0, a]$, $n \geq 1$, be a sequence of measurable functions satisfying*

- (a) $f_1 \geq f_2 \geq \dots \geq 0$;
- (b) $\lim_{n \rightarrow \infty} f_n(x) = 0$, $\forall x \in X$.

Then one has the equality

$$(1) \quad \lim_{n \rightarrow \infty} \int_X f_n d\mu = 0.$$

PROOF. Let us define, for each $\varepsilon > 0$, and each integer $n \geq 1$, the set

$$A_n^\varepsilon = \{x \in X : f_n(x) \geq \varepsilon\}.$$

Obviously, we have $A_n^\varepsilon \in \mathcal{A}$, $\forall \varepsilon > 0$, $n \geq 1$. One key fact we are going to use is the following.

Claim 1: For every $\varepsilon > 0$, one has the equality

$$\lim_{n \rightarrow \infty} \mu(A_n^\varepsilon) = 0.$$

Fix $\varepsilon > 0$. Let us first observe that, using (a), we have the inclusions

$$(2) \quad A_1^\varepsilon \supset A_2^\varepsilon \supset \dots$$

Second, using (b), we clearly have the equality $\bigcap_{k=1}^{\infty} A_k^{\varepsilon} = \emptyset$. Since μ is finite, using the Continuity Property (Lemma III.4.1), we have

$$\lim_{n \rightarrow \infty} \mu(A_n^{\varepsilon}) = \mu\left(\bigcap_{n=1}^{\infty} A_n^{\varepsilon}\right) = \mu(\emptyset) = 0.$$

Claim 2: For every $\varepsilon > 0$ and every integer $n \geq 1$, one has the inequality

$$0 \leq \int_X f_n d\mu \leq a\mu(A_n^{\varepsilon}) + \varepsilon\mu(X).$$

Fix ε and n , and let us consider the elementary function

$$h_n^{\varepsilon} = a\chi_{A_n^{\varepsilon}} + \varepsilon\chi_{B_n^{\varepsilon}},$$

where $B_n^{\varepsilon} = X \setminus A^{\varepsilon}$. Obviously, since $\mu(X) < \infty$, the function h_n^{ε} is elementary integrable. By construction, we clearly have $0 \leq f_n \leq h_n^{\varepsilon}$, so using the properties of integration, we get

$$0 \leq \int_X f_n d\mu \leq \int_X h_n^{\varepsilon} d\mu = a\mu(A_n^{\varepsilon}) + \varepsilon\mu(B^{\varepsilon}) \leq a\mu(A^{\varepsilon}) + \varepsilon\mu(X).$$

Using Claims 1 and 2, it follows immediately that

$$0 \leq \liminf_{n \rightarrow \infty} \int_X f_n d\mu \leq \limsup_{n \rightarrow \infty} \int_X f_n d\mu \leq \varepsilon\mu(X).$$

Since the last inequality holds for arbitrary $\varepsilon > 0$, the desired equality (1) immediately follows. $\square$

We now turn our attention to a weaker notion of limit, for sequences of measurable functions.

DEFINITION. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbf{K}$ be one of the symbols $\mathbb{R}$, $\mathbb{R}$, or $\mathbb{C}$. Suppose $f_n : X \rightarrow \mathbf{K}$, $n \geq 1$, are measurable functions. Given a measurable function $f : X \rightarrow \mathbf{K}$, we say that the sequence $(f_n)_{n=1}^{\infty}$ *converges μ -almost everywhere to f* , if there exists some set $N \in \mathcal{A}$, with $\mu(N) = 0$, such that

$$\lim_{n \rightarrow \infty} f_n(x) = f(x), \quad \forall x \in X \setminus N.$$

In this case we write

$$f = \mu\text{-a.e.-}\lim_{n \rightarrow \infty} f_n.$$

REMARK 2.1. This notion of convergence has, among other things, a certain uniqueness feature. One way to describe this is to say that the limit of a μ -a.e. convergent sequence is μ -almost unique, in the sense that if f and g are measurable functions which satisfy the equalities $f = \mu\text{-a.e.-}\lim_{n \rightarrow \infty} f_n$ and $g = \mu\text{-a.e.-}\lim_{n \rightarrow \infty} f_n$, then $f = g$, μ -a.e. This is quite obvious, because there exist sets $M, N \in \mathcal{A}$, with $\mu(M) = \mu(N) = 0$, such that

$$\begin{aligned} \lim_{n \rightarrow \infty} f_n(x) &= f(x), \quad \forall x \in X \setminus M, \\ \lim_{n \rightarrow \infty} f_n(x) &= g(x), \quad \forall x \in X \setminus N, \end{aligned}$$

then it is obvious that $\mu(M \cup N) = 0$, and

$$f(x) = g(x), \quad \forall x \in X \setminus [M \cup N].$$

COMMENT. The above definition makes sense if $\mathbf{K}$ is an arbitrary metric space. Any of the spaces $\mathbb{R}$, $\mathbb{R}$, and $\mathbb{C}$ is in fact a *complete* metric space. There are instances where the requirement that f is measurable is in fact redundant. This is somehow clarified by the the next two exercises.

*Exercise 1**. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbf{K}$ be a complete separable metric space, and let $f_n : X \rightarrow \mathbf{K}$, $n \geq 1$, be measurable functions.

(i) Prove that the set

$$L = \{x \in X : (f_n(x))_{n=1}^{\infty} \subset \mathbf{K} \text{ is convergent} \}$$

belongs to $\mathcal{A}$.

(ii) If we fix some point $\alpha \in \mathbf{K}$, and we define $\ell : X \rightarrow \mathbf{K}$ by

$$\ell(x) = \begin{cases} \lim_{n \rightarrow \infty} f_n(x) & \text{if } x \in L \\ \alpha & \text{if } x \in X \setminus L \end{cases}$$

then ℓ is measurable.

In particular, if $\mu(X \setminus L) = 0$, then $\ell = \mu$ -a.e.- $\lim_{n \rightarrow \infty} f_n$.

HINTS: If d denotes the metric on $\mathbf{K}$, then prove first that, for every $\varepsilon > 0$ and every $m, n \geq 1$, the set

$$D_{mn}^{\varepsilon} = \{x \in X : d(f_m(x), f_n(x)) < \varepsilon\}$$

belongs to $\mathcal{A}$ (use the results from III.3). Based on this fact, prove that, for every $p, k \geq 1$, the set

$$E_k^p = \{x \in X : d(f_m(x), f_n(x)) < \frac{1}{p}, \forall m, n \geq k\}$$

belongs to $\mathcal{A}$. Finally, use completeness to prove that

$$L = \bigcap_{p=1}^{\infty} \left(\bigcup_{k=1}^{\infty} E_k^p \right).$$

Exercise 2. Use the setting from Exercise 1. Prove that Let $(X, \mathcal{A}, \mu)$, $\mathbf{K}$, and $(f_n)_{n=1}^{\infty}$ be as in Exercise 1. Assume $f : X \rightarrow \mathbf{K}$ is an *arbitrary* function, for which there exists some set $N \in \mathcal{A}$ with $\mu(N) = 0$, and

$$\lim_{n \rightarrow \infty} f_n(x) = f(x), \quad \forall x \in X \setminus N.$$

Prove that, when μ is a *complete* measure on $\mathcal{A}$ (see III.5), the function f is automatically measurable.

HINT: Use the results from Exercise 1. We have $X \setminus N \subset L$, and $f(x) = \ell(x)$, $\forall x \in X \setminus N$. Prove that, for a Borel set $B \subset \mathbf{K}$, one has the equality $f^{-1}(B) = \ell^{-1}(B) \Delta M$, for some $M \subset N$. By completeness, we have $M \in \mathcal{A}$, so $f^{-1}(B) \in \mathcal{A}$.

The following fundamental result is a generalization of Lemma 2.1.

THEOREM 2.1 (Lebesgue Monotone Convergence Theorem). *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $(f_n)_{n=1}^{\infty} \subset \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$ be a sequence with:*

- $f_n \leq f_{n+1}$, μ -a.e., $\forall n \geq 1$;
- $\sup \left\{ \int_X f_n d\mu : n \geq 1 \right\} < \infty$.

Assume $f : X \rightarrow [0, \infty]$ is a measurable function, with $f = \mu$ -a.e.- $\lim_{n \rightarrow \infty} f_n$. Then $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and $\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu$.

PROOF. Define $\alpha_n = \int_X f_n d\mu$, $n \geq 1$. First of all, we clearly have

$$0 \leq \alpha_1 \leq \alpha_2 \leq \dots,$$

so the sequence $(\alpha_n)_{n=1}^\infty$ has a limit $\alpha = \lim_{n \rightarrow \infty} \alpha_n$, and we have in fact the equality

$$\alpha = \sup \{ \alpha_n : n \geq 1 \} < \infty.$$

With these notations, all we need to prove is the fact that $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and that we have

$$(3) \quad \int_X f d\mu = \alpha.$$

Fix a set $M \in \mathcal{A}$, with $\mu(M) = 0$, and such that $\lim_{n \rightarrow \infty} f_n(x) = f(x)$, $\forall x \in X \setminus M$. For each n , we define the set

$$M_n = \{x \in X : f_n(x) > f_{n+1}(x)\}.$$

Obviously $M_n \in \mathcal{A}$, and by assumption, we have $\mu(M_n) = 0$, $\forall n \geq 1$. Define the set $N = M \cup (\bigcup_{n=1}^\infty M_n)$. It is clear that $\mu(N) = 0$, and

- $0 \leq f_1(x) \leq f_2(x) \leq \dots \leq f(x)$, $\forall x \in X \setminus N$;
- $f(x) = \lim_{n \rightarrow \infty} f_n(x)$, $\forall x \in X \setminus N$.

So if we put $A = X \setminus N$, and if we define the measurable functions $g_n = f_n \chi_A$, $n \geq 1$, and $g = f \chi_A$, then we have

- (a) $0 \leq g_1 \leq g_2 \leq \dots \leq g$ (everywhere!);
- (b) $\lim_{n \rightarrow \infty} g_n(x) = g(x)$, $\forall x \in X$;
- (c) $g_n = f_n$, μ -a.e., $\forall n \geq 1$;
- (d) $g = f$, μ -a.e.;

Notice that property (c) gives $g_n \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$ and $\int_X g_n d\mu = \alpha_n$, $\forall n \geq 1$. By property (d), we see that we have the equivalence

$$f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu) \iff f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu).$$

Moreover, if $g \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, then we will have $\int_X g d\mu = \int_X f d\mu$. These observations show that it suffices to prove the theorem with g 's in place of the f 's. The advantage is now the fact that we have the slightly stronger properties (a) and (b) above. The first step in the proof is the following.

Claim 1: For every $t \in (0, \infty)$, one has the inequality $\mu(g^{-1}((t, \infty))) \leq \frac{\alpha}{t}$.

Denote the set $g^{-1}((t, \infty))$ simply by A_t . For each $n \geq 1$, we also define the set $A_t^n = g_n^{-1}((t, \infty))$. Using property (a) above, it is clear that we have the inclusions

$$(4) \quad A_1^t \subset A_t^2 \subset \dots \subset A_t.$$

Using property (b) above, we also have the equality $A_t = \bigcup_{n=1}^\infty A_t^n$. Using the continuity Lemma 4.1, we then have

$$\mu(A_t) = \lim_{n \rightarrow \infty} \mu(A_t^n),$$

so in order to prove the Claim, it suffices to prove the inequalities

$$(5) \quad \mu(A_t^n) \leq \frac{\alpha_n}{t}, \quad \forall n \geq 1.$$

But the above inequality is pretty obvious, since we clearly have $0 \leq t \chi_{A_t^n} \leq g_n$, which gives

$$t \mu(A_t^n) = \int_X t \chi_{A_t^n} d\mu \leq \int_X g_n d\mu = \alpha_n.$$

Claim 2: For any elementary function $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with $0 \leq h \leq g$, one has

- (i) $h \in \mathfrak{L}_{\mathbb{R},elem}^1(X, \mathcal{A}, \mu)$;
- (ii) $\int_X h d\mu \leq \alpha$.

Start with some elementary function h , with $0 \leq h \leq g$. Assume h is not identically zero, so we can write it as

$$h = \beta_1 \chi_{B_1} + \cdots + \beta_p \chi_{B_p},$$

with $(B_j)_{j=1}^p \subset \mathcal{A}$ pairwise disjoint, and $0 < \beta_1 < \cdots < \beta_p$. Define the set $B = B_1 \cup \cdots \cup B_p$. It is obvious that, if we put $t = \beta_1/2$, we have the inclusion $B \subset g^{-1}((t, \infty])$, so by Claim 1, we get $\mu(B) < \infty$. This gives, of course $\mu(B_j) < \infty$, $\forall j = 1, \dots, p$, so h is indeed elementary integrable. To prove the estimate (ii), we define the measurable functions $h_n : X \rightarrow [0, \infty]$ by $h_n = \min\{g_n, h\}$, $\forall n \geq 1$. Since $0 \leq h_n \leq g_n$, $\forall n \geq 1$, it follows that, $h_n \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, $\forall n \geq 1$, and we have the inequalities

$$(6) \quad \int_X h_n d\mu \leq \int_X g_n d\mu = \alpha_n, \quad \forall n \geq 1.$$

It is obvious that we have

- (*) $0 \leq h_1 \leq h_2 \leq \cdots \leq h \leq \beta_p \chi_B$ (everywhere);
- (**) $h(x) = \lim_{n \rightarrow \infty} h_n(x)$, $\forall x \in X$.

Let us restrict everything to B . We consider the σ -algebra $\mathcal{B} = \mathcal{A}|_B$, and the measure $\nu = \mu|_B$. Consider the elementary function $\psi = h|_B \in \mathcal{B}\text{-Elem}_{\mathbb{R}}(B)$, as well as the measurable functions $\psi_n = h_n|_B : B \rightarrow [0, \infty]$, $n \geq 1$. It is clear that $\psi \in \mathfrak{L}_{\mathbb{R},elem}^1(B, \mathcal{B}, \nu)$, and we have the equality

$$(7) \quad \int_B \psi d\nu = \int_X h d\mu.$$

Likewise, using (*), which clearly forces $h_n|_{X \setminus B} = 0$, it follows that, for each $n \geq 1$, the function ψ_n belongs to $\mathfrak{L}_+^1(B, \mathcal{B}, \nu)$, and by (6), we have

$$(8) \quad \int_B \psi_n d\nu = \int_X h_n d\mu, \quad \forall n \geq 1.$$

Let us analyze the differences $\varphi_n = \psi - \psi_n$. On the one hand, using (*), we have $\varphi_n(x) \in [0, \beta_p]$, $\forall x \in B$, $n \geq 1$. On the other hand, again by (*), we have $\varphi_1 \geq \varphi_2 \geq \cdots$. Finally, by (**) we have $\lim_{n \rightarrow \infty} \varphi_n(x) = 0$, $\forall x \in B$. We can apply Lemma 2.1, and we will get $\lim_{n \rightarrow \infty} \int_B \varphi_n d\nu = 0$. This clearly gives,

$$\int_B \psi d\nu = \lim_{n \rightarrow \infty} \int_B \psi_n d\nu,$$

and then using (7) and (8), we get the equality

$$\int_X h d\mu = \lim_{n \rightarrow \infty} \int_X h_n d\mu.$$

Combining this with (6), immediately gives the desired estimate $\int_X h d\mu \leq \alpha$.

Having proven Claim 2, let us observe now that, using the definition of the positive integral, it follows immediately that $g \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and we have the inequality

$$\int_X g d\mu \leq \alpha.$$

The other inequality is pretty obvious, because the inequality $g \geq g_n$ forces

$$\int_X g \, d\mu \geq \int_X g_n \, d\mu = \alpha_n, \quad \forall n \geq 1,$$

so we immediately get

$$\int_X g \, d\mu \geq \sup\{\alpha_n; n \geq 1\} = \alpha. \quad \square$$

COMMENT. In the previous section we introduced the convention which defines $\int_X f \, d\mu = \infty$, if $f : X \rightarrow [0, \infty]$ is measurable, but $f \notin \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. Using this convention, the Lebesgue Monotone Convergence Theorem has the following general version.

THEOREM 2.2 (General Lebesgue Monotone Convergence Theorem). *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f, f_n : X \rightarrow [0, \infty]$, $n \geq 1$, be measurable functions, such that*

- $f_n \leq f_{n+1}$, μ -a.e., $\forall n \geq 1$;
- $f = \mu$ -a.e.- $\lim_{n \rightarrow \infty} f_n$.

Then

$$(9) \quad \int_X f \, d\mu = \lim_{n \rightarrow \infty} \int_X f_n \, d\mu.$$

PROOF. As before, the sequence $(\alpha_n)_{n=1}^\infty \subset [0, \infty]$, defined by $\alpha_n = \int_X f_n \, d\mu$, $\forall n \geq 1$, is non-decreasing, and it has a limit

$$\alpha = \lim_{n \rightarrow \infty} \alpha_n = \sup \left\{ \int_X f_n \, d\mu : n \geq 1 \right\} \in [0, \infty].$$

There are two cases to analyze.

Case I: $\alpha = \infty$.

In this case the inequalities $f \geq f_n \geq 0$, μ -a.e. will force

$$\int_X f \, d\mu \geq \int_X f_n \, d\mu = \alpha_n, \quad \forall n \geq 1,$$

which will force $\int_X f \, d\mu \geq \alpha$, so we indeed get

$$\int_X f \, d\mu = \infty = \alpha.$$

Case II: $\alpha < \infty$.

In this case we apply directly Theorem 2.1. $\square$

The following result provides an equivalent definition of integrability for non-negative functions (compare to the construction in Section 1).

COROLLARY 2.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f : X \rightarrow [0, \infty]$ be a measurable function. The following are equivalent:*

- (i) $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$;
- (ii) *there exists a sequence $(h_n)_{n=1}^\infty \subset \mathfrak{L}_{\mathbb{R}, \text{elem}}^1(X, \mathcal{A}, \mu)$, with*
 - $0 \leq h_1 \leq h_2 \leq \dots$;
 - $\lim_{n \rightarrow \infty} h_n(x) = f(x)$, $\forall x \in X$;
 - $\sup \left\{ \int_X h_n \, d\mu : n \geq 1 \right\} < \infty$.

Moreover, if $(h_n)_{n=1}^\infty$ is as in (ii), then one has the equality

$$(10) \quad \int_X f d\mu = \lim_{n \rightarrow \infty} \int_X h_n d\mu.$$

PROOF. (i) $\Rightarrow$ (ii). Assume $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. Using Theorem III.3.2, we know there exists a sequence $(h_n)_{n=1}^\infty \subset \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with

- (a) $0 \leq h_1 \leq h_2 \leq \dots \leq f$;
- (b) $\lim_{n \rightarrow \infty} h_n(x) = f(x), \forall x \in X$.

Note the (a) forces $h_n \in \mathfrak{L}_{\mathbb{R}, \text{elem}}^1(X, \mathcal{A}, \mu)$, as well as the inequalities $\int_X h_n d\mu \leq \int_X f d\mu < \infty, \forall n \geq 1$, so the sequence $(h_n)_{n=1}^\infty$ clearly satisfies condition (ii).

The implication (ii) $\Rightarrow$ (i), and the equality (10) immediately follow from the General Lebesgue Monotone Convergence Theorem. $\square$

COROLLARY 2.2 (Fatou Lemma). *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f_n : X \rightarrow [0, \infty], n \geq 1$, be a sequence of measurable functions. Define the function $f : X \rightarrow [0, \infty]$ by*

$$f(x) = \liminf_{n \rightarrow \infty} f_n(x), \quad \forall x \in X.$$

Then f is measurable, and one has the inequality

$$\int_X f d\mu \leq \liminf_{n \rightarrow \infty} \int_X f_n d\mu.$$

PROOF. The fact that f is measurable is already known (see Corollary III.3.5). Define the sequence $(\alpha_n)_{n=1}^\infty \subset [0, \infty]$ by $\alpha_n = \int_X f_n d\mu, \forall n \geq 1$.

Define, for each integer $n \geq 1$, the function $g_n : X \rightarrow [0, \infty]$ by

$$g_n(x) = \inf \{f_k(x) : k \geq n\}, \quad \forall x \in X.$$

By Corollary III.3.4, we know that $g_n, n \geq 1$ are all measurable. Moreover, it is clear that

- $0 \leq g_1 \leq g_2 \leq \dots$;
- $f(x) = \lim_{n \rightarrow \infty} g_n(x), \forall x \in X$.

By the General Lebesgue Monotone Convergence Theorem 2.2, it follows that

$$(11) \quad \int_X f d\mu = \lim_{n \rightarrow \infty} \int_X g_n d\mu.$$

Notice that, if we define the sequence $(\beta_n)_{n=1}^\infty \subset [0, \infty]$, by $\beta_n = \int_X g_n d\mu, \forall n \geq 1$, then the obvious inequalities $0 \leq g_n \leq f_n$ give $\int_X g d\mu \leq \int_X f_n d\mu$, so we get

$$\beta_n \leq \alpha_n, \quad \forall n \geq 1.$$

Using (11), we then get

$$\int_X f d\mu = \lim_{n \rightarrow \infty} \beta_n = \liminf_{n \rightarrow \infty} \beta_n \leq \liminf_{n \rightarrow \infty} \alpha_n. \quad \square$$

The following is an important application of Theorem 2.1, that deals with Riemann integration.

COROLLARY 2.3. *Let $a < b$ be real numbers. Denote by λ the Lebesgue measure, and consider the Lebesgue space $([a, b], \mathfrak{M}_\lambda([a, b]), \lambda)$, where $\mathfrak{M}_\lambda([a, b])$ denotes the*

σ -algebra of all Lebesgue measurable subsets of $[a, b]$. Then every Riemann integrable function $f : [a, b] \rightarrow \mathbb{R}$ belongs to $\mathfrak{L}_{\mathbb{R}}^1([a, b], \mathfrak{M}_{\lambda}([a, b]), \lambda)$, and one has the equality

$$(12) \quad \int_{[a, b]} f d\lambda = \int_a^b f(x) dx.$$

PROOF. We are going to use the results from III.6. First of all, the fact that f is Lebesgue integrable, i.e. f belongs to $\mathfrak{L}_{\mathbb{R}}^1([a, b], \mathfrak{M}_{\lambda}([a, b]), \lambda)$, is clear since f is Lebesgue measurable, and bounded. (Here we use the fact that the measure space $([a, b], \mathfrak{M}_{\lambda}([a, b]), \lambda)$ is finite.)

Next we prove the equality between the Riemann integral and the Lebesgue integral. Adding a constant, if necessary, we can assume that $f \geq 0$. For every partition $\Delta = (a = x_0 < x_1 < \dots < x_n = b)$ of $[a, b]$, we define the numbers

$$m_k = \inf_{t \in [x_{k-1}, x_k]} f(t), \quad \forall k = 1, \dots, n,$$

and we define the function

$$f_{\Delta} = m_1 \chi_{[x_0, x_1]} + m_2 \chi_{(x_1, x_2]} + \dots + m_n \chi_{(x_{n-1}, x_n]}.$$

Fix a sequence of partitions $(\Delta_p)_{p=1}^{\infty}$, with $\Delta_1 \subset \Delta_2 \subset \dots$, and $\lim_{p \rightarrow \infty} |\Delta_p| = 0$. We know (see III.6) that we have

$$f = \lambda\text{-a.e.} \lim_{p \rightarrow \infty} f_{\Delta_p}.$$

Clearly we have $0 \leq f_{\Delta_1} \leq f_{\Delta_2} \leq \dots \leq f$, so by Theorem 2.1, we get

$$(13) \quad \int_{[a, b]} f d\lambda = \lim_{p \rightarrow \infty} \int_{[a, b]} f_{\Delta_p} d\lambda.$$

Notice however that

$$\int_{[a, b]} f_{\Delta_p} d\lambda = L(f, \Delta_p), \quad \forall p \geq 1,$$

where $L(f, \Delta_p)$ denotes the lower Darboux sum. Combining this with (13), and with the well known properties of Riemann integration, we immediately get (12). $\square$

The following is another important convergence theorem.

THEOREM 2.3 (Lebesgue Dominated Convergence Theorem). *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbf{K}$ be one of the symbols $\mathbb{R}$, $\mathbb{R}$, or $\mathbb{C}$, and let $(f_n)_{n=1}^{\infty} \subset \mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu)$. Assume $f : X \rightarrow \mathbf{K}$ is a measurable function, such that*

- (i) $f = \mu\text{-a.e.} \lim_{n \rightarrow \infty} f_n$;
- (ii) *there exists some function $g \in \mathfrak{L}_{+}^1(X, \mathcal{A}, \mu)$, such that*

$$|f_n| \leq g, \quad \mu\text{-a.e.}, \quad \forall n \geq 1.$$

Then $f \in \mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu)$, and one has the equality

$$(14) \quad \int_X f d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu.$$

PROOF. The fact that f is integrable follows from the following

Claim: $|f| \leq g, \mu\text{-a.e.}$

To prove this fact, we define, for each $n \geq 1$, the set

$$M_n = \{x \in X : |f_n(x)| > g(x)\}.$$

It is clear that $M_n \in \mathcal{A}$, and $\mu(M_n) = 0$, $\forall n \geq 1$. If we choose $M \in \mathcal{A}$ such that $\mu(M) = 0$, and

$$f(x) = \lim_{n \rightarrow \infty} f_n(x), \quad \forall x \in X \setminus M,$$

then the set $N = M \cup (\bigcup_{n=1}^{\infty} M_n) \in \mathcal{A}$ will satisfy

- $\mu(N) = 0$;
- $|f_n(x)| \leq g(x)$, $\forall x \in X \setminus N$;
- $f(x) = \lim_{n \rightarrow \infty} f_n(x)$, $\forall x \in X \setminus N$.

We then clearly get

$$|f(x)| \leq g(x), \quad \forall x \in X \setminus N,$$

and the Claim follows.

Having proven that f is integrable, we now concentrate on the equality (14).

Case I: $\mathbf{K} = \mathbb{R}$.

First of all, without any loss of generality, we can assume that $0 \leq g(x) < \infty$, $\forall x \in X$. (See Lemma 1.2.) Let us define the functions $g_n = \min\{f_n, g\}$ and $h_n = \max\{f_n, -g\}$, $n \geq 1$. Since we have $-g \leq f_n \leq g$, μ -a.e., we immediately get

$$(15) \quad g_n = h_n = f_n, \quad \mu\text{-a.e.}, \quad \forall n \geq 1,$$

thus giving the fact that $g_n, h_n \in \mathcal{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, $\forall n \geq 1$, as well as the equalities

$$(16) \quad \int_X g_n d\mu = \int_X h_n d\mu = \int_X f_n d\mu, \quad \forall n \geq 1.$$

Define the measurable functions $\varphi, \psi : X \rightarrow \mathbb{R}$ by

$$\varphi(x) = \liminf_{n \rightarrow \infty} h_n(x) \quad \text{and} \quad \psi(x) = \limsup_{n \rightarrow \infty} g_n(x), \quad \forall x \in X.$$

Using (15), we clearly have $f = \varphi = \psi$, μ -a.e., so we get

$$(17) \quad \int_X f d\mu = \int_X \varphi d\mu = \int_X \psi d\mu.$$

Remark also that we have equalities

$$(18) \quad g(x) - \varphi(x) = \liminf_{n \rightarrow \infty} [g(x) - g_n(x)] \quad \text{and} \quad g(x) + \psi(x) = \liminf_{n \rightarrow \infty} [g(x) + h_n(x)], \quad \forall x \in X.$$

Since we clearly have

$$g - g_n \geq 0 \quad \text{and} \quad g + h_n \geq 0, \quad \forall n \geq 1,$$

using (18), and Fatou Lemma (Corollary 2.2) and we get the inequalities

$$\begin{aligned} \int_X (g - \varphi) d\mu &\leq \liminf_{n \rightarrow \infty} \int_X (g - g_n) d\mu, \\ \int_X (g + \psi) d\mu &\leq \liminf_{n \rightarrow \infty} \int_X (g + h_n) d\mu, \end{aligned}$$

In other words, we get

$$\begin{aligned}\int_X g \, d\mu - \int_X \varphi \, d\mu &\leq \liminf_{n \rightarrow \infty} \left[\int_X g \, d\mu - \int_X g_n \, d\mu \right] = \int_X g \, d\mu - \limsup_{n \rightarrow \infty} \int_X g_n \, d\mu, \\ \int_X g \, d\mu + \int_X \psi \, d\mu &\leq \liminf_{n \rightarrow \infty} \left[\int_X g \, d\mu + \int_X h_n \, d\mu \right] = \int_X g \, d\mu + \liminf_{n \rightarrow \infty} \int_X h_n \, d\mu.\end{aligned}$$

Using the equalities (16) and (17), the above inequalities give

$$\begin{aligned}\int_X f \, d\mu &= \int_X \varphi \, d\mu \geq \limsup_{n \rightarrow \infty} \int_X g_n \, d\mu = \limsup_{n \rightarrow \infty} \int_X f_n \, d\mu, \\ \int_X f \, d\mu &= \int_X \psi \, d\mu \leq \liminf_{n \rightarrow \infty} \int_X h_n \, d\mu = \liminf_{n \rightarrow \infty} \int_X f_n \, d\mu.\end{aligned}$$

In other words, we have

$$\int_X f \, d\mu \leq \liminf_{n \rightarrow \infty} \int_X f_n \, d\mu \leq \limsup_{n \rightarrow \infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu,$$

thus giving the equality (14)

The case $\mathbf{K} = \mathbb{R}$ is trivial (it is in fact contained in case $\mathbf{K} = \bar{\mathbb{R}}$).

The case $\mathbf{K} = \mathbb{C}$ is also pretty clear, using real and imaginary parts, since for each $n \geq 1$, we clearly have

$$\begin{aligned}|\operatorname{Re} f_n| &\leq g, \quad \mu\text{-a.e.}, \\ |\operatorname{Im} f_n| &\leq g, \quad \mu\text{-a.e.},\end{aligned} \quad \square$$

Exercise 3. Give an example of a sequence of continuous functions $f_n : [0, 1] \rightarrow [0, \infty)$, such that

- (a) $\lim_{n \rightarrow \infty} f_n(x) = 0, \forall n \geq 1;$
- (b) $\int_{[0,1]} f_n \, d\lambda = 1, \forall n \geq 1.$

(Here λ denotes the Lebesgue measure). This shows that the Lebesgue Dominated Convergence Theorem fails, without the dominance condition (ii).

HINT: Consider the functions f_n defined by

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x \leq 1/n \\ n(2 - nx) & \text{if } 1/n \leq x \leq 2/n \\ 0 & \text{if } 2/n \leq x \leq 1 \end{cases}$$

The Lebesgue Convergence Theorems 2.2 and 2.3 have many applications. They are among the most important results in Measure Theory. In many instances, these theorem are employed during proofs, at key steps. The next two results are good illustrations.

PROPOSITION 2.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f : X \rightarrow [0, \infty]$ be a measurable function. Then the map*

$$\nu : \mathcal{A} \ni A \longmapsto \int_A f \, d\mu \in [0, \infty]$$

defines a measure on $\mathcal{A}$.

PROOF. It is clear that $\nu(\emptyset) = 0$. To prove σ -additivity, start with a pairwise disjoint sequence $(A_n)_{n=1}^\infty \subset \mathcal{A}$, and put $A = \bigcup_{n=1}^\infty A_n$. For each integer $n \geq 1$, define the set $B_n = \bigcup_{k=1}^n A_k$, and the measurable function $g_n = f \chi_{B_n}$. Define also the function $g = f \chi_A$. It is obvious that

- $0 \leq g_1 \leq g_2 \leq \dots \leq g$ (everywhere),

- $\lim_{n \rightarrow \infty} g_n(x) = g(x), \forall x \in X$.

Using the General Lebesgue Monotone Convergence Theorem, it follows that

$$(19) \quad \nu(A) = \int_X f \chi_A d\mu = \int_X g d\mu = \lim_{n \rightarrow \infty} \int_X g_n d\mu.$$

Notice now that, for each $n \geq 1$, one has the equality

$$g_n = f \chi_{A_1} + \cdots + f \chi_{A_n},$$

so using Remark 1.7.C, we get

$$\int_X g_n d\mu = \sum_{k=1}^n \int_X f \chi_{A_k} d\mu = \sum_{k=1}^n \nu(A_k),$$

so the equality (19) immediately gives $\nu(A) = \sum_{n=1}^{\infty} \nu(A_n)$. $\square$

The next result is a version of the previous one for $\mathbb{K}$ -valued functions.

PROPOSITION 2.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbf{K}$ be one of the symbols $\mathbb{R}, \mathbb{R}$, or $\mathbb{C}$, and let $(A_n)_{n=1}^{\infty} \subset \mathcal{A}$ be a pairwise disjoint sequence with $\bigcup_{n=1}^{\infty} A_n = X$. For a function $f : X \rightarrow \mathbf{K}$, the following are equivalent.*

- (i) $f \in \mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu)$;
- (ii) $f|_{A_n} \in \mathfrak{L}_{\mathbf{K}}^1(A_n, \mathcal{A}|_{A_n}, \mu|_{A_n}), \forall n \geq 1$, and

$$\sum_{n=1}^{\infty} \int_{A_n} |f| d\mu < \infty.$$

Moreover, if f satisfies these equivalent conditions, then

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_{A_n} f d\mu.$$

PROOF. (i) $\Rightarrow$ (ii). Assume $f \in \mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu)$. Applying Proposition 2.1, to $|f|$, we immediately get

$$\sum_{n=1}^{\infty} \int_{A_n} |f| d\mu = \int_X |f| d\mu < \infty,$$

which clearly proves (ii).

(ii) $\Rightarrow$ (i). Assume f satisfies condition (ii). Define, for each $n \geq 1$, the set $B_n = A_1 \cup \cdots \cup A_n$. First of all, since $(A_n)_{n=1}^{\infty} \subset \mathcal{A}$, and $\bigcup_{n=1}^{\infty} A_n = X$, it follows that f is measurable. Consider the functions $f_n = f \chi_{B_n}$ and $g_n = f \chi_{A_n}, n \geq 1$. Notice that, since $f|_{A_n} \in \mathfrak{L}_{\mathbf{K}}^1(A_n, \mathcal{A}|_{A_n}, \mu|_{A_n})$, it follows that $g_n \in \mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu), \forall n \geq 1$, and we also have

$$\int_X g_n d\mu = \int_{A_n} f d\mu, \quad \forall n \geq 1.$$

In fact we also have

$$\int_X |g_n| d\mu = \int_{A_n} |f| d\mu, \quad \forall n \geq 1.$$

Notice that we obviously have $f_n = g_1 + \cdots + g_n$, and $|f_n| = |g_1| + \cdots + |g_n|$, so if we define

$$S = \sum_{n=1}^{\infty} \int_{A_n} |f| d\mu,$$

we get

$$\int_X |f_n| d\mu = \sum_{k=1}^n \int_{A_k} |f| d\mu \leq S < \infty, \quad \forall n \geq 1.$$

Notice however that we have $0 \leq |f_1| \leq |f_2| \leq \dots \leq |f|$, as well as the equality $\lim_{n \rightarrow \infty} f_n(x) = f(x)$, $\forall x \in X$. On the one hand, using the General Lebesgue Monotone Convergence Theorem, we will get

$$\int_X |f| d\mu = \lim_{n \rightarrow \infty} \int_X |f_n| d\mu = \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \int_{A_k} |f| d\mu \right] = \sum_{n=1}^{\infty} \int_{A_n} |f| d\mu = S < \infty,$$

which proves that $|f| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, so in particular f belongs to $\mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu)$. On the other hand, since we have $|f_n| \leq |f|$, by the Lebesgue Dominated Convergence Theorem, we get

$$\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu = \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \int_{A_k} f d\mu \right] = \sum_{n=1}^{\infty} \int_{A_n} f d\mu. \quad \square$$

COROLLARY 2.4. *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbf{K}$ be one of the symbols $\mathbb{R}$, $\mathbb{R}$, or $\mathbb{C}$, and let $(X_n)_{n=1}^{\infty} \subset \mathcal{A}$ be sequence with $\bigcup_{n=1}^{\infty} X_n = X$, and $X_1 \subset X_2 \subset \dots$. For a function $f : X \rightarrow \mathbf{K}$ be a measurable function, the following are equivalent.*

- (i) $f \in \mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu)$;
- (ii) $f|_{X_n} \in \mathfrak{L}_{\mathbf{K}}^1(X_n, \mathcal{A}|_{X_n}, \mu|_{X_n})$, $\forall n \geq 1$, and

$$\sup \left\{ \int_{X_n} |f| d\mu : n \geq 1 \right\} < \infty.$$

Moreover, if f satisfies these equivalent conditions, then

$$\int_X f d\mu = \lim_{n \rightarrow \infty} \int_{X_n} f d\mu.$$

PROOF. Apply the above result to the sequence $(A_n)_{n=1}^{\infty}$ given by $A_1 = X_1$ and $A_n = X_n \setminus X_{n-1}$, $\forall n \geq 2$. $\square$

REMARK 2.2. Suppose $(X, \mathcal{A}, \mu)$ is a measure space, $\mathbb{K}$ is one of the fields $\mathbb{R}$ or $\mathbb{C}$, and $f \in \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$. By Proposition 2.2, we get the fact that the map

$$\nu : \mathcal{A} \ni A \longmapsto \int_A f d\mu \in \mathbb{K}$$

is a $\mathbb{K}$ -valued measure on $\mathcal{A}$. By Proposition 2.1, we also know that

$$\omega : \mathcal{A} \ni A \longmapsto \int_A |f| d\mu \in \mathbb{K}$$

is a finite “honest” measure on $\mathcal{A}$. Using Proposition 1.6, we clearly have

$$|\nu(A)| = \left| \int_A f d\mu \right| \leq \int_A |f| d\mu = \omega(A), \quad \forall A \in \mathcal{A},$$

which by the results from III.8 gives the inequality $|\nu| \leq \omega$. (Here $|\nu|$ denotes the variation measure of ν .) Later on (see Section 4) we are going to see that in fact we have the equality $|\nu| = \omega$.

COMMENT. It is important to understand the “sequential” nature of the convergence theorems discussed here. If we examine for instance the Monotone Convergence Theorem, we could easily formulate a “series” version, which states the equality

$$\int_X \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu,$$

for any sequence measurable functions $f_n : X \rightarrow [0, \infty]$.

Suppose now we have an arbitrary family $f_j : X \rightarrow [0, \infty]$, $j \in J$ of measurable functions, and we define

$$f(x) = \sum_{j \in J} f_j(x), \quad \forall x \in X.$$

(Here we use the summability convention which defines the sum as the supremum of all finite sums.) In general, f is not always measurable. But if it is, one still cannot conclude that

$$\int_X f d\mu = \sum_{j \in J} \int_X f_j d\mu.$$

The following example illustrates this anomaly.

EXAMPLE 2.1. Take the measure space $([0, 1], \mathfrak{M}_\lambda([0, 1]), \lambda)$, and fix $J \subset [0, 1]$ and arbitrary set. For each $j \in J$ we consider the characteristic function $f_j = \chi_{\{j\}}$. It is obvious that the function $f : X \rightarrow [0, \infty]$, defined by

$$f(x) = \sum_{j \in J} f_j(x), \quad \forall x \in [0, 1],$$

is equal to χ_J . If J is non-measurable, this already gives an example when $f = \sum_{j \in J} f_j$ is non-measurable. But even if J were measurable, it would be impossible to have the equality

$$\int_X f d\lambda = \sum_{j \in J} \int_X f_j d\lambda,$$

simply because the right hand side is zero, while the left hand side is equal to $\lambda(J)$.

The next two exercises illustrate straightforward (but nevertheless interesting) applications of the convergence theorems to quite simple situations.

Exercise 4. Let $\mathcal{A}$ be a σ -algebra on a (non-empty) set X , and let $(\mu_n)_{n=1}^{\infty}$ be a sequence of signed measures on $\mathcal{A}$. Assume that, for each $A \in \mathcal{A}$, the sequence $(\mu_n(A))_{n=1}^{\infty}$ has a limit denoted $\mu(A) \in [-\infty, \infty]$. Prove that the map $\mu : \mathcal{A} \rightarrow [0, \infty]$ defines a measure on $\mathcal{A}$, if the sequence $(\mu_n)_{n=1}^{\infty}$ satisfies one of the following hypotheses:

- A. $0 \leq \mu_1(A) \leq \mu_2(A) \leq \dots, \forall A \in \mathcal{A}$;
- B. there exists a *finite* measure ω on $\mathcal{A}$, such that $|\mu_n(A)| \leq \omega(A), \forall n \geq 1, A \in \mathcal{A}$.

HINT: To prove σ -additivity, fix a pairwise disjoint sequence $(A_k)_{k=1}^{\infty} \subset \mathcal{A}$, and put $A = \bigcup_{k=1}^{\infty} A_k$. Treat the problem of proving the equality $\mu(A) = \sum_{k=1}^{\infty} \mu(A_k)$ as a convergence problem on the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu)$ - with ν the counting measure - for the sequence of functions $f_n : \mathbb{N} \rightarrow [0, \infty]$ defined by $f_n(k) = \mu_n(A_k), \forall k \in \mathbb{N}$.

Exercise 5.* Let $\mathcal{A}$ be a σ -algebra on a (non-empty) set X , and let $(\mu_j)_{j \in J}$ be a family of signed measures on $\mathcal{A}$. Assume either of the following is true:

A. $\mu_j(A) \geq 0, \forall j \in J, A \in \mathcal{A}$.

B. There exists a *finite* measure ω on $\mathcal{A}$, such that $\sum_{j \in J} |\mu_j(A)| \leq \omega(A)$, $\forall A \in \mathcal{A}$.

Define the map $\mu : \mathcal{A} \rightarrow [0, \infty]$ by $\mu(A) = \sum_{j \in J} \mu_j(A)$, $\forall A \in \mathcal{A}$. (In Case A, the sum is defined as the supremum over finite sums. In case B, it follows that the family $(\mu_j(A))_{j \in J}$ is summable.) Prove that μ is a measure on $\mathcal{A}$.

HINT: To prove σ -additivity, fix a pairwise disjoint sequence $(A_k)_{k=1}^\infty \subset \mathcal{A}$, and put $A = \bigcup_{k=1}^\infty A_k$. To prove the equality $\mu(A) = \sum_{k=1}^\infty \mu(A_k)$, analyze the following cases: (I) There is some $k \geq 1$, such that $\mu(A_k) = \infty$; (II) $\mu(A_k) < \infty, \forall k \geq 1$. The first case is quite trivial. In the second case reduce the problem to the previous exercise, by observing that, for each $k \geq 1$, the set $J(A_k) = \{j \in J : \mu_j(A_k) > 0\}$ must be countable. Then the set $J(A) = \{j \in J : \mu_j(A) > 0\}$ is also countable.

COMMENT. One of the major drawbacks of the theory of Riemann integration is illustrated by the approach to improper integration. Recall that for a function $h : [a, b) \rightarrow \mathbb{R}$ the improper Riemann integral is defined as

$$\int_a^{b-} h(t) dt = \lim_{x \rightarrow b-} \int_a^x f(t) dt,$$

provided that

- (a) $h|_{[a, x]}$ is Riemann integrable, $\forall x \in (a, b)$, and
- (b) the above limit exists.

The problem is that although the improper integral may exist, and the function is actually defined on $[a, b]$, it may fail to be Riemann integrable, for example when it is unbounded.

In contrast to this situation, by Corollary 2.4, we see that if for example $h \geq 0$, then the Lebesgue integrability of h on $[a, b]$ is equivalent to the fact that

- (i) $h|_{[a, x]}$ is Lebesgue integrable, $\forall x \in (a, b)$, and
- (ii) $\lim_{x \rightarrow b-} \int_{[a, x]} h d\lambda$ exists.

Going back to the discussion on improper Riemann integral, we can see that a sufficient condition for $h : [a, b) \rightarrow \mathbb{R}$ to be Riemann integrable in the improper sense, is the fact that h has property (a) above, and h is Lebesgue integrable on $[a, b)$. In fact, if $h \geq 0$, then by Corollary 2.4, this is also necessary.

NOTATION. Let $-\infty \leq a < b \leq \infty$, and let f be a Lebesgue integrable function, defined on some interval J which is one of (a, b) , $[a, b)$, $(a, b]$, or $[a, b]$. Then the Lebesgue integral $\int_J f d\lambda$ will be denoted simply by $\int_a^b f d\lambda$.

*Exercise 6**. Let $(X, \mathcal{A}, \mu)$ be a *finite* measure space. Prove that for every $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, one has the equality

$$\int_X f d\mu = \int_0^\infty \mu(f^{-1}([t, \infty))) dt,$$

where the second term is defined as improper Riemann integral.

HINT: The function $\varphi : [0, \infty) \rightarrow [0, \infty)$ defined by $\varphi(t) = \mu(f^{-1}([t, \infty)))$, $\forall t \geq 0$, is non-increasing, so it is Riemann integrable on every interval $[0, a]$, $a > 0$. Prove the inequalities

$$\int_{X_a} f d\mu \leq \int_0^a \varphi(t) dt \leq \int_X f d\mu, \quad \forall a > 0,$$

where $X_a = f^{-1}([0, a))$, by analyzing lower and upper Darboux sums of $\varphi|_{[0, a]}$. Use Corollary 2.4 to get $\lim_{a \rightarrow \infty} \int_{X_a} f d\mu = \int_X f d\mu$.

LECTURE 35

3. Banach spaces of integrable functions I: the L^p spaces

In this section we discuss an important construction, which is extremely useful in virtually all branches of Analysis. In Section 1, we have already introduced the space $\mathfrak{L}^1$. The first construction deals with a generalization of this space.

DEFINITIONS. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$.

A. For a number $p \in (1, \infty)$, we define the space

$$\mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu) = \{f : X \rightarrow \mathbb{K} : f \text{ measurable, and } \int_X |f|^p d\mu < \infty\}.$$

Here we use the convention introduced in Section 1, which defines $\int_X h d\mu = \infty$, for those measurable functions $h : X \rightarrow [0, \infty]$, that are not integrable.

Of course, in this definition we can allow also the value $p = 1$, and in this case we get the familiar definition of $\mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$.

B. For $p \in [1, \infty)$, we define the map $Q_p : \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu) \rightarrow [0, \infty)$ by

$$Q_p(f) = \int_X |f|^p d\mu, \quad \forall f \in \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu).$$

REMARK 3.1. The space $\mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ was studied earlier (see Section 1). It has the following features:

- (i) $\mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ is a $\mathbb{K}$ -vector space.
- (ii) The map $Q_1 : \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu) \rightarrow [0, \infty)$ is a *seminorm*, i.e.
 - (a) $Q_1(f + g) \leq Q_1(f) + Q_1(g)$, $\forall f, g \in \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$;
 - (b) $Q_1(\alpha f) = |\alpha| \cdot Q_1(f)$, $\forall f \in \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, $\alpha \in \mathbb{K}$.
- (iii) $|\int_X f d\mu| \leq Q_1(f)$, $\forall f \in \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$.

Property (b) is clear. Property (a) immediately follows from the inequality $|f + g| \leq |f| + |g|$, which after integration gives

$$\int_X |f + g| d\mu \leq \int_X [|f| + |g|] d\mu = \int_X |f| d\mu + \int_X |g| d\mu.$$

In what follows, we aim at proving similar features for the spaces $\mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$ and Q_p , $1 < p < \infty$.

The following will help us prove that $\mathfrak{L}^p$ is a vector space.

Exercise 1 $\diamond$. Let $p \in (1, \infty)$. Then one has the inequality

$$(s + t)^p \leq 2^{p-1}(s^p + t^p), \quad \forall s, t \in [0, \infty).$$

HINT: The inequality is trivial, when $s = t = 0$. If $s + t > 0$, reduce the problem to the case $t + s = 1$, and prove, using elementary calculus techniques that

$$\min_{t \in [0,1]} [t^p + (1-t)^p] = 2^{1-p}.$$

PROPOSITION 3.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $p \in (1, \infty)$. When equipped with pointwise addition and scalar multiplication, $\mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$ is a $\mathbb{K}$ -vector space.*

PROOF. It $f, g \in \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$, then by Exercise 1 we have

$$\int_X |f + g|^p d\mu \leq \int_X (|f| + |g|)^p d\mu \leq 2^{p-1} \left[\int_X |f|^p d\mu + \int_X |g|^p d\mu \right] < \infty,$$

so $f + g$ indeed belongs to $\mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$.

It $f \in \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$, and $\alpha \in \mathbb{K}$, then the equalities

$$\int_X |\alpha f|^p d\mu = \int_X |\alpha|^p \cdot |f|^p d\mu = |\alpha|^p \cdot \int_X |f|^p d\mu$$

clearly prove that αf also belongs to $\mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$. $\square$

Our next task will be to prove that Q_p is a seminorm, for all $p > 1$. In this direction, the following is a key result. (The above mentioned convention will be used throughout this entire section.)

THEOREM 3.1 (Hölder's Inequality for integrals). *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f, g : X \rightarrow [0, \infty]$ be measurable functions, and let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then one has the inequality¹⁵*

$$(1) \quad \int_X fg d\mu \leq \left[\int_X f^p d\mu \right]^{1/p} \cdot \left[\int_X g^q d\mu \right]^{1/q}.$$

PROOF. If either $\int_X f^p d\mu = \infty$, or $\int_X g^q d\mu = \infty$, then the inequality (1) is trivial, because in this case, the right hand side is ∞ . For the remainder of the proof we will assume that $\int_X f^p d\mu < \infty$ and $\int_X g^q d\mu < \infty$.

Use Corollary 2.1 to find two sequences $(\varphi_n)_{n=1}^\infty, (\psi_n)_{n=1}^\infty \subset \mathfrak{L}_{\mathbb{R}, \text{elem}}^1(X, \mathcal{A}, \mu)$, such that

- $0 \leq \varphi_1 \leq \varphi_2 \leq \dots$ and $0 \leq \psi_1 \leq \psi_2 \leq \dots$;
- $\lim_{n \rightarrow \infty} \varphi_n(x) = f(x)^p$ and $\lim_{n \rightarrow \infty} \psi_n(x) = g(x)^q, \forall x \in X$.

By the Lebesgue Dominated Convergence Theorem, we will also get the equalities

$$(2) \quad \int_X f^p d\mu = \lim_{n \rightarrow \infty} \int_X \varphi_n d\mu \text{ and } \int_X g^q d\mu = \lim_{n \rightarrow \infty} \int_X \psi_n d\mu.$$

Remark that the functions $f_n = \varphi_n^{1/p}, g_n \psi_n^{1/q}, n \geq 1$ are also elementary (because they obviously have finite range). It is obvious that we have

- $0 \leq f_1 \leq f_2 \leq \dots$, and $0 \leq g_1 \leq g_2 \leq \dots$;
- $\lim_{n \rightarrow \infty} f_n(x) = f(x)$, and $\lim_{n \rightarrow \infty} g_n(x) = g(x), \forall x \in X$.

With these notations, the equalities (2) read

$$(3) \quad \int_X f^p d\mu = \lim_{n \rightarrow \infty} \int_X (f_n)^p d\mu \text{ and } \int_X g^q d\mu = \lim_{n \rightarrow \infty} \int_X (g_n)^q d\mu.$$

Of course, the products $f_n g_n, n \geq 1$ are again elementary, and satisfy

¹⁵ Here we use the convention $\infty^{1/p} = \infty^{1/q} = \infty$.

- $0 \leq f_1 g_1 \leq f_2 g_2 \leq \dots$;
- $\lim_{n \rightarrow \infty} [f_n(x) g_n(x)] = f(x) g(x), \forall x \in X$.

Using the General Lebesgue Monotone Convergence Theorem, we then get

$$\int_X f g \, d\mu = \lim_{n \rightarrow \infty} \int_X f_n g_n \, d\mu.$$

Using (3) we now see that, in order to prove (1), it suffices to prove the inequalities

$$\int_X f_n g_n \, d\mu \leq \left[\int_X (f_n)^p \, d\mu \right]^{1/p} \cdot \left[\int_X (g_n)^q \, d\mu \right]^{1/q}, \quad \forall n \geq 1.$$

In other words, it suffices to prove (1), under the extra assumption that *both f and g are elementary integrable*.

Suppose f and g are elementary integrable. Then (see III.1) there exist pairwise disjoint sets $(D_j)_{j=1}^m \subset \mathcal{A}$, with $\mu(D_j) < \infty, \forall j = 1, \dots, m$, and numbers $\alpha_1, \beta_1, \dots, \alpha_m, \beta_m \in [0, \infty)$, such that

$$\begin{aligned} f &= \alpha_1 \chi_{D_1} + \dots + \alpha_m \chi_{D_m} \\ g &= \beta_1 \chi_{D_1} + \dots + \beta_m \chi_{D_m} \end{aligned}$$

Notice that we have

$$f g = \alpha_1 \beta_1 \chi_{D_1} + \dots + \alpha_m \beta_m \chi_{D_m},$$

so the left hand side of (1) is the given by

$$\int_X f g \, d\mu = \sum_{j=1}^m \alpha_j \beta_j \mu(D_j).$$

Define the numbers $x_j = \alpha_j \mu(D_j)^{1/p}, y_j = \beta_j \mu(D_j)^{1/q}, j = 1, \dots, m$. Using these numbers, combined with $\frac{1}{p} + \frac{1}{q} = 1$, we clearly have

$$(4) \quad \int_X f g \, d\mu = \sum_{j=1}^m (x_j y_j).$$

At this point we are going to use the Hölder inequality for finite sequences (Lemma II.2.3), which gives

$$\sum_{j=1}^m (x_j y_j) \leq \left[\sum_{j=1}^m (x_j)^p \right]^{1/p} \cdot \left[\sum_{j=1}^m (y_j)^q \right]^{1/q},$$

so the equality (4) continues with

$$\begin{aligned} \int_X f g \, d\mu &\leq \left[\sum_{j=1}^m (x_j)^p \right]^{1/p} \cdot \left[\sum_{j=1}^m (y_j)^q \right]^{1/q} = \\ &= \left[\sum_{j=1}^m (\alpha_j)^p \mu(D_j) \right]^{1/p} \cdot \left[\sum_{j=1}^m (\beta_j)^q \mu(D_j) \right]^{1/q} = \\ &= \left[\int_X f^p \, d\mu \right]^{1/p} \cdot \left[\int_X g^q \, d\mu \right]^{1/q}. \quad \square \end{aligned}$$

COROLLARY 3.1. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. For any two functions $f \in \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$ and $g \in \mathfrak{L}_{\mathbb{K}}^q(X, \mathcal{A}, \mu)$, the product fg belongs to $\mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ and one has the inequality

$$\left| \int_X fg \, d\mu \right| \leq Q_p(f) \cdot Q_q(g).$$

PROOF. By Hölder's inequality, applied to $|f|$ and $|g|$, we get

$$\int_X |fg| \, d\mu \leq Q_p(f) \cdot Q_q(g) < \infty,$$

so $|fg|$ belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, i.e. fg belongs to $\mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$. The desired inequality then follows from the inequality $\left| \int_X fg \, d\mu \right| \leq \int_X |fg| \, d\mu$. $\square$

NOTATION. Suppose $(X, \mathcal{A}, \mu)$ is a measure space, $\mathbb{K}$ is one of the fields $\mathbb{R}$ or $\mathbb{C}$, and $p, q \in (1, \infty)$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. For any pair of functions $f \in \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$, $g \in \mathfrak{L}_{\mathbb{K}}^q(X, \mathcal{A}, \mu)$, we shall denote the number $\int_X fg \, d\mu \in \mathbb{K}$ simply by $\langle f, g \rangle$. With this notation, Corollary 3.1 reads:

$$|\langle f, g \rangle| \leq Q_p(f) \cdot Q_q(g), \forall f \in \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu), g \in \mathfrak{L}_{\mathbb{K}}^q(X, \mathcal{A}, \mu).$$

The following result gives an alternative description of the maps Q_p , $p \in (1, \infty)$.

PROPOSITION 3.2. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. and let $f \in \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$. Then one has the equality

$$(5) \quad Q_p(f) = \sup \{ |\langle f, g \rangle| : g \in \mathfrak{L}_{\mathbb{K}}^q(X, \mathcal{A}, \mu), Q_q(g) \leq 1 \}.$$

PROOF. Let us denote the right hand side of (5) simply by $P(f)$. By Corollary 3.1, we clearly have the inequality

$$P(f) \leq Q_p(f).$$

To prove the other inequality, let us first observe that in the case when $Q_p(f) = 0$, there is nothing to prove, because the above inequality already forces $P(f) = 0$. Assume then $Q_p(f) > 0$, and define the function $h : x \rightarrow \mathbb{K}$ by

$$h(x) = \begin{cases} \frac{|f(x)|^p}{f(x)} & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \end{cases}$$

It is obvious that h is measurable. Moreover, one has the equality $|h| = |f|^{p-1}$, which using the equality $qp = p + q$ gives $|h|^q = |f|^{qp-q} = |f|^p$. This proves that $h \in \mathfrak{L}_{\mathbb{K}}^q(X, \mathcal{A}, \mu)$, as well as the equality

$$Q_q(h) = \left[\int_X |h|^q \, d\mu \right]^{1/q} = \left[\int_X |f|^p \, d\mu \right]^{1/q} = Q_p(f)^{p/q}.$$

If we define the number $\alpha = Q_p(f)^{-p/q}$, then the function $g = \alpha h$ has $Q_q(g) = 1$, so we get

$$P(f) \geq \left| \int_X fg \, d\mu \right| = \frac{1}{Q_p(f)^{p/q}} \left| \int_X fh \, d\mu \right|.$$

Notice that $fh = |f|^p$, so the above inequality can be continued with

$$P(f) \geq \frac{1}{Q_p(f)^{p/q}} \int_X |f|^p d\mu = \frac{Q_p(f)^p}{Q_p(f)^{p/q}} = Q_p(f). \quad \square$$

COROLLARY 3.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $p \in (1, \infty)$. Then the Q_p is a seminorm on $\mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$, i.e.*

- (a) $Q_p(f_1 + f_2) \leq Q_p(f_1) + Q_p(f_2)$, $\forall f_1, f_2 \in \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$;
- (b) $Q_p(\alpha f) = |\alpha| \cdot Q_p(f)$, $\forall f \in \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$, $\alpha \in \mathbb{K}$.

PROOF. (a). Take $q = \frac{p}{p-1}$, so that $\frac{1}{p} + \frac{1}{q} = 1$. Start with some arbitrary $g \in \mathfrak{L}_{\mathbb{K}}^q(X, \mathcal{A}, \mu)$, with $Q_q(g) \leq 1$. Then the functions f_1g and f_2g belong to $\mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, and so $f_1g + f_2g$ also belongs to $\mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$. We then get

$$\begin{aligned} |\langle f_1 + f_2, g \rangle| &= \left| \int_X (f_1g + f_2g) d\mu \right| = \left| \int_X f_1g d\mu + \int_X f_2g d\mu \right| \leq \\ &\leq \left| \int_X f_1g d\mu \right| + \left| \int_X f_2g d\mu \right| = |\langle f_1, g \rangle| + |\langle f_2, g \rangle|. \end{aligned}$$

Using Proposition 3.2, the above inequality gives

$$|\langle f_1 + f_2, g \rangle| \leq Q_p(f_1) + Q_p(f_2).$$

Since the above inequality holds for all $g \in \mathfrak{L}_{\mathbb{K}}^q(X, \mathcal{A}, \mu)$, with $Q_q(g) \leq 1$, again by Proposition 3.2, we get

$$Q_p(f_1 + f_2) \leq Q_p(f_1) + Q_p(f_2).$$

Property (b) is obvious. $\square$

REMARKS 3.2. Let $(X, \mathcal{A}, \mu)$ be a measure space, and $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $p \in [1, \infty)$.

A. If $f \in \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$ and if $g : X \rightarrow \mathbb{K}$ is a measurable function, with $g = f$, μ -a.e., then $g \in \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$, and $Q_p(g) = Q_p(f)$.

B. If we define the space

$$\mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \mu) = \{f : X \rightarrow \mathbb{K} : f \text{ measurable, } f = 0, \mu\text{-a.e.}\},$$

then $\mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \mu)$ is a linear subspace of $\mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$. In fact one has the equality

$$\mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \mu) = \{f \in \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu) : Q_p(f) = 0\}.$$

The inclusion " $\subset$ " is trivial. Conversely, $f \in \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$ has $Q_p(f) = 0$, then the measurable function $g : X \rightarrow [0, \infty)$ defined by $g = |f|^p$ will have $\int_X g d\mu = 0$. By Exercise 2.3 this forces $g = 0$, μ -a.e., which clearly gives $f = 0$, μ -a.e.

DEFINITION. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $p \in [1, \infty)$. We define

$$L_{\mathbb{K}}^p(X, \mathcal{A}, \mu) = \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu) / \mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \mu).$$

In other words, $L_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$ is the collection of equivalence classes associated with the relation " $=$, μ -a.e." For a function $f \in \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$ we denote by $[f]$ its equivalence class in $L_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$. So the equality $[f] = [g]$ is equivalent to $f = g$, μ -a.e. By the above Remark, there exists a (unique) map $\|\cdot\|_p : L_{\mathbb{K}}^p(X, \mathcal{A}, \mu) \rightarrow [0, \infty)$, such that

$$\|[f]\|_p = Q_p(f), \quad \forall f \in \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu).$$

By the above Remark, it follows that $\|\cdot\|_p$ is a *norm* on $L_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$. When $\mathbb{K} = \mathbb{C}$ the subscript $\mathbb{C}$ will be omitted.

CONVENTIONS. Let $(X, \mathcal{A}, \mu)$, $\mathbb{K}$, and p be as above. We are going to abuse a bit the notation, by writing

$$f \in L_{\mathbb{K}}^p(X, \mathcal{A}, \mu),$$

if f belongs to $\mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$. (We will always have in mind the fact that this notation signifies that f is almost uniquely determined.) Likewise, we are going to replace $Q_p(f)$ with $\|f\|_p$.

Given $p, q \in (1, \infty)$, with $\frac{1}{p} + \frac{1}{q} = 1$, we use the same notation for the (correctly defined) map

$$\langle \cdot, \cdot \rangle : L_{\mathbb{K}}^p(X, \mathcal{A}, \mu) \times L_{\mathbb{K}}^q(X, \mathcal{A}, \mu) \rightarrow \mathbb{K}.$$

REMARK 3.3. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, and let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Given $f \in L_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$, we define the map

$$\Lambda_f : L_{\mathbb{K}}^q(X, \mathcal{A}, \mu) \ni g \mapsto \langle f, g \rangle \in \mathbb{K}.$$

According to Proposition 3.2, the map Λ_f is linear, continuous, and has norm $\|\Lambda_f\| = \|f\|_p$. If we denote by $L_{\mathbb{K}}^q(X, \mathcal{A}, \mu)^*$ the Banach space of all linear continuous maps $L_{\mathbb{K}}^q(X, \mathcal{A}, \mu) \rightarrow \mathbb{K}$, then we have a correspondence

$$(6) \quad L_{\mathbb{K}}^p(X, \mathcal{A}, \mu) \ni f \mapsto \Lambda_f \in L_{\mathbb{K}}^q(X, \mathcal{A}, \mu)^*$$

which is *linear and isometric*. This correspondence will be analyzed later in Section 5.

NOTATION. Given a sequence $(f_n)_{n=1}^{\infty}$, and a function f , in $L_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$, we are going to write

$$f = L^p\text{-}\lim_{n \rightarrow \infty} f_n,$$

if $(f_n)_{n=1}^{\infty}$ converges to f in the norm topology, i.e. $\lim_{n \rightarrow \infty} \|f_n - f\|_p = 0$.

The following technical result is very useful in the study of L^p spaces.

THEOREM 3.2 (L^p Dominated Convergence Theorem). *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, let $p \in [1, \infty)$ and let $(f_n)_{n=1}^{\infty}$ be a sequence in $L_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$. Assume $f : X \rightarrow \mathbb{K}$ is a measurable function, such that*

- (i) $f = \mu\text{-a.e.}\text{-}\lim_{n \rightarrow \infty} f_n$;
- (ii) *there exists some function $g \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, such that*

$$|f_n| \leq |g|, \quad \mu\text{-a.e.}, \quad \forall n \geq 1.$$

Then $f \in L_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$, and one has the equality

$$f = L^p\text{-}\lim_{n \rightarrow \infty} f_n.$$

PROOF. Consider the functions $\varphi_n = |f_n|^p$, $n \geq 1$, and $\varphi = |f|^p$, and $\psi = |g|^p$. Notice that

- $\varphi = \mu\text{-a.e.}\text{-}\lim_{n \rightarrow \infty} \varphi_n$;
- $|\varphi_n| \leq \psi$, $\mu\text{-a.e.}$, $\forall n \geq 1$;
- $\psi \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$.

We can apply the Lebesgue Dominated Convergence Theorem, so we get the fact that $\varphi \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, which gives the fact that $f \in L_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$. Now if we consider the functions $\eta_n = |f_n - f|^p$, and $\eta = 2^{p-1}(|g|^p + |f|^p)$, then we have (use Exercise 1):

- $0 = \mu\text{-a.e.}\text{-}\lim_{n \rightarrow \infty} \eta_n$;

- $|\eta_n| \leq \eta$, μ -a.e., $\forall n \geq 1$;
- $\eta \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$.

Again using the Lebesgue Dominated Convergence Theorem, we get

$$\lim_{n \rightarrow \infty} \int_X \eta_n d\mu = 0,$$

which means that

$$\lim_{n \rightarrow \infty} \int_X |f_n - f|^p d\mu,$$

which reads $\lim_{n \rightarrow \infty} (\|f_n - f\|_p)^p = 0$, so we clearly have $f = L^p$ - $\lim_{n \rightarrow \infty} f_n$. $\square$

Our main goal is to prove that the L^p spaces are Banach spaces. The key result which gives this, but also has some other interesting consequences, is the following.

THEOREM 3.3. *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, let $p \in [1, \infty)$ and let $(f_k)_{k=1}^\infty$ be a sequence in $L_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$, such that*

$$\sum_{k=1}^{\infty} \|f_k\|_p < \infty.$$

Consider the sequence $(g_n)_{n=1}^\infty \subset L_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$ of partial sums:

$$g_n = \sum_{k=1}^n f_k, \quad n \geq 1.$$

Then there exists a function $g \in L_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$, such that

- (a) $g = \mu$ -a.e.- $\lim_{n \rightarrow \infty} g_n$;
- (b) $g = L^p$ - $\lim_{n \rightarrow \infty} g_n$.

PROOF. Denote the sum $\sum_{k=1}^\infty \|f_k\|_p$ simply by S . For each integer $n \geq 1$, define the function $h_n : X \rightarrow [0, \infty]$, by

$$h_n(x) = \sum_{k=1}^n |f_k(x)|, \quad \forall x \in X.$$

It is clear that $h_n \in L_{\mathbb{R}}^p(X, \mathcal{A}, \mu)$, and we also have

$$(7) \quad \|h_n\|_p \leq \sum_{k=1}^n \|f_k\|_p \leq S, \quad \forall n \geq 1.$$

Notice also that $0 \leq h_1 \leq h_2 \leq \dots$. Define then the function $h : X \rightarrow [0, \infty]$ by

$$h(x) = \lim_{n \rightarrow \infty} h_n(x), \quad \forall x \in X.$$

Claim: $h \in L_{\mathbb{R}}^p(X, \mathcal{A}, \mu)$.

To prove this fact, we define the functions $\varphi = h^p$ and $\varphi_n = (h_n)^p$, $n \geq 1$. Notice that, we have

- $0 \leq \varphi_1 \leq \varphi_2 \leq \dots$;
- $\varphi_n \in L_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, $\forall n \geq 1$;
- $\lim_{n \rightarrow \infty} \varphi_n(x) = \varphi(x)$, $\forall x \in X$;
- $\sup \left\{ \int_X \varphi_n d\mu : n \geq 1 \right\} \leq M^p$.

Using the Lebesgue Monotone Convergence Theorem, it then follows that $h^p = \varphi \in L^1_{\mathbb{R}}(X, \mathcal{A}, \mu)$, so h indeed belongs to $L^p_{\mathbb{R}}(X, \mathcal{A}, \mu)$. (7) gives

Let us consider now the set $N = \{x \in X : h(x) = \infty\}$. On the one hand, since we also have

$$N = \{x \in X : \varphi(x) < \infty\},$$

and φ is integrable, it follows that $N \in \mathcal{A}$, and $\mu(N) = 0$. On the other hand, since

$$\sum_{k=1}^{\infty} |f_n(x)| = h(x) < \infty, \quad \forall x \in X \setminus N,$$

it follows that, for each $x \in X \setminus N$, the series $\sum_{k=1}^{\infty} f_k(x)$ is convergent. Let us define then $g : X \rightarrow \mathbb{K}$ by

$$g(x) = \begin{cases} \sum_{k=1}^{\infty} f_k(x) & \text{if } x \in X \setminus N \\ 0 & \text{if } x \in N \end{cases}$$

It is obvious that g is measurable, and we have

$$g = \mu\text{-a.e.-}\lim_{n \rightarrow \infty} g_n.$$

Since we have

$$|g_n| = \left| \sum_{k=1}^n f_k \right| \leq \sum_{k=1}^n |f_k| = h_n \leq h, \quad \forall n \geq 1,$$

using the Claim, and Theorem 3.2, it follows that g indeed belongs to $L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ and we also have the equality $g = L^p\text{-}\lim_{n \rightarrow \infty} g_n$. $\square$

COROLLARY 3.3. *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. Then $L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ is a Banach space, for each $p \in [1, \infty)$.*

PROOF. This is immediate from the above result, combined with the completeness criterion given by Remark II.3.1. $\square$

Another interesting consequence of Theorem 3.3 is the following.

COROLLARY 3.4. *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, let $p \in [1, \infty)$, and let $f \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$. Any sequence $(f_n)_{n=1}^{\infty} \subset L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$, with $f = L^p\text{-}\lim_{n \rightarrow \infty} f_n$, has a subsequence $(f_{n_k})_{k=1}^{\infty}$ such that $f = \mu\text{-a.e.-}\lim_{k \rightarrow \infty} f_{n_k}$.*

PROOF. Without any loss of generality, we can assume that $f = 0$, so that we have

$$\lim_{n \rightarrow \infty} \|f_n\|_p = 0.$$

Choose then integers $1 \leq n_1 < n_2 < \dots$, such that

$$\|f_{n_k}\|_p \leq \frac{1}{2^k}, \quad \forall k \geq 1.$$

If we define the functions

$$g_m = \sum_{k=1}^m f_{n_k},$$

then by Theorem 3.3, it follows that there exists some $g \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$, such that

$$g = \mu\text{-a.e.-}\lim_{m \rightarrow \infty} g_m.$$

This means that there exists some $N \in \mathcal{A}$, with $\mu(N) = 0$, such that

$$\lim_{m \rightarrow \infty} g_m(x) = g(x), \quad \forall x \in X \setminus N.$$

In other words, for each $x \in X \setminus N$, the series $\sum_{k=1}^{\infty} f_{n_k}(x)$ is convergent (to some number $g(x) \in \mathbb{K}$). In particular, it follows that

$$\lim_{k \rightarrow \infty} f_{n_k}(x) = 0, \quad \forall x \in X \setminus N,$$

so we indeed have $0 = \mu\text{-a.e.}-\lim_{k \rightarrow \infty} f_{n_k}$. $\square$

The following result collects some properties of L^p spaces in the case when the underlying measure space is finite.

PROPOSITION 3.3. *Suppose $(X, \mathcal{A}, \mu)$ is a finite measure space, and $\mathbb{K}$ is one of the fields $\mathbb{R}$ or $\mathbb{C}$.*

- (i) *If $f : X \rightarrow \mathbb{K}$ is a bounded measurable function, then $f \in L_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$, $\forall p \in [1, \infty)$.*
- (ii) *For any $p, q \in [1, \infty)$, with $p < q$, one has the inclusion $\mathfrak{L}_{\mathbb{K}}^q(X, \mathcal{A}, \mu) \subset \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$. So taking quotients by $\mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \mu)$, one gets an inclusion of vector spaces*

$$(8) \quad L_{\mathbb{K}}^q(X, \mathcal{A}, \mu) \hookrightarrow L_{\mathbb{K}}^p(X, \mathcal{A}, \mu).$$

Moreover the above inclusion is a continuous linear map.

PROOF. The key property that we are going to use here is the fact that the constant function $1 = \chi_X$ is μ -integrable (being elementary μ -integrable).

(i). This part is pretty clear, because if we start with a bounded measurable function $f : X \rightarrow \mathbb{K}$ and we take $M = \sup_{x \in X} |f(x)|$, then the inequality $|f|^p \leq M^p \cdot 1$, combined with the integrability of 1, will force the integrability of $|f|^p$, i.e. $f \in L_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$.

(ii). Fix $1 \leq p < q < \infty$, as well as a function $f \in \mathfrak{L}_{\mathbb{K}}^q(X, \mathcal{A}, \mu)$. Consider the number $r = \frac{q}{p} > 1$, and $s = \frac{r}{r-1}$, so that we have $\frac{1}{r} + \frac{1}{s} = 1$. Since $f \in \mathfrak{L}_{\mathbb{K}}^q(X, \mathcal{A}, \mu)$, the function $g = |f|^q$ belongs to $\mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$. If we define then the function $h = |f|^p$, then we obviously have $g = h^r$, so we get the fact that h belongs to $\mathfrak{L}_{\mathbb{K}}^s(X, \mathcal{A}, \mu)$. Using part (i), we get the fact that $1 \in \mathfrak{L}_{\mathbb{K}}^s(X, \mathcal{A}, \mu)$, so by Corollary 3.1, it follows that $h = 1 \cdot h$ belongs to $\mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, and moreover, one has the inequality

$$\begin{aligned} \int_X |f|^p d\mu &= \int_X h d\mu \leq \|1\|_s \cdot \|h\|_r = \left[\int_X 1 d\mu \right]^{1/s} \cdot \left[\int_X h^r d\mu \right]^{1/r} = \\ &= \mu(X)^{1/s} \cdot \left[\int_X |f|^q d\mu \right]^{1/r} = \mu(X)^{1/s} \cdot (\|f\|_q)^{q/r}. \end{aligned}$$

On the one hand, this inequality proves that $f \in \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$. On the other hand, this also gives the inequality

$$(\|f\|_p)^p \leq \mu(X)^{1/s} \cdot (\|f\|_q)^{q/r} = \mu(X)^{1-\frac{p}{q}} \cdot (\|f\|_q)^p,$$

which yields

$$\|f\|_p \leq \mu(X)^{\frac{1}{p} - \frac{1}{q}} \cdot \|f\|_q.$$

This proves that the linear map (8) is continuous (and has norm no greater than $\mu(X)^{\frac{1}{p} - \frac{1}{q}}$). $\square$

Exercise 2. Give an example of a sequence of continuous functions $f_n : [0, 1] \rightarrow [0, \infty)$, $n \geq 1$, such that $L^p\text{-}\lim_{n \rightarrow \infty} f_n = 0$, $\forall p \in [1, \infty)$, but for which it is not true that $0 = \mu\text{-a.e.}\text{-}\lim_{n \rightarrow \infty} f_n$. (Here we work on the measure space $([0, 1], \mathfrak{M}_\lambda([0, 1]), \lambda)$.)

Exercise 3. Let $\Omega \subset \mathbb{R}^n$ be an open set. Prove that $C_c^\mathbb{K}(\Omega)$ is dense in $L_\mathbb{K}^p(\Omega, \mathfrak{M}_\lambda(\Omega), \lambda)$, for every $p \in [1, \infty)$. (Here λ denotes the n -dimensional Lebesgue measure, and $\mathfrak{M}_\lambda(\Omega)$ denotes the collection of all Lebesgue measurable subsets of Ω .)

NOTATIONS. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. We define the space

$$\mathfrak{N}_{\mathbb{K}, \text{elem}}(X, \mathcal{A}, \mu) = \mathfrak{L}_{\mathbb{K}, \text{elem}}^1(X, \mathcal{A}, \mu) \cap \mathfrak{N}_\mathbb{K}(X, \mathcal{A}, \mu),$$

and we define the quotient space

$$L_{\mathbb{K}, \text{elem}}^1(X, \mathcal{A}, \mu) = \mathfrak{L}_{\mathbb{K}, \text{elem}}^1(X, \mathcal{A}, \mu) / \mathfrak{N}_{\mathbb{K}, \text{elem}}(X, \mathcal{A}, \mu).$$

In other words, if one considers the quotient map

$$\Pi_1 : \mathfrak{L}_\mathbb{K}^1(X, \mathcal{A}, \mu) \rightarrow L_\mathbb{K}^1(X, \mathcal{A}, \mu),$$

then $L_{\mathbb{K}, \text{elem}}^1(X, \mathcal{A}, \mu) = \Pi_1(\mathfrak{L}_{\mathbb{K}, \text{elem}}^1(X, \mathcal{A}, \mu))$. Notice that we have the obvious inclusion

$$\mathfrak{L}_{\mathbb{K}, \text{elem}}^1(X, \mathcal{A}, \mu) \subset \mathfrak{L}_\mathbb{K}^p(X, \mathcal{A}, \mu), \quad \forall p \in [1, \infty),$$

so we consider the quotient map

$$\Pi_p : \mathfrak{L}_\mathbb{K}^p(X, \mathcal{A}, \mu) \rightarrow L_\mathbb{K}^p(X, \mathcal{A}, \mu),$$

we can also define the subspace

$$L_{\mathbb{K}, \text{elem}}^p(X, \mathcal{A}, \mu) = \Pi_p(\mathfrak{L}_{\mathbb{K}, \text{elem}}^p(X, \mathcal{A}, \mu)), \quad \forall p \in [1, \infty).$$

Remark that, as vector spaces, the spaces $L_{\mathbb{K}, \text{elem}}^p(X, \mathcal{A}, \mu)$ are identical, since

$$\text{Ker } \Pi_p = \mathfrak{N}_\mathbb{K}(X, \mathcal{A}, \mu), \quad \forall p \in [1, \infty).$$

With these notations we have the following fact.

PROPOSITION 3.4. $L_{\mathbb{K}, \text{elem}}^p(X, \mathcal{A}, \mu)$ is dense in $L_\mathbb{K}^p(X, \mathcal{A}, \mu)$, for each $p \in [1, \infty)$.

PROOF. Fix $p \in [1, \infty)$, and start with some $f \in L_\mathbb{K}^p(X, \mathcal{A}, \mu)$. What we need to prove is the existence of a sequence $(f_n)_{n=1}^\infty \subset \mathfrak{L}_{\mathbb{K}, \text{elem}}^1(X, \mathcal{A}, \mu)$, such that $f = L^p\text{-}\lim_{n \rightarrow \infty} f_n$. Taking real and imaginary parts (in the case $\mathbb{K} = \mathbb{C}$), it suffices to consider the case when f is real valued. Since $|f|$ also belongs to L^p , it follows that $f^+ = \max\{f, 0\} = \frac{1}{2}(|f| + f)$, and $f^- = \max\{-f, 0\} = \frac{1}{2}(|f| - f)$ both belong to L^p , so in fact we can assume that f is non-negative. Consider the function $g = f^p \in \mathfrak{L}_\mathbb{K}^1(X, \mathcal{A}, \mu)$. Use the definition of the integral, to find a sequence $(g_n)_{n=1}^\infty \subset \mathfrak{L}_{\mathbb{K}, \text{elem}}^1(X, \mathcal{A}, \mu)$, such that

- $0 \leq g_n \leq g$, $\forall n \geq 1$;
- $\lim_{n \rightarrow \infty} \int_X g_n d\mu = \int_X g d\mu$.

This gives the fact that $g = L^1\text{-}\lim_{n \rightarrow \infty} g_n$. Using Corollary 3.4, after replacing $(g_n)_{n=1}^\infty$ with a subsequence, we can also assume that $g = \mu\text{-a.e.}\text{-}\lim_{n \rightarrow \infty} g_n$. If we put $f_n = (g_n)^{1/p}$, $\forall n \geq 1$, we now have

- $0 \leq f_n \leq f$, $\forall n \geq 1$;
- $f = \mu\text{-a.e.}\text{-}\lim_{n \rightarrow \infty} f_n$.

Obviously, the f_n 's are still elementary integrable, and by the L^p Dominated Convergence Theorem, we indeed get $f = L^p\text{-}\lim_{n \rightarrow \infty} f_n$. $\square$

COMMENTS. A. The above result gives us the fact that $L_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$ is the *completion* of $L_{\mathbb{K}, \text{elem}}^p(X, \mathcal{A}, \mu)$. This allows for the following alternative construction of the $\mathcal{L}^p$ spaces.

B. For a measurable function $f : X \rightarrow \mathbb{K}$, by the (proof of the) above result, it follows that the condition $f \in L_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$ is equivalent to the equality $f = \mu\text{-a.e.}\text{-}\lim_{n \rightarrow \infty} f_n$, for some sequence $(f_n)_{n=1}^{\infty}$ of elementary integrable functions, which is *Cauchy in the L^p norm*, i.e.

(c) *for every $\varepsilon > 0$, there exists N_{ε} , such that*

$$\|f_m - f_n\|_p < \varepsilon, \quad \forall m, n \geq N_{\varepsilon}.$$

One key feature, which will be heavily exploited in the next section, deals with the Banach space $p = 2$, for which we have the following.

PROPOSITION 3.5. *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$.*

(i) *The map $(\cdot | \cdot) : L_{\mathbb{K}}^2(X, \mathcal{A}, \mu) \times L_{\mathbb{K}}^2(X, \mathcal{A}, \mu) \rightarrow \mathbb{K}$, given by*

$$(f | g) = \langle \bar{f}, g \rangle = \int_X \bar{f}g \, d\mu, \quad \forall f, g \in L_{\mathbb{K}}^2(X, \mathcal{A}, \mu),$$

defines an inner product on $L_{\mathbb{K}}^2(X, \mathcal{A}, \mu)$.

(ii) *One has the equality*

$$\|f\|_2 = \sqrt{(f | f)}, \quad \forall f \in L_{\mathbb{K}}^2(X, \mathcal{A}, \mu).$$

Consequently, $L_{\mathbb{K}}^2(X, \mathcal{A}, \mu)$ is a Hilbert space.

PROOF. The properties of the inner product are immediate, from the properties of integration. The second property is also clear. $\square$

REMARK 3.4. The main biproduct of the above feature is the fact that the correspondence (6) is an *isometric isomorphism*, in the case $p = q = 2$. This follows from Riesz Theorem (only the surjectivity is the issue here; the rest has been discussed in Remark 3.3). If $\phi : L_{\mathbb{K}}^2(X, \mathcal{A}, \mu) \rightarrow \mathbb{K}$ is a linear continuous map, then there exists some $h \in L_{\mathbb{K}}^2(X, \mathcal{A}, \mu)$, such that

$$\phi(g) = (h | g), \quad \forall g \in L_{\mathbb{K}}^2(X, \mathcal{A}, \mu).$$

If we put $f = \bar{h}$, then the above equality gives

$$\phi(g) = \langle f, g \rangle, \quad \forall g \in L_{\mathbb{K}}^2(X, \mathcal{A}, \mu).$$

i.e. $\phi = \Lambda_f$.

COMMENTS. Eventually (see Section 5) we shall prove that the correspondence (6) is surjective also in the general case.

The correspondence (6) also has a version for $q = 1$. This would require the definition of an L^p space for the case $p = \infty$. We shall postpone this until we reach Section 5. The next exercise hints towards such a construction.

Exercise 4 $^{\diamond}$. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $f : X \rightarrow \mathbb{K}$ be a *bounded* measurable function. Define $M = \sup_{x \in X} |f(x)|$. Prove the following.

- (i) Whenever $g \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$, it follows that the function fg also belongs to $L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$, and one has the inequality

$$\|fg\|_1 \leq M \cdot \|g\|_1.$$

- (ii) The map

$$\Lambda_f : L^1_{\mathbb{K}}(X, \mathcal{A}, \mu) \ni g \longmapsto \int_X fg \, d\mu \in \mathbb{K}$$

is linear and continuous. Moreover, one has the inequality $\|\Lambda_f\| \leq M$.

REMARK 3.5. If we apply the above Exercise to the constant function $f = 1$, we get the (already known) fact that the integration map

$$(9) \quad \Lambda_1 : L^1_{\mathbb{K}}(X, \mathcal{A}, \mu) \ni g \longmapsto \int_X g \, d\mu \in \mathbb{K}$$

is linear and continuous, and has norm $\|\Lambda_1\| \leq 1$. The following exercise gives the exact value of the norm.

Exercise 5. With the notations above, prove that the following are equivalent:

- (i) the measure space $(X, \mathcal{A}, \mu)$ is *non-degenerate*, i.e. there exists $A \in \mathcal{A}$ with $0 < \mu(A) < \infty$;
- (ii) $L^1_{\mathbb{K}}(X, \mathcal{A}, \mu) \neq \{0\}$;
- (ii) the integration map (9) has norm $\|\Lambda_1\| = 1$.

LECTURES 36-37

4. Radon-Nikodym Theorems

In this section we discuss a very important property which has many important applications.

DEFINITION. Let X be a non-empty set, and let $\mathcal{A}$ be a σ -algebra on X . Given two measures μ and ν on $\mathcal{A}$, we say that ν *has the Radon-Nikodym property relative to μ* , if there exists a measurable function $f : X \rightarrow [0, \infty]$, such that

$$(1) \quad \nu(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}.$$

Here we use the convention which defines the integral in the right hand side by

$$\int_A f d\mu = \begin{cases} \int_X f \chi_A d\mu & \text{if } f \chi_A \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu) \\ \infty & \text{if } f \chi_A \notin \mathfrak{L}_+^1(X, \mathcal{A}, \mu) \end{cases}$$

In this case, we say that f is a *density for ν relative to μ* .

The Radon-Nikodym property has an equivalent useful formulation.

PROPOSITION 4.1 (Change of Variables). *Let X be a non-empty set, and let $\mathcal{A}$ be a σ -algebra on X , let μ and ν be measures on $\mathcal{A}$, and let $f : X \rightarrow [0, \infty]$ be a measurable function.*

A. *The following are equivalent*

- (i) *ν has the Radon-Nikodym property relative to μ , and f is a density for ν relative to μ ;*
- (ii) *for every measurable function $h : X \rightarrow [0, \infty]$, one has the equality¹⁶*

$$(2) \quad \int_X h d\nu = \int_X hf d\mu.$$

B. *If ν and f are as above, and $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, then the equality (2) also holds for those measurable functions $h : X \rightarrow \mathbb{K}$ with $h \in L_{\mathbb{K}}^1(X, \mathcal{A}, \nu)$ and $hf \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$.*

PROOF. A. (i) $\Rightarrow$ (ii). Assume property (i) holds, which means that we have (1). Fix a measurable function $h : X \rightarrow [0, \infty]$, and use Theorem III.3.2, to find a sequence $(h_n)_{n=1}^{\infty} \subset \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with

- (a) $0 \leq h_1 \leq h_2 \leq \cdots \leq h$;
- (b) $\lim_{n \rightarrow \infty} h_n(x) = h(x)$, $\forall x \in X$.

Of course, we also have

$$(a') \quad 0 \leq h_1 f \leq h_2 f \leq \cdots \leq hf;$$

¹⁶ For the product hf we use the conventions $0 \cdot \infty = \infty \cdot 0 = 0$, and $t \cdot \infty = \infty \cdot t = \infty$, $\forall t \in (0, \infty]$.

$$(b') \lim_{n \rightarrow \infty} h_n(x)f(x) = h(x)f(x), \forall x \in X.$$

Using the Monotone Convergence Theorem, we then get the equalities

$$(3) \quad \int_X h \, d\nu = \lim_{n \rightarrow \infty} \int_X h_n \, d\nu \text{ and } \int_X hf \, d\mu = \lim_{n \rightarrow \infty} \int_X h_n f \, d\nu$$

Notice that, if we fix n and we write $h_n = \sum_{k=1}^p \alpha_k \chi_{A_k}$, for some $A_1, \dots, A_p \in \mathcal{A}$, and $\alpha_1 > \dots > \alpha_p > 0$, then

$$\int_X h_n \, d\nu = \sum_{k=1}^p \alpha_k \nu(A_k) = \sum_{k=1}^p \int_X \alpha_k \chi_{A_k} f \, d\mu = \int_X h_n f \, d\mu,$$

so using (3), we immediately get (2).

The implication (ii) $\Rightarrow$ (i) is trivial, using functions of the form $h = \chi_A$, $A \in \mathcal{A}$.

B. Suppose ν has the Radon-Nikodym property relative to μ , and f is a density for ν relative to μ , and let $h : X \rightarrow \mathbb{K}$ be a measurable function with $h \in L^1_{\mathbb{K}}(X, \mathcal{A}, \nu)$ and $hf \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$. In the complex case, using the inequalities $|\operatorname{Re} h| \leq |h|$ and $|\operatorname{Im} h| \leq |h|$, it is clear that both functions $\operatorname{Re} h$ and $\operatorname{Im} h$ belong to $L^1(X, \mathcal{A}, \nu)$, and also the products $(\operatorname{Re} h)f$ and $(\operatorname{Im} h)f$ belong to $L^1(X, \mathcal{A}, \mu)$. This shows that it suffices to prove (2) under the additional hypothesis that h is real-valued. In this case we consider the functions $h^{\pm}$, defined by

$$h^+ = \max\{h, 0\} \text{ and } h^- = \max\{-h, 0\}.$$

Since we have $0 \leq h^{\pm} \leq |h|$, it follows that $h^{\pm} \in \mathcal{L}^1_+(X, \mathcal{A}, \nu)$, as well as $h^{\pm}f \in \mathcal{L}^1_+(X, \mathcal{A}, \mu)$. In particular, we get the equalities

$$(4) \quad \int_X h \, d\nu = \int_X h^+ \, d\nu - \int_X h^- \, d\nu \text{ and } \int_X hf \, d\mu = \int_X h^+ f \, d\mu - \int_X h^- f \, d\mu.$$

Since $h^{\pm} \geq 0$, we can use property A.(ii) above, and we have

$$\int_X h^{\pm} \, d\nu = \int_X h^{\pm} f \, d\mu,$$

and then the desired equality (2) immediately follows from (4). $\square$

One important issue is the uniqueness of the density. For this purpose, it will be helpful to introduce the following.

DEFINITION. Let T be one of the spaces $[-\infty, \infty]$ or $\mathbb{C}$, and let R be some relation on T (in our case R will be either “=,” or “ $\geq$,” or “ $\leq$,” on $[-\infty, \infty]$). Given a measurable space $(X, \mathcal{A}, \mu)$, and two measurable functions $f_1, f_2 : X \rightarrow T$,

$$f_1 R f_2, \mu\text{-l.a.e.}$$

if the set

$$A = \{x \in X : f_1(x) R f_2(x)\}$$

belongs to $\mathcal{A}$, and it has locally μ -null complement in X , i.e. $\mu([X \setminus A] \cap F) = 0$, for every set $F \in \mathcal{A}$ with $\mu(F) < \infty$. (If R is one of the relations listed above, the set A automatically belongs to $\mathcal{A}$, so all intersections $[X \setminus A] \cap F$, $F \in \mathcal{A}$, also belong to $\mathcal{A}$.) The abbreviation “ μ -l.a.e.” stands for “ μ -locally-almost everywhere.” Remark that one has the implication

$$f_1 R f_2, \mu\text{-a.e.} \Rightarrow f_1 R f_2, \mu\text{-l.a.e.}$$

Remark that, when μ is σ -finite, then the other implication also holds:

$$f_1 R f_2, \mu\text{-l.a.e.} \Rightarrow f_1 R f_2, \mu\text{-a.e.}$$

With this terminology, one has the following uniqueness result.

PROPOSITION 4.2. *Suppose $\mathcal{A}$ is a σ algebra on some non-empty set X , and μ and ν are measures on $\mathcal{A}$, such that ν has the Radon-Nikodym property relative to μ . If $f, g : X \rightarrow [0, \infty]$ are densities for ν relative to μ , then*

$$f = g, \mu\text{-l.a.e.}$$

In particular, if μ is σ -finite, then

$$f = g, \mu\text{-a.e.}$$

PROOF. Consider the set $B = \{x \in X : f(x) \neq g(x)\}$, which belongs to $\mathcal{A}$. We need to prove that B is locally μ -null, i.e. one has $\mu(B \cap F) = 0$, for all $F \in \mathcal{A}$ with $\mu(F) < \infty$. Fix $F \in \mathcal{A}$ with $\mu(F) < \infty$, and let us write $B \cap F = D \cup E$, where

$$D = \{x \in B \cap F : f(x) < g(x)\} \text{ and } E = \{x \in B \cap F : f(x) > g(x)\}.$$

If we define, for each integer $n \geq 1$, the sets

$$D_n = \{x \in B \cap F : f(x) + \frac{1}{n} \leq g(x)\} \text{ and } E_n = \{x \in B \cap F : f(x) \geq g(x) + \frac{1}{n}\},$$

then it is clear that

$$B \cap F = D \cup E = \bigcup_{n=1}^{\infty} (D_n \cup E_n),$$

so in order to prove that $\mu(B \cap F) = 0$, it suffices to show that $\mu(D_n) = \mu(E_n) = 0$, $\forall n \geq 1$.

Fix $n \geq 1$. It is obvious that $f(x) < \infty$, $\forall x \in D_n$, so if we define the sequence $(D_n^k)_{k=1}^{\infty} \subset \mathcal{A}$, by

$$D_n^k = \{x \in D_n : f(x) \leq k\}, \quad \forall k \geq 1,$$

we have the equality $D_n = \bigcup_{k=1}^{\infty} D_n^k$, so in order to prove that $\mu(D_n) = 0$, it suffices to show that $\mu(D_n^k) = 0$, $\forall k \geq 1$. On the one hand, since $f(x) \leq k$, $\forall k \geq 1$, using the inclusion $D_n^k \subset F$, we get

$$\nu(D_n^k) = \int_{D_n^k} f d\mu \leq \int_X k \chi_{D_n^k} d\mu = k\mu(D_n^k) \leq k\mu(F) < \infty.$$

On the other hand, since $g(x) \geq f(x) + \frac{1}{n}$, $\forall x \in D_n^k$, we get

$$\begin{aligned} \nu(D_n^k) &= \int_{D_n^k} g d\mu \geq \int_X (f \chi_{D_n^k} + \frac{1}{n} \chi_{D_n^k}) d\mu = \\ &= \int_X f \chi_{D_n^k} d\mu + \int_X \frac{1}{n} \chi_{D_n^k} d\mu = \nu(D_n^k) + \frac{1}{n} \mu(D_n^k). \end{aligned}$$

Since $\nu(D_n^k) < \infty$, the above inequality forces $\mu(D_n^k) = 0$.

The fact that $\mu(E_n) = 0$, $\forall n \geq 1$, is proven the exact same way. $\square$

In general, the uniqueness of the density does not hold μ -a.e., as it is seen from the following.

EXAMPLE 4.1. Take X to be some non-empty set, put $\mathcal{A} = \{\emptyset, X\}$, and define the measure μ on $\mathcal{A}$, by $\mu(\emptyset) = 0$ and $\mu(X) = \infty$. It is clear that μ has the Radon-Nikodym property relative to itself, but as sensitivities one can choose for instance the constant functions $f = 1$ and $g = 2$. Clearly, the equality $f = g$, μ -a.e. is not true.

REMARK 4.1. The local almost uniqueness result, given in Proposition 4.2, holds under slightly weaker assumptions. Namely, if $(X, \mathcal{A}, \mu)$ is a measure space, and if $f, g : X \rightarrow [0, \infty]$ are measurable functions for which we have the equality

$$\int_A f d\mu = \int_A g d\mu,$$

for all $A \in \mathcal{A}$ with $\mu(A) < \infty$, then we still have the equality $f = g$, μ -l.a.e. This follows actually from Proposition 4.2, applied to functions of the form $f|_A$ and $g|_A$.

Let us introduce the following.

NOTATIONS. For a measure space $(X, \mathcal{A}, \mu)$ we define

$$\begin{aligned}\mathcal{A}_0^\mu &= \{N \in \mathcal{A} : \mu(N) = 0\}; \\ \mathcal{A}_{\text{fin}}^\mu &= \{F \in \mathcal{A} : \mu(F) < \infty\}; \\ \mathcal{A}_{0,\text{loc}}^\mu &= \{A \in \mathcal{A} : \mu(A \cap F) = 0, \forall F \in \mathcal{A}_{\text{fin}}^\mu\}.\end{aligned}$$

With these notations, we have the inclusions

$$\mathcal{A}_0^\mu = \mathcal{A}_{0,\text{loc}}^\mu \cap \mathcal{A}_{\text{fin}}^\mu \subset \mathcal{A}_{0,\text{loc}}^\mu \subset \mathcal{A},$$

and $\mathcal{A}_0^\mu$ and $\mathcal{A}_{0,\text{loc}}^\mu$ are in fact σ -rings.

COMMENT. The “locally-almost everywhere” terminology is actually designed to “hide some pathologies under the rug.” For instance, if $(X, \mathcal{A}, \mu)$ is a *degenerate* measure space, i.e. $\mu(A) \in \{0, \infty\}$, $\forall A \in \mathcal{A}$, then “anything happens locally almost-everywhere,” which means that we have the equality $\mathcal{A}_{0,\text{loc}}^\mu = \mathcal{A}$.

At the other end, there is a particular type of measure spaces on which, even in the absence of σ -finiteness, the notions of “locally-almost everywhere” and “almost everywhere” coincide, i.e. we have the equality $\mathcal{A}_{0,\text{loc}}^\mu = \mathcal{A}_0^\mu$. Such spaces are described by the following.

DEFINITION. A measure space $(X, \mathcal{A}, \mu)$ is said to be *nowhere degenerate*, or with *finite subset property*, if

- (F) *for every set $A \in \mathcal{A}$ with $\mu(A) > 0$, there exists some set $F \in \mathcal{A}$, with $F \subset A$, and $0 < \mu(F) < \infty$.*

With this terminology, one has the following result.

PROPOSITION 4.3. *For a measure space $(X, \mathcal{A}, \mu)$, the following are equivalent:*

- (i) $\mathcal{A}_{0,\text{loc}}^\mu = \mathcal{A}_0^\mu$;
- (ii) $(X, \mathcal{A}, \mu)$ has the finite subset property.

PROOF. (i) $\Rightarrow$ (ii). Assume $\mathcal{A}_{0,\text{loc}}^\mu = \mathcal{A}_0^\mu$, and let us prove that $(X, \mathcal{A}, \mu)$ has the finite subset property. We argue by contradiction, so let us assume there exists some set $A \in \mathcal{A}$, with $\mu(A) = \infty$, such that $\mu(B) \in \{0, \infty\}$, for every $B \in \mathcal{A}$, with $B \subset A$. In particular, if we start with some arbitrary $F \in \mathcal{A}_{\text{fin}}^\mu$, using the fact that $\mu(A \cap F) \leq \mu(F) < \infty$, we see that we must have $\mu(A \cap F) = 0$. This proves precisely that $A \in \mathcal{A}_{0,\text{loc}}^\mu$. By assumption, it follows that $A \in \mathcal{A}_0^\mu$, i.e. $\mu(A) = 0$, which is impossible.

(ii) $\Rightarrow$ (i). Assume that $(X, \mathcal{A}, \mu)$ has the finite subset property, and let us prove the equality (i). Since one inclusion is always true, all we need to prove is the inclusion $\mathcal{A}_{0,\text{loc}}^\mu \subset \mathcal{A}_0^\mu$, which equivalent to the inclusion $\mathcal{A}_{0,\text{loc}}^\mu \subset \mathcal{A}_{\text{fin}}^\mu$. Start with some set $A \in \mathcal{A}_{0,\text{loc}}^\mu$, but assume $\mu(A) = \infty$. On the one hand, using the finite

subset property, there exists some set $F \in \mathcal{A}$ with $F \subset A$ and $\mu(F) > 0$. On the other hand, since $A \in \mathcal{A}_{0,\text{loc}}^\mu$, we have $\mu(F) = 0$, which is impossible. $\square$

EXAMPLE 4.2. Take X be an uncountable set, let $\mathcal{A} = \mathcal{P}(X)$, and let μ be the counting measure, i.e.

$$\mu(A) = \begin{cases} \text{Card } A & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite} \end{cases}$$

Then $(X, \mathcal{P}(X), \mu)$ has the finite subset property, but is not σ -finite.

When we restrict to *integrable* functions, the two notions μ -l.a.e. and μ -a.e. coincide. More precisely, we have the following.

PROPOSITION 4.4. *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $p \in [1, \infty)$. For a function $f \in \mathcal{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$, the following are equivalent:*

- (i) $f = 0$, μ -l.a.e.
- (ii) $f = 0$, μ -a.e.

PROOF. Of course, we only need to prove the implication (i) $\Rightarrow$ (ii). Assume $f = 0$, μ -l.a.e. Using the function $g = |f|^p$, we can assume that $p = 1$ and $f(x) \geq 0$, $\forall x \in X$. Consider then the set $N = \{x \in X : f(x) > 0\}$, and write it as a union $N = \bigcup_{n=1}^{\infty} N_n$, where

$$N_n = \{x \in X : f(x) \geq \frac{1}{n}\}, \quad \forall n \geq 1.$$

Of course, all we need is the fact that $\mu(N_n) = 0$, $\forall n \geq 1$. Fix $n \geq 1$. On the one hand, the assumption on f , it follows that $N_n \in \mathcal{A}_{0,\text{loc}}^\mu$. On the other hand, the inequality $\frac{1}{n} \chi_{N_n} \leq f$, forces the elementary function $\frac{1}{n} \chi_{N_n}$ to be μ -integrable, i.e. $\mu(N_n) < \infty$. Consequently we have

$$N \in \mathcal{A}_{0,\text{loc}}^\mu \cap \mathcal{A}_{\text{fin}}^\mu = \mathcal{A}_0^\mu. \quad \square$$

COMMENT. In what follows we will discuss several results, which all have as conclusion the fact that one measure has the Radon-Nikodym property with respect to another one. All such results will be called “Radon-Nikodym Theorems.”

The first result is in fact quite general, in the sense that it works for finite signed or complex measures.

THEOREM 4.1 (“Easy” Radon-Nikodym Theorem). *Let $(X, \mathcal{A}, \mu)$ be a finite measure space, let $\mathbb{K}$ denote one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $C > 0$ be some constant. Suppose ν is a $\mathbb{K}$ -valued measure on $\mathcal{A}$, such that*

$$|\nu(A)| \leq C\mu(A), \quad \forall A \in \mathcal{A}.$$

Then there exists some function $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, such that

$$(5) \quad \nu(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}.$$

Moreover:

- (i) *Any function $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, satisfying (5) has the property $|f| \leq C$, μ -a.e. If ν is an “honest” measure, then one also has the inequality $|f| \geq 0$, μ -a.e.*
- (ii) *A function satisfying (5) is essentially unique, in the sense that, whenever $f_1, f_2 \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ satisfy (5), it follows that $f_1 = f_2$, μ -a.e.*

PROOF. The idea is to somehow make sense of $\int_X h d\nu$, for suitable measurable functions h , and to examine the properties of such a number relative to the integral $\int_X h d\mu$. The second integral is of course defined, for instance for $h \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$, but the first integral is not, because ν is not an “honest” measure. The proof will be carried on in several steps.

Step 1: There exist four “honest” finite measures ν_k , $k = 1, 2, 3, 4$, and numbers α_k , $k = 1, 2, 3, 4$, such that $\nu = \alpha_1\nu_1 + \alpha_2\nu_2 + \alpha_3\nu_3 + \alpha_4\nu_4$, and

$$(6) \quad \nu_k \leq C\mu, \quad \forall k = 1, 2, 3, 4.$$

In the case $\mathbb{K} = \mathbb{R}$ we use the Hahn-Jordan decomposition $\nu = \nu^+ - \nu^-$. We also know that $\nu^\pm \leq |\nu|$, the variation measure of ν . In this case we take $\alpha_1 = 1$, $\nu_1 = \nu^+$, $\alpha_2 = -1$, $\nu_2 = \nu^-$, and we set $\nu_3 = \nu_4 = 0$, $\alpha_3 = \alpha_4 = 0$.

In the case $\mathbb{K} = \mathbb{C}$, we write $\nu = \eta + i\lambda$, with η and λ finite signed measures, and we use the Hahn-Jordan decompositions $\eta = \eta^+ - \eta^-$ and $\lambda = \lambda^+ - \lambda^-$. We also know that the variation measures of η and λ satisfy $|\eta| \leq |\nu|$ and $|\lambda| \leq |\nu|$, so we also have $\eta^\pm \leq |\nu|$ and $\lambda^\pm \leq |\nu|$. In this case we can then take $\alpha_1 = 1$, $\nu_1 = \eta^+$, $\alpha_2 = -1$, $\nu_2 = \eta^-$, $\alpha_3 = i$, $\nu_3 = \lambda^+$, $\alpha_4 = -i$, $\nu_4 = \lambda^-$.

Notice that in either case we have

$$\nu_k \leq |\nu|, \quad \forall k = 1, 2, 3, 4.$$

By Remark III.8.5 it follows that we have $|\nu| \leq C\mu$, so we immediately get the inequalities (6).

Step 2: For any measurable function $h : X \rightarrow [0, \infty]$, one has the inequality

$$(7) \quad \int_X h d\nu_k \leq C \int_X h d\mu, \quad \forall k = 1, 2, 3, 4.$$

To prove this, we choose a sequence of elementary functions $(h_n)_{n=1}^\infty \subset \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with

- $0 \leq h_1 \leq h_2 \leq \dots$ (everywhere),
- $\lim_{n \rightarrow \infty} h_n(x) = h(x)$, $\forall x \in X$,

so that by the General Monotone Convergence Theorem, we get the equalities

$$\int_X h d\mu = \lim_{n \rightarrow \infty} \int_X h_n d\mu \quad \text{and} \quad \int_X h d\nu_k = \lim_{n \rightarrow \infty} \int_X h_n d\nu_k, \quad \forall k = 1, 2, 3, 4.$$

This means that, in order to prove (7), it suffices to prove it under the extra assumption that h is elementary. In this case, we have

$$h = \beta_1 \chi_{B_1} + \dots + \beta_p \chi_{B_p},$$

with $\beta_1, \dots, \beta_p \geq 0$ and $B_1, \dots, B_p \in \mathcal{A}$. The inequality is then immediate, from (6) since we have

$$\int_X h d\nu_k = \sum_{j=1}^p \beta_j \nu_k(B_j) \leq C \sum_{j=1}^p \mu(B_j) = C \int_X h d\mu.$$

As a consequence of Step 2, we get the fact that, for every $k = 1, 2, 3, 4$, one has the inclusions

$$\mathfrak{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu) \subset \mathfrak{L}^1_{\mathbb{K}}(X, \mathcal{A}, \nu_k) \quad \text{and} \quad \mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \mu) \subset \mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \nu_k).$$

Taking quotients, this gives rise to correctly defined linear maps

$$(8) \quad \Phi_k : L^1_{\mathbb{K}}(X, \mathcal{A}, \mu) \ni h \longmapsto h \in L^1_{\mathbb{K}}(X, \mathcal{A}, \nu_k), \quad k = 1, 2, 3, 4.$$

(Here we use the abusive notation that identifies an element in L^1 with a function in $\mathfrak{L}^1$, which is defined almost uniquely.) Moreover, one has the inequality

$$\int_X |h| d\nu_k \leq C \int_X |h| d\mu, \quad \forall h \in \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu), \quad k = 1, 2, 3, 4,$$

in other words, the linear maps (8) are all continuous. For every $k = 1, 2, 3, 4$, let ϕ_k denote the integration map

$$\phi_k : L_{\mathbb{K}}^1(X, \mathcal{A}, \nu_k) \ni h \longmapsto \int_X h d\nu_k \in \mathbb{K}.$$

We know (see Remark 3.5) that the ϕ_k 's are continuous. In particular, the compositions $\psi_k = \phi_k \circ \Phi_k : L_{\mathbb{K}}^1(X, \mathcal{A}, \mu) \rightarrow \mathbb{K}$, which are defined by

$$\psi_k : L_{\mathbb{K}}^1(X, \mathcal{A}, \mu) \ni h \longmapsto \int_X h d\nu_k, \quad k = 1, 2, 3, 4,$$

are linear and continuous.

We now use Proposition 3.3 which states that one has an inclusion

$$(9) \quad \Theta : L_{\mathbb{K}}^2(X, \mathcal{A}, \mu) \hookrightarrow L_{\mathbb{K}}^1(X, \mathcal{A}, \mu),$$

which is in fact a linear continuous map. So if we consider the compositions $\theta_k = \psi_k \circ \Theta$, which are defined by

$$\theta_k : L_{\mathbb{K}}^1(X, \mathcal{A}, \mu) \ni h \longmapsto \int_X h d\nu_k, \quad k = 1, 2, 3, 4,$$

then these compositions are linear and continuous. Apply then Riesz Theorem (in the form given in Remark 3.4), to find functions $f_1, f_2, f_3, f_4 \in L_{\mathbb{K}}^2(X, \mathcal{A}, \mu)$, such that

$$\theta_k(h) = \langle f_k, h \rangle, \quad \forall h \in L_{\mathbb{K}}^2(X, \mathcal{A}, \mu), \quad k = 1, 2, 3, 4.$$

In particular, using functions of the form $h = \chi_A$, $A \in \mathcal{A}$ (which all belong to $L_{\mathbb{K}}^2(X, \mathcal{A}, \mu)$, due to the finiteness of μ), we get

$$\nu_k(A) = \int_X \chi_A d\nu_k = \int_X f_k \chi_A d\mu, \quad \forall A \in \mathcal{A}, \quad k = 1, 2, 3, 4.$$

Finally, if we define the function $f = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \alpha_4 f_4 \in L_{\mathbb{K}}^2(X, \mathcal{A}, \mu)$, then the above equalities immediately give the equality (5).

At this point we only know that f belongs to $L_{\mathbb{K}}^2(X, \mathcal{A}, \mu)$. Using the inclusion (9), it turns out that f indeed belongs to $L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$.

Let us prove now the additional properties (i) and (ii).

To prove the first assertion in (i), we start off by fixing some function $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, which satisfies (5), and we define the set

$$A = \{x \in X : |f(x)| > C\},$$

for which we must prove that $\mu(A) = 0$. Since f is measurable, it follows that A belongs to $\mathcal{A}$. Consider the “rational unit sphere” $S_{\mathbb{Q}}^1$ in $\mathbb{K}$, defined as

$$(10) \quad S_{\mathbb{Q}}^1 = \begin{cases} \{-1, 1\} & \text{if } \mathbb{K} = \mathbb{R} \\ \{e^{2\pi i t} : t \in \mathbb{Q}\} & \text{if } \mathbb{K} = \mathbb{C} \end{cases}$$

The point is that $S_{\mathbb{Q}}^1$ is dense in the unit sphere S^1 in $\mathbb{K}$:

$$S^1 = \{\alpha \in \mathbb{K} : |\alpha| = 1\},$$

so we immediately have the equality $A = \bigcup_{\alpha \in S_{\mathbb{Q}}^1} A_{\alpha}$, where

$$A_{\alpha} = \{x \in X : \operatorname{Re}[\alpha f(x)] > C\}.$$

Since $S_{\mathbb{Q}}^1$ is countable, in order to prove that $\mu(A) = 0$, it then suffices to show that $\mu(A_{\alpha}) = 0$, $\forall \alpha \in S_{\mathbb{Q}}^1$. Fix then $\alpha \in S_{\mathbb{Q}}^1$, and consider the $\mathbb{K}$ -valued measure $\eta = \alpha\nu$. It is clear that we still have

$$(11) \quad |\eta(A)| = |\nu(A)| \leq C\mu(A), \quad \forall A \in \mathcal{A},$$

as well as the equality

$$(12) \quad \eta(A) = \int_A \alpha f d\mu, \quad \forall A \in \mathcal{A}.$$

For each integer $n \geq 1$, let us define the set

$$A_{\alpha}^n = \{x \in X : \operatorname{Re}[\alpha f(x)] \geq C + \frac{1}{n}\},$$

so that we obviously have the equality $A_{\alpha} = \bigcup_{n=1}^{\infty} A_{\alpha}^n$. In particular, in order to prove $\mu(A_{\alpha}) = 0$, it suffices to prove that $\mu(A_{\alpha}^n) = 0$, $\forall n \geq 1$. Fix for the moment $n \geq 1$. Using (12), it follows that

$$\operatorname{Re} \eta(A_{\alpha}^n) = \operatorname{Re} \left[\int_{A_{\alpha}^n} \alpha f d\mu \right] = \int_{A_{\alpha}^n} \operatorname{Re}[\alpha f] d\mu = \int_X \operatorname{Re}[\alpha f] \chi_{A_{\alpha}^n} d\mu.$$

Since we have $\operatorname{Re}[\alpha f] \chi_{A_{\alpha}^n} \geq (C + \frac{1}{n}) \chi_{A_{\alpha}^n}$, the above inequality can be continued with

$$\operatorname{Re} \eta(A_{\alpha}^n) \geq \int_X (C + \frac{1}{n}) \chi_{A_{\alpha}^n} d\mu = (C + \frac{1}{n}) \mu(A_{\alpha}^n).$$

Of course, this will give

$$|\eta(A_{\alpha}^n)| \geq \operatorname{Re} \eta(A_{\alpha}^n) \geq (C + \frac{1}{n}) \mu(A_{\alpha}^n).$$

Note now that, using (11), this will finally give

$$C\mu(A_{\alpha}^n) \geq (C + \frac{1}{n}) \mu(A_{\alpha}^n),$$

which clearly forces $\mu(A_{\alpha}^n) = 0$.

Having proven that $|f| \leq C$, μ -a.e., let us turn our attention now to the uniqueness property (ii). Suppose $f_1, f_2 \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ are such that

$$\nu(A) = \int_A f_1 d\mu = \int_A f_2 d\mu, \quad \forall A \in \mathcal{A}.$$

Consider then the difference $f = f_1 - f_2$ and the trivial measure $\nu_0 = 0$. Obviously we have

$$|\nu_0(A)| \leq \frac{1}{n} \mu(A), \quad \forall A \in \mathcal{A},$$

for every integer $n \geq 1$, as well as

$$\nu_0(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}.$$

By the first assertion in (i), it follows that

$$|f_1 - f_2| = |f| \leq \frac{1}{n}, \quad \mu\text{-a.e.},$$

for every $n \geq 1$. So if we take the sets $(N_n)_{n=1}^{\infty} \subset \mathcal{A}$ defined by

$$N_n = \{x \in X : |f_1(x) - f_2(x)| > \frac{1}{n}\},$$

then $\mu(N_n) = 0, \forall n \geq 1$. Of course, if we put $N = \bigcup_{n=1}^{\infty} N_n$, then on the one hand we have $\mu(N) = 0$, and on the other hand, we have

$$f_1(x) - f_2(x) = 0, \quad \forall x \in X \setminus N,$$

which means that we indeed have $f_1 = f_2, \mu$ -a.e.

Finally, let us prove the second assertion in (i), which starts with the assumption that ν is an “honest” measure. Let $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ satisfy (5). By the uniqueness property (ii), it follows immediately that

$$f = \operatorname{Re} f, \quad \mu\text{-a.e.},$$

so we can assume that f is already real-valued. Consider the “honest” measure $\omega = C\mu - \nu$, and notice that the function $g : X \rightarrow \mathbb{R}$ defined by

$$g(x) = C - f(x), \quad \forall x \in X,$$

clearly has the property

$$\omega(A) = \int_A g d\mu, \quad \forall A \in \mathcal{A}.$$

Since we obviously have

$$0 \leq \omega(A) \leq C\mu(A), \quad \forall A \in \mathcal{A},$$

by the first assertion of (i), applied to the measure ω and the function g , it follows that $|g| \leq C, \mu$ -a.e. In other words, we have now a combined inequality:

$$\max\{|f|, |C - f|\} \leq C, \quad \mu\text{-a.e.}$$

Of course, since f is real valued, this forces $f \geq 0, \mu$ -a.e. □

In what follows we are going to offer various generalizations of Theorem 4.1. There are several directions in which Theorem 4.1 can be generalized. The main direction, which we present here, will aim at weakening the condition $|\nu| \leq C\mu$. The following result explains that in fact the case of $\mathbb{K}$ -valued measures can be always reduced to the case of “honest” finite ones.

PROPOSITION 4.5 (Polar Decomposition). *Let $\mathcal{A}$ be a σ -algebra on a non-empty set X , let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let ν be a $\mathbb{K}$ -valued measure on $\mathcal{A}$. Let $|\nu|$ denote the variation measure of ν . There exists some function $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, |\nu|)$, such that*

$$(13) \quad \nu(A) = \int_A f d|\nu|, \quad \forall A \in \mathcal{A}.$$

Moreover

- (i) *Any function $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, |\nu|)$, satisfying (13) has the property $|f| = 1, |\nu|$ -a.e.*
- (ii) *A function satisfying (13) is essentially unique, in the sense that, whenever $f_1, f_2 \in L_{\mathbb{K}}^1(X, \mathcal{A}, |\nu|)$ satisfy (13), it follows that $f_1 = f_2, |\nu|$ -a.e.*

PROOF. We know that

$$|\nu(A)| \leq |\nu|(A), \quad \forall A \in \mathcal{A}.$$

So if we apply Theorem 4.1 for the finite measure $\mu = |\nu|$ and $C = 1$, we immediately get the existence of $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, |\nu|)$, satisfying (13). Again by Theorem 4.1, the

uniqueness property (ii) is automatic, and we also have $|f| \leq 1$, $|\nu|$ -a.e. To prove the fact that we have in fact the equality $|f| = 1$, $|\nu|$ -a.e., we define the set

$$A = \{x \in X : |f(x)| < 1\},$$

which belongs to $\mathcal{A}$, and we prove that $|\nu|(A) = 0$. If we define the sequence of sets $(A_n)_{n=1}^\infty \subset \mathcal{A}$, by

$$A_n = \{x \in X : |f(x)| \leq 1 - \frac{1}{n}\}, \quad \forall n \geq 1,$$

then we clearly have $A = \bigcup_{n=1}^\infty A_n$, so all we have to show is the fact that $|\nu|(A_n) = 0$, $\forall n \geq 1$. Fix $n \geq 1$. For every $B \in \mathcal{A}$, with $B \subset A_n$, we have

$$|f(x)| \leq 1 - \frac{1}{n}, \quad \forall x \in B,$$

so using (13) we get

$$|\nu(B)| = \left| \int_B f d|\nu| \right| \leq \int_B |f| d|\nu| \leq \int_B (1 - \frac{1}{n}) d|\nu| = (1 - \frac{1}{n})|\nu|(B).$$

Now if we take an arbitrary pairwise disjoint sequence $(B_k)_{k=1}^\infty \subset \mathcal{A}$, with $\bigcup_{k=1}^\infty B_k = A_n$, then the above estimate will give

$$\sum_{k=1}^\infty |\nu(B_k)| \leq (1 - \frac{1}{n}) \sum_{k=1}^\infty |\nu|(B_k) = (1 - \frac{1}{n})|\nu|(A_n).$$

Taking supremum in the left hand side, and using the definition of the variation measure, the above estimate will finally give

$$|\nu|(A_n) \leq (1 - \frac{1}{n})|\nu|(A_n),$$

which clearly forces $|\nu|(A_n) = 0$. $\square$

REMARK 4.2. The case $\mathbb{K} = \mathbb{R}$ can be slightly generalized, to include the case of infinite signed measures. If ν is a signed measure on $\mathcal{A}$ and if we consider the Hahn-Jordan set decomposition (X^+, X^-) , then the density f is simply the function

$$f(x) = \begin{cases} 1 & \text{if } x \in X^+ \\ -1 & \text{if } x \in X^- \end{cases}$$

The equality (13) will then hold only for those sets $A \in \mathcal{A}$ with $|\nu|(A) < \infty$. Since $|\nu|$ is allowed to be infinite, as explained in Example 4.1, the only version of uniqueness property (ii) will hold with “ $|\nu|$ -l.a.e” in place of “ $|\nu|$ -a.e” Likewise, the absolute value property (i) will have to be replaced with “ $|f| = 1$, $|\nu|$ -l.a.e”

COMMENT. Up to this point, it seems that the hypotheses from Theorem 4.1 are essential, particularly the dominance condition $|\nu| \leq C\mu$. It is worth discussing this property in a bit more detail, especially having in mind that we plan to weaken it as much as possible.

NOTATION. Suppose $\mathcal{A}$ is a σ -algebra on some non-empty set X , and suppose μ and ν are “honest” (not necessarily finite) measures on $\mathcal{A}$. We shall write

$$\nu \Subset \mu,$$

if there exists some constant $C > 0$, such that

$$|\nu|(A) \leq C\mu(A), \quad \forall A \in \mathcal{A}.$$

A few steps in the proof of Theorem 4.1 hold even without the finiteness assumption, as indicated by the following.

Exercise 1.* Suppose $\mathcal{A}$ is a σ algebra on some non-empty set X , and suppose μ and ν are “honest” measures on $\mathcal{A}$. Prove the following.

(i) If $\nu \subseteq \mu$, then one has the inclusions

$$\mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \mu) \subset \mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \nu) \text{ and } \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu) \subset \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \nu), \quad \forall p \in [1, \infty).$$

Consequently (see the proof of Theorem 4.1) one has linear maps

$$L_{\mathbb{K}}^p(X, \mathcal{A}, \mu) \ni h \longmapsto h \in L_{\mathbb{K}}^p(X, \mathcal{A}, \nu), \quad \forall p \in [1, \infty).$$

Show that these linear maps are continuous.

(ii) Conversely, assuming one has the inclusion

$$\mathfrak{L}_{\mathbb{K}}^{p_0}(X, \mathcal{A}, \mu) \subset \mathfrak{L}_{\mathbb{K}}^{p_0}(X, \mathcal{A}, \nu),$$

for *some* $p_0 \in [1, \infty)$, prove that $\nu \subseteq \mu$.

HINT: To prove (ii) show first one has the inclusion $\mathfrak{L}_+^1(X, \mathcal{A}, \mu) \subset \mathfrak{L}_+^1(X, \mathcal{A}, \nu)$. Then show that the quantity

$$C = \sup \left\{ \int_X h d\nu : h \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu), \int_X h d\mu \leq 1 \right\}$$

is finite. If $C = \infty$, there exists some sequence $(h_n)_{n=1}^\infty \subset \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, with

$$\int_X h_n d\mu \leq 1 \text{ and } \int_X h d\nu \geq 4^n, \quad \forall n \geq 1.$$

Consider then the series $\sum_{n=1}^\infty \frac{1}{2^n} h_n$, and get a contradiction. Finally prove that $\nu(A) \leq C\mu(A)$, $\forall A \in \mathcal{A}$.

It is the moment now to introduce the following relation, which is a highly non-trivial weakening of the relation $\subseteq$.

DEFINITION. Let $\mathcal{A}$ is a σ -algebra on some non-empty set X , and suppose μ and ν are “honest” (not necessarily finite) measures on $\mathcal{A}$. We say that ν is *absolutely continuous with respect to* μ , if for every $A \in \mathcal{A}$, one has the implication

$$(14) \quad \mu(A) = 0 \implies \nu(A) = 0.$$

In this case we are going to use the notation

$$\nu \ll \mu.$$

It is obvious that one always has the implication

$$\nu \subseteq \mu \implies \nu \ll \mu.$$

REMARKS 4.3. Let $(X, \mathcal{A}, \mu)$ be a measure space. A. If ν is an “honest” measure on $\mathcal{A}$, which has the Radon-Nikodym property relative to μ , then $\nu \ll \mu$. This is pretty obvious, since if we pick $f : X \rightarrow [0, \infty]$ to be a density for ν relative to μ , then for every $A \in \mathcal{A}$ with $\mu(A) = 0$, we have $f\chi_A = 0$, μ -a.e., so we get

$$\nu(A) = \int_A f d\mu = \int_X f\chi_A d\mu = 0.$$

B. For an “honest” measure ν on $\mathcal{A}$, the relation $\nu \ll \mu$ is equivalent to the inclusion

$$\mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \mu) \subset \mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \nu).$$

By Exercise 1, this already suggests that the relation $\ll$ is much weaker than $\subseteq$ (see Exercise 2 below).

C. If ν is either a signed or a complex measure on $\mathcal{A}$, then the following are equivalent:

(i) the variation measure $|\nu|$ is absolutely continuous with respect to μ ;

(ii) for every $A \in \mathcal{A}$, one has the implication (14)

The implication (i) $\Rightarrow$ (ii) is trivial, since one has

$$|\nu(A)| \leq |\nu|(A), \quad \forall A \in \mathcal{A}.$$

The implication (ii) $\Rightarrow$ (i) is also clear, since if we start with some $A \in \mathcal{A}$ with $\mu(A) = 0$, then we get $|\nu(B)| = 0$, for all $B \in \mathcal{A}$ with $B \subset A$, and then arguing exactly as in the proof of Proposition 4.3, we get $|\nu|(A) = 0$.

CONVENTION. Using Remark 4.2.A, we extend the definition of absolute continuity, and the notation $\nu \ll \mu$ to include the case when ν is either a signed measure, or a complex measure on $\mathcal{A}$. In other words, the notation $\nu \ll \mu$ means that $|\nu| \ll \mu$.

The following technical result is key for the second Radon-Nikodym Theorem.

LEMMA 4.1. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let ν be an "honest" measure on $\mathcal{A}$, with $\nu \ll \mu$. Then there exists a sequence $(\nu_n)_{n=1}^\infty$ of "honest" measures on $\mathcal{A}$, such that*

- (i) $\nu_n \ll \mu, \forall n \geq 1$; in particular the measures $\nu_n, n \geq 1$ are all finite;
- (ii) $\nu_1 \leq \nu_2 \leq \dots$;
- (iii) $\lim_{n \rightarrow \infty} \nu_n(A) = \nu(A), \forall A \in \mathcal{A}$.

PROOF. Let us define

$$\nu_n = (n\mu) \wedge \nu, \quad \forall n \geq 1.$$

Recall (see III.8, the Lattice Property; it is essential here that one of the measures, namely $n\mu$, is finite) that by construction ν_n has the following properties:

- (a) $\nu_n \leq n\mu$ and $\nu_n \leq \nu$;
- (b) whenever ω is a measure with $\omega \leq n\mu$ and $\omega \leq \nu$, it follows that $\omega \leq \nu_n$.

Property (a) above already gives condition (i). It will be helpful to notice that property (a) also gives the inequality

$$(15) \quad \nu_n \leq \nu, \quad \forall n \geq 1.$$

The monotonicity condition is now trivial, since by (b) the inequalities $\nu_{n-1} \leq (n-1)\mu \leq n\mu$ and $\nu_{n-1} \leq \nu$, imply $\nu_{n-1} \leq (n\mu) \wedge \nu = \nu_n$.

To derive property (iii), it will be helpful to recall the actual definition of the operation $\wedge$. Fix for the moment $n \geq 1$. One first considers the signed measure $\lambda_n = n\mu - \nu$, and its Hahn-Jordan decomposition $\lambda_n = \lambda_n^+ - \lambda_n^-$. In our case, we get $\lambda_n^+ \leq n\mu$ and $\lambda_n^- \leq \nu$. With these notations the measures ν_n are defined by $\nu_n = n\mu - \lambda_n^+, \forall n \geq 1$. If we fix, for each $n \geq 1$, a Hahn-Jordan set decomposition (X_n^+, X_n^-) for λ_n relative to λ_n , then we have

$$(16) \quad \nu_n(A) = \nu(A \cap X_n^+) + n\mu(A \cap X_n^-), \quad \forall A \in \mathcal{A}, n \geq 1.$$

Consider then the sets $X_\infty^+ = \bigcup_{n=1}^\infty X_n^+$ and $X_\infty^- = \bigcap_{n=1}^\infty X_n^-$. It is clear that $X_\infty^\pm \in \mathcal{A}$, and $X_\infty^- = X \setminus X_\infty^+$.

Fix now a set $A \in \mathcal{A}$, and let us prove the equality (iii). On the one hand, the obvious inclusions $X_n^- \supset X_\infty^-$, combined with (16), give the inequalities

$$(17) \quad \nu_n(A) \geq \nu(A \cap X_n^+) + n\mu(A \cap X_\infty^-), \quad \forall n \geq 1.$$

On the other hand, since $\lambda_{n+1} = \mu + \lambda_n, \forall n \geq 1$, using Lemma III.8.2, we get the relations

$$X_1^+ \subset_\mu X_2^+ \subset_\mu \dots$$

(Recall that the notation $D \underset{\mu}{\subset} E$ stands for $\mu(D \setminus E) = 0$.) Since $\nu \ll \mu$, we also have the relations

$$A \cap X_1^+ \underset{\nu}{\subset} A \cap X_2^+ \underset{\nu}{\subset} \dots,$$

so using Proposition III.4.3, one gets the equality

$$\nu(A \cap X_\infty^+) = \lim_{n \rightarrow \infty} \nu(A \cap X_n^+).$$

Combining this with the inequalities (15) and (17) then yields the inequality

$$(18) \quad \nu(A) \geq \limsup_{n \rightarrow \infty} \nu_n(A) \geq \liminf_{n \rightarrow \infty} \nu_n(A) \geq \nu(A \cap X_\infty^+) + \lim_{n \rightarrow \infty} [n\mu(A \cap X_\infty^-)].$$

There are two possibilities here.

Case I: $\mu(A \cap X_\infty^-) > 0$.

In this case, the estimate (18) forces

$$\nu(A) = \limsup_{n \rightarrow \infty} \nu_n(A) = \liminf_{n \rightarrow \infty} \nu_n(A) = \infty.$$

Case II: $\mu(A \cap X_\infty^-) = 0$.

In this case, using absolute continuity, we get $\nu(A \cap X_\infty^-) = 0$, and the equality $A = (A \cap X_\infty^+) \cup (A \cap X_\infty^-)$ yields

$$\nu(A) = \nu(A \cap X_\infty^+).$$

Then (18) forces

$$\limsup_{n \rightarrow \infty} \nu_n(A) = \liminf_{n \rightarrow \infty} \nu_n(A) = \nu(A).$$

In either case, the conclusion is the same: $\lim_{n \rightarrow \infty} \nu_n(A) = \nu(A)$. □

After the above preparation, we are now in position to prove the following.

THEOREM 4.2 (Radon-Nikodym Theorem: the finite case). *Let $(X, \mathcal{A}, \mu)$ be a finite measure space.*

A. If ν is an “honest” measure on $\mathcal{A}$, with $\nu \ll \mu$, then there exists a measurable function $f : X \rightarrow [0, \infty]$, such that

$$(19) \quad \nu(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}.$$

Moreover, such a function is essentially unique, in the sense that, whenever $f_1, f_2 : X \rightarrow [0, \infty]$ are measurable functions, that satisfy (19), it follows that $f_1 = f_2$, μ -a.e.

B. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. If λ is a $\mathbb{K}$ -valued measure on $\mathcal{A}$, with $\lambda \ll \mu$, then there exists a function $f \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$, such that

$$(20) \quad \lambda(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}.$$

Moreover:

- (i) *A function $f \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$ satisfying (20) is essentially unique, in the sense that, whenever $f_1, f_2 \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$ satisfy (20), it follows that $f_1 = f_2$, μ -a.e.*
- (ii) *If $f \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$ is any function satisfying (20), then the variation measure $|\lambda|$ of λ is given by*

$$|\lambda|(A) = \int_A |f| d\mu, \quad \forall A \in \mathcal{A}.$$

PROOF. A. Use Lemma 4.1 to find a sequence $(\nu_n)_{n=1}^\infty$ of “honest” measures on $\mathcal{A}$, such that

- $\nu_n \in \mu$, $\forall n \geq 1$; in particular the measures ν_n , $n \geq 1$ are all finite;
- $\nu_1 \leq \nu_2 \leq \dots$;
- $\lim_{n \rightarrow \infty} \nu_n(A) = \nu(A)$, $\forall A \in \mathcal{A}$.

For each $n \geq 1$, we apply the “Easy” Radon-Nikodym Theorem 4.1, to find some measurable function $f_n : X \rightarrow \mathbb{R}$, such that

$$\nu_n(A) = \int_A f_n d\mu, \quad \forall A \in \mathcal{A}.$$

Claim: The sequence $(f_n)_{n=1}^\infty$ satisfies

$$0 \leq f_n \leq f_{n+1}, \quad \mu\text{-a.e.}, \quad \forall n \geq 1.$$

Fix $n \geq 1$. On the one hand, since the ν_n ’s are “honest” finite measures, and $\nu_n \in \mu$, by part (i) of Theorem 4.1, it follows that $f_n \geq 0$, μ -a.e. On other hand, since $\nu_{n+1} - \nu_n$ is also an “honest” finite measure with $\nu_{n+1} - \nu_n \in \mu$, and with density $f_{n+1} - f_n$, again by part (i) of Theorem 4.1, it follows that $f_{n+1} - f_n \geq 0$, μ -a.e.

Having proven the above Claim, let us define the function $f : X \rightarrow [0, \infty]$, by

$$f(x) = \liminf_{n \rightarrow \infty} [\max\{f_n(x), 0\}] \quad \forall x \in X.$$

It is obvious that f is measurable. By the Claim, we have in fact the equality

$$f = \mu\text{-a.e.} \lim_{n \rightarrow \infty} f_n.$$

Since we also have

$$f\chi_A = \mu\text{-a.e.} \lim_{n \rightarrow \infty} f_n\chi_A, \quad \forall A \in \mathcal{A},$$

using the Claim and the Monotone Convergence Theorem, we get

$$\begin{aligned} \int_A f d\mu &= \int_X f\chi_A d\mu = \lim_{n \rightarrow \infty} \int_X f_n\chi_A d\mu = \lim_{n \rightarrow \infty} \int_A f_n d\mu = \\ &= \lim_{n \rightarrow \infty} \nu_n(A) = \nu(A), \quad \forall A \in \mathcal{A}. \end{aligned}$$

Having shown that f satisfies (19), let us observe that the uniqueness property stated in part A is a consequence of Proposition 4.2.

B. Let λ be a $\mathbb{K}$ -valued. In particular, the variation measure $|\lambda|$ is finite, so by the Polar Decomposition (Proposition 4.3) there exists some measurable function $h : X \rightarrow \mathbb{K}$, such that

$$(21) \quad \lambda(A) = \int_A h d|\lambda|, \quad \forall A \in \mathcal{A},$$

and such that $|h| = 1$, $|\lambda|$ -a.e. Replacing h with the measurable function $h' : X \rightarrow \mathbb{K}$, defined by

$$h'(x) = \begin{cases} h(x) & \text{if } |h(x)| = 1 \\ 1 & \text{if } |h(x)| \neq 1 \end{cases}$$

we can assume that in fact we have

$$|h(x)| = 1, \quad \forall x \in X.$$

Apply then part A, to the measure $|\lambda|$, which is again absolutely continuous with respect to μ , to find some measurable function $g : X \rightarrow [0, \infty]$, such that

$$|\lambda|(A) = \int_A g \, d\mu, \quad \forall A \in \mathcal{A}.$$

Remark that, since

$$\int_X g \, d\mu = |\lambda|(X) < \infty,$$

it follows that $g \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. Fix for the moment some set $A \in \mathcal{A}$. On the one hand, since

$$(22) \quad |h\kappa_A| \leq 1,$$

and $|\lambda|$ is finite, it follows that $h\kappa_A \in L_{\mathbb{K}}^1(X, \mathcal{A}, |\lambda|)$. On the other hand, since $g \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, using (22) we get the fact that $h\kappa_A g \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$. Using the Change of Variable formula (Proposition 4.1) we then get the equality

$$\int_X h\kappa_A \, d|\lambda| = \int_X h\kappa_A g \, d\mu,$$

which by (21) reads:

$$\lambda(A) = \int_A hg \, d\mu.$$

Now the function $f_0 = hg$ (which has $|f_0| = g$) belongs to $L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, and clearly satisfies (20).

To prove the uniqueness property (i), we start with two functions $f_1, f_2 \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ which satisfy

$$\int_A f_1 \, d\mu = \int_A f_2 \, d\mu = \lambda(A), \quad \forall A \in \mathcal{A}.$$

If we define the function $\varphi = f_1 - f_2 \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, then we clearly have

$$\int_A \varphi \, d\mu = \int_A 0 \, d\mu = \omega(A), \quad \forall A \in \mathcal{A},$$

where ω is the zero measure. Since $\omega \leq \mu$, using Theorem 4.1 it follows that $\varphi = 0$, μ -a.e.

To prove (ii) we start with some $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ that satisfies (20), and we use the uniqueness property (i) to get the equality $f = f_0$, μ -a.e., where f_0 is the function constructed above. In particular, using the construction of f_0 , the fact that $|f_0| = g$, and the fact that g is a density for $|\lambda|$ relative to μ , we get

$$\int_A |f| \, d\mu = \int_A |f_0| \, d\mu = \int_A g \, d\mu = |\lambda|(A), \quad \forall A \in \mathcal{A}. \quad \square$$

At this point we would like to go further, beyond the finite case. The following generalization of Theorem 4.2 is pretty straightforward.

COROLLARY 4.1 (Radon-Nikodym Theorem: the σ -finite case). *Let $(X, \mathcal{A}, \mu)$ be a σ -finite measure space.*

A. If ν is an "honest" measure on $\mathcal{A}$, with $\nu \ll \mu$, then there exists a measurable function $f : X \rightarrow [0, \infty]$, such that

$$(23) \quad \nu(A) = \int_A f \, d\mu, \quad \forall A \in \mathcal{A}.$$

Moreover, such a function is essentially unique, in the sense that, whenever $f_1, f_2 : X \rightarrow [0, \infty]$ are measurable functions, that satisfy (19), it follows that $f_1 = f_2$, μ -a.e.

B. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. If λ is a $\mathbb{K}$ -valued measure on $\mathcal{A}$, with $\lambda \ll \mu$, then there exists a function $f \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$, such that

$$(24) \quad \lambda(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}.$$

Moreover:

- (i) A function $f \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$ satisfying (20) is essentially unique, in the sense that, whenever $f_1, f_2 \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$ satisfy (24), it follows that $f_1 = f_2$, μ -a.e.
- (ii) If $f \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$ is any function satisfying (24), then the variation measure $|\lambda|$ of λ is given by

$$|\lambda|(A) = \int_A |f| d\mu, \quad \forall A \in \mathcal{A}.$$

PROOF. Since μ is σ -finite, there exists a sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{A}_{\text{fin}}^{\mu}$, with $\bigcup_{n=1}^{\infty} A_n = X$. Put $X_1 = A_1$ and $X_n = A_n \setminus (A_1 \cup \dots \cup A_{n-1})$, $\forall n \geq 2$. Then $(X_n)_{n=1}^{\infty} \subset \mathcal{A}_{\text{fin}}^{\mu}$ is pairwise disjoint, and we still have $\bigcup_{n=1}^{\infty} X_n = X$. The Corollary follows then immediately from Theorem 4.2, applied to the measure spaces $(X_n, \mathcal{A}|_{X_n}, \mu|_{X_n})$ and the measures $\nu|_{X_n}$ and $\lambda|_{X_n}$ respectively. What is used here is the fact that, if $\mathbf{K}$ denotes one of the sets $[0, \infty]$, $\mathbb{R}$ or $\mathbb{C}$, then for a function $f : X \rightarrow \mathbf{K}$ the fact that f is measurable, is equivalent to the fact that $f|_{X_n}$ is measurable for each $n \geq 1$. Moreover, given two functions $f_1, f_2 : X \rightarrow \mathbf{K}$, the condition $f_1 = f_2$, μ -a.e. is equivalent to the fact that $f_1|_{X_n} = f_2|_{X_n}$, μ -a.e., $\forall n \geq 1$. Finally, for $f : X \rightarrow \mathbb{K} (= \mathbb{R}, \mathbb{C})$, the condition $f \in \mathfrak{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$, is equivalent to the fact that $f|_{X_n} \in \mathfrak{L}^1_{\mathbb{K}}(X_n, \mathcal{A}|_{X_n}, \mu|_{X_n})$, $\forall n \geq 1$, and

$$\sum_{n=1}^{\infty} \int_{X_n} (f|_{X_n}) d(\mu|_{X_n}) < \infty. \quad \square$$

COMMENT. The σ -finite case of the Radon-Nikodym Theorem, given above, is in fact a particular case of a more general version (Theorem 4.3 below). In order to formulate this, we need a concept which has already appeared earlier in III.5. Recall that a measure space $(X, \mathcal{A}, \mu)$ is said to be *decomposable*, if there exists a pairwise disjoint subcollection $\mathcal{F} \subset \mathcal{A}_{\text{fin}}^{\mu}$, such that

- (i) $\bigcup_{F \in \mathcal{F}} F = X$;
- (ii) for a set $A \subset X$, the condition $A \in \mathcal{A}$ is equivalent to the condition

$$A \cap F \in \mathcal{A}, \quad \forall F \in \mathcal{F};$$

- (iii) one has the equality

$$\mu(A) = \sum_{F \in \mathcal{F}} \mu(A \cap F), \quad \forall A \in \mathcal{A}_{\text{fin}}^{\mu}.$$

Such a collection $\mathcal{F}$ is then called a *decomposition* of $(X, \mathcal{A}, \mu)$. Condition (ii) is referred to as the *patching property*, because it characterizes measurability as follows.

- (P) Given a measurable space $(Y, \mathcal{B})$, a function $f : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ is measurable, if and only if all restrictions $f|_F : (F, \mathcal{A}|_F) \rightarrow (Y, \mathcal{B})$, $F \in \mathcal{F}$, are measurable.

THEOREM 4.3 (Radon-Nikodym Theorem: the decomposable case). *Let $(X, \mathcal{A}, \mu)$ be a decomposable measure space. Let $\mathcal{A}_{\sigma\text{-fin}}^\mu$ be the collection of all μ - σ -finite sets in $\mathcal{A}$, that is,*

$$\mathcal{A}_{\sigma\text{-fin}}^\mu = \left\{ A \in \mathcal{A} : \text{there exists } (A_n)_{n=1}^\infty \subset \mathcal{A}_{\text{fin}}^\mu, \text{ with } A = \bigcup_{n=1}^\infty A_n \right\}.$$

A. If ν is an “honest” measure on $\mathcal{A}$, with $\nu \ll \mu$, then there exists a measurable function $f : X \rightarrow [0, \infty]$, such that

$$(25) \quad \nu(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}_{\sigma\text{-fin}}^\mu.$$

Moreover, such a function is locally essentially unique, in the sense that, whenever $f_1, f_2 : X \rightarrow [0, \infty]$ are measurable functions, that satisfy (25), it follows that $f_1 = f_2$, μ -l.a.e.

B. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. If λ is a $\mathbb{K}$ -valued measure on $\mathcal{A}$, with $\lambda \ll \mu$, then there exists a function $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, such that

$$(26) \quad \lambda(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}_{\sigma\text{-fin}}^\mu.$$

Moreover:

- (i) A function $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ satisfying (26) is essentially unique, in the sense that, whenever $f_1, f_2 \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ satisfy (26), it follows that $f_1 = f_2$, μ -a.e.
- (ii) If $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ is any function satisfying (26), then the variation measure $|\lambda|$, of λ , satisfies

$$|\lambda|(A) = \int_A |f| d\mu, \quad \forall A \in \mathcal{A}_{\sigma\text{-fin}}^\mu.$$

PROOF. Fix $\mathcal{F}$ to be a decomposition for $(X, \mathcal{A}, \mu)$.

A. For every $F \in \mathcal{F}$, we apply Theorem 4.2 to the measure space $(F, \mathcal{A}|_F, \mu|_F)$ and the measure $\nu|_F$, to find some measurable function $f_F : F \rightarrow [0, \infty]$, such that

$$\nu(A) = \int_A f_F d\mu, \quad \forall A \in \mathcal{A}|_F.$$

Using the patching property, there exists a measurable function $f : X \rightarrow [0, \infty]$, such that $f|_F = f_F$, $\forall F \in \mathcal{F}$. The key feature we are going to prove is a particular case of (25).

Claim 1: $\nu(A) = \int_A f d\mu$, $\forall A \in \mathcal{A}_{\text{fin}}^\mu$.

Fix $A \in \mathcal{A}_{\text{fin}}^\mu$. On the one hand, we know that

$$\mu(A) = \sum_{F \in \mathcal{F}} \mu(A \cap F).$$

Since the sum is finite, it follows that the subcollection

$$\mathcal{F}(A) = \{F \in \mathcal{F} : \mu(A \cap F) > 0\}$$

is at most countable. We then form the set $\tilde{A} = \bigcup_{F \in \mathcal{F}(A)} [A \cap F]$, which is clearly a subset of A . The difference $D = A \setminus \tilde{A}$ has again $\mu(D) < \infty$, so its measure is also given as

$$\mu(D) = \sum_{F \in \mathcal{F}} \mu(D \cap F).$$

Notice however that we have $\mu(D \cap F) = 0$, $\forall F \in \mathcal{F}$. (If $F \in \mathcal{F}(A)$, we already have $D \cap F = \emptyset$, whereas if $F \in \mathcal{F} \setminus \mathcal{F}(A)$, we have $D \cap F \subset A \cap F$, with $\mu(A \cap F) = 0$.) Using then the above equality, we get $\mu(D) = 0$. By absolute continuity we also get $\nu(D) = 0$. Using the equality $A = \tilde{A} \cup D$, and σ -additivity (it is essential here that $\mathcal{F}(A)$ is countable), it follows that

$$\nu(A) = \nu(\tilde{A}) = \sum_{F \in \mathcal{F}(A)} \nu(A \cap F).$$

Using the hypothesis, we then get

$$(27) \quad \nu(A) = \sum_{F \in \mathcal{F}(A)} \int_{A \cap F} f \, d\mu.$$

Now if we list $\mathcal{F}(A) = \{F_k\}_{k=1}^\infty$, and if we take a partial sum, we have

$$\sum_{k=1}^n \int_{A \cap F_k} f \, d\mu = \int_{G_n} f \, d\mu = \int_X f \chi_{G_n} \, d\mu,$$

where

$$G_n = \bigcup_{k=1}^n [A \cap F_k], \quad \forall n \geq 1.$$

It is clear that we have

- $f \chi_{G_1} \leq f \chi_{G_2} \leq \dots$,
- $\lim_{n \rightarrow \infty} (f \chi_{G_n})(x) = (f \chi_{\tilde{A}})(x)$, $\forall x \in X$,

so using the Monotone Convergence Theorem, it follows that

$$\lim_{n \rightarrow \infty} \int_X f \chi_{G_n} \, d\mu = \int_X f \chi_{\tilde{A}} \, d\mu = \int_{\tilde{A}} f \, d\mu.$$

Using (27) we then get

$$\nu(A) = \lim_{n \rightarrow \infty} \int_X f \chi_{G_n} \, d\mu = \int_{\tilde{A}} f \, d\mu.$$

On the other hand, since $\mu(A \setminus \tilde{A}) = 0$, it follows that

$$\int_A f \, d\mu = \int_{\tilde{A}} f \, d\mu,$$

so the preceding equality immediately gives the desired equality

$$\nu(A) = \int_A f \, d\mu.$$

At this point let us remark that the local almost uniqueness of f already follows from Remark 4.1.

Let us prove now the equality (25). Start with some set $A \in \mathcal{A}_{\sigma\text{-fin}}^\mu$, and choose a sequence $(A_n)_{n=1}^\infty \subset \mathcal{A}_{\text{fin}}^\mu$, such that $A = \bigcup_{n=1}^\infty A_n$. Define the sequence $(B_n)_{n=1}^\infty$ by

$$B_n = A_1 \cup \dots \cup A_n, \quad \forall n \geq 1,$$

so that we still have $B_n \in \mathcal{A}_{\text{fin}}^\mu$, $\forall n \geq 1$, as well as $A = \bigcup_{n=1}^\infty B_n$, but moreover we have $B_1 \subset B_2 \subset \dots$. For each $n \geq 1$, using Claim 1, we have the equality

$$\nu(B_n) = \int_{B_n} f d\mu.$$

Using these equalities, combined with

- $0 \leq f \chi_{B_1} \leq f \chi_{B_2} \leq \dots$,
- $\lim_{n \rightarrow \infty} (f \chi_{B_n})(x) = (f \chi_A)(x)$, $\forall x \in X$,

the Monotone Convergence Theorem, combined with continuity yields

$$\int_B f d\mu = \int_X f \chi_B d\mu = \lim_{n \rightarrow \infty} \int_X f \chi_{B_n} d\mu = \lim_{n \rightarrow \infty} \int_{B_n} f d\mu = \lim_{n \rightarrow \infty} \nu(B_n) = \nu(A).$$

B. We start off by choosing a measurable function $h : X \rightarrow \mathbb{K}$, with $|h| = 1$, such that

$$\lambda(A) = \int_A h d|\lambda|, \quad \forall A \in \mathcal{A}.$$

Using part A, there exists some measurable function $g_0 : X \rightarrow [0, \infty]$, such that

$$(28) \quad |\lambda|(A) = \int_A g_0 d\mu, \quad \forall A \in \mathcal{A}_{\sigma\text{-fin}}^\mu.$$

At this point, g_0 may not be integrable, but we have the freedom to perturb it (μ -l.a.e.) to try to make it integrable. This is done as follows. Consider the collection

$$\mathcal{F}_0 = \{F \in \mathcal{F} : |\lambda|(F) > 0\}.$$

Since $|\lambda|$ is *finite*, it follows that $\mathcal{F}_0$ is at most countable. Define then the set $X_0 = \bigcup_{F \in \mathcal{F}_0} F \in \mathcal{A}_{\sigma\text{-fin}}^\mu$. Since X_0 is μ - σ -finite, every set $A \in \mathcal{A}$ with $A \subset X_0$, is μ - σ -finite, so we have

$$|\lambda|(A) = \int_A g_0 d\mu, \quad \forall A \in \mathcal{A}|_{X_0}.$$

Applying the σ -finite version of the Radon-Nikodym Theorem to the σ -finite measure space $(X_0, \mathcal{A}|_{X_0}, \mu|_{X_0})$ and the finite measure $\lambda|_{X_0}$, it follows that the density $g_0|_{X_0}$ belongs to $\mathfrak{L}_+^1(X_0, \mathcal{A}|_{X_0}, \mu|_{X_0})$, which means that the function $g = g_0 \chi_{X_0}$ belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. With this choice of g , let us prove now that the equality (28) still holds, with g in place of g_0 . Exactly as in the proof of part A, it suffices to prove only the equality

$$(29) \quad |\lambda|(A) = \int_A g d\mu, \quad \forall A \in \mathcal{A}_{\sigma\text{-fin}}^\mu.$$

Claim 2: $|\lambda|(A) = |\lambda|(A \cap X_0)$, $\forall A \in \mathcal{A}_{\sigma\text{-fin}}^\mu$.

Since (use the fact that $|\lambda|$ is finite) the equality is equivalent to

$$|\lambda|(A \setminus X_0) = 0, \quad \forall A \in \mathcal{A}_{\sigma\text{-fin}}^\mu,$$

it suffices to prove it only for $A \in \mathcal{A}_{\text{fin}}^\mu$. If $A \in \mathcal{A}_{\text{fin}}^\mu$, using the properties of the decomposition $\mathcal{F}$, we have

$$\begin{aligned} |\lambda|(A) &= \sum_{F \in \mathcal{F}} |\lambda|(A \cap F) = \sum_{F \in \mathcal{F}_0} |\lambda|(A \cap F) + \sum_{F \in \mathcal{F} \setminus \mathcal{F}_0} |\lambda|(A \cap F) = \\ &= |\lambda|\left(\bigcup_{F \in \mathcal{F}_0} [A \cap F]\right) + \sum_{F \in \mathcal{F} \setminus \mathcal{F}_0} |\lambda|(A \cap F) = \\ &= |\lambda|(A \cap X_0) + \sum_{F \in \mathcal{F}'(A)} |\lambda|(A \cap F). \end{aligned}$$

Notice now that, for $F \in \mathcal{F} \setminus \mathcal{F}_0$, we have $|\lambda|(F) = 0$, which gives $|\lambda|(A \cap F) = 0$, so the Claim follows immediately from the above computation.

Having proven the above Claim, let us prove now (29). Fix $A \in \mathcal{A}_{\text{fin}}^\mu$. The desired equality is now immediate from Claim 2, combined with (28):

$$\begin{aligned} |\lambda|(A) &= |\lambda|(A \cap X_0) = \int_{A \cap X_0} g_0 d\mu = \int_X g_0 \chi_{A \cap X_0} d\mu = \\ &= \int_X g_0 \chi_{X_0} \chi_A d\mu = \int_X g \chi_A d\mu = \int_A g d\mu. \end{aligned}$$

Define now the function $f_0 = hg$. Since $|f_0| = g \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, it follows that $f_0 \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$. Let us prove that f_0 satisfies the equality (26). Start with some $A \in \mathcal{A}_{\sigma\text{-fin}}^\mu$. On the one hand, using Claim 2, we have

$$|\lambda|(A \setminus X_0) \leq |\lambda|(A \setminus X_0) = 0,$$

so we get $\lambda(A) = \lambda(A \cap X_0)$. Using the σ -finite version of the Radon-Nikodym Theorem for $(X_0, \mathcal{A}|_{X_0}, \mu|_{X_0})$ and $\lambda|_{X_0}$, we then have

$$\begin{aligned} \lambda(A) &= \lambda(A \cap X_0) = \int_{A \cap X_0} hg_0 d\mu = \int_X hg_0 \chi_{A \cap X_0} d\mu = \\ &= \int_X hg_0 \chi_{X_0} \chi_A d\mu = \int_X hg \chi_A d\mu = \int_A hg d\mu = \int_A f_0 d\mu. \end{aligned}$$

We now prove the uniqueness property (i) of f (μ -a.e.). Assume $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ is another function, such that

$$\lambda(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}_{\sigma\text{-fin}}^\mu.$$

Claim 3: $f = f_0$, μ -l.a.e.

What we need to show here is the fact that

$$f \chi_B = f_0 \chi_B, \quad \mu\text{-a.e.}, \quad \forall B \in \mathcal{A}_{\text{fin}}^\mu.$$

But this follows immediately from the uniqueness from part B of Theorem 4.2, applied to the finite measure space $(B, \mathcal{A}|_B, \mu|_B)$ and the measure $\lambda|_B$, which has both $f|_B$ and $f_0|_B$ as densities.

Using Claim 3, we now have $f - f_0 \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, with $f - f_0 = 0$, μ -l.a.e., so we can apply Proposition 4.4, which forces $f - f_0 = 0$, μ -a.e., so we indeed get $f = f_0$, μ -a.e.

Property (ii) is obvious, since by (i), any function $f \in \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, that satisfies (26), automatically satisfies $|f| = |f_0| = g$, μ -a.e. $\square$

COMMENT. One should be aware of the (severe) limitations of Theorem 4.3, notably the fact that the equalities (25) and (26) hold *only* for $A \in \mathcal{A}_{\sigma\text{-fin}}^\mu$. For example, if one considers the measure space $(X, \mathcal{P}(X), \mu)$, with X uncountable, and μ defined by

$$\mu(A) = \begin{cases} \infty & \text{if } A \text{ is uncountable} \\ 0 & \text{if } A \text{ is countable} \end{cases}$$

This measure space is decomposable, with a decomposition consisting of singletons: $\mathcal{F} = \{\{x\} : x \in X\}$. For a measure ν on $\mathcal{P}(X)$, the condition $\nu \ll \mu$ means precisely that $\nu(A) = 0$ for all countable subsets $A \subset X$. In this case the equality (25) says practically nothing, since it is restricted solely to countable sets $A \subset X$, when both sides are zero.

In this example, it is also instructive to analyze the case when ν is finite (see part B in Theorem 4.3). If we follow the proof of the Theorem, we see that at some point we have constructed a certain set $X_0 = \bigcup_{F \in \mathcal{F}_0} F$, where $\mathcal{F}_0 = \{F \in \mathcal{F} : \nu(F) > 0\}$. In our situation however it turns out that $X_0 = \emptyset$. This example brings up a very interesting question, which turns out to sit at the very foundation of set theory.

Question: Does there exist an uncountable set X , and a finite measure ν on $\mathcal{P}(X)$, such that $\nu(X) > 0$, but $\nu(A) = 0$, for every countable subset $A \subset X$?

(The above vanishing condition is of course equivalent to the fact that $\nu(\{x\}) = 0$, $\forall x \in X$.) It turns out that, not only that the answer of this question is unknown, but in fact several mathematicians are seriously thinking of proposing it as an axiom to be added to the current system of axioms used in set theory!

The limitations of Theorem 4.3 also force limitations in the Change of Variables property (see Proposition 4.1), which in this case has the following statement.

PROPOSITION 4.6 (Local Change of Variables). *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let ν be a measure on $\mathcal{A}$, and let $f : X \rightarrow [0, \infty]$ be a measurable function.*

A. *The following are equivalent:*

(i) *one has*

$$\nu(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}_{\sigma\text{-fin}}^\mu;$$

(ii) *for every measurable function $h : X \rightarrow [0, \infty]$, with the property that the set $E_h = \{x \in X : h(x) \neq 0\}$ belongs to $\mathcal{A}_{\sigma\text{-fin}}^\mu$, one has the equality*

$$(30) \quad \int_X h d\nu = \int_X hf d\mu.$$

B. *If ν and f are as above, and $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, then the equality (30) also holds for those measurable functions $h : X \rightarrow \mathbb{K}$ with $E_h \in \mathcal{A}_{\sigma\text{-fin}}^\mu$, for which $h \in L_{\mathbb{K}}^1(X, \mathcal{A}, \nu)$ and $hf \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$.*

PROOF. A. (i) $\Rightarrow$ (ii). Assume (i) holds. Start with some measurable function $h : X \rightarrow [0, \infty]$, such that the set $E_h = \{x \in X : h(x) \neq 0\}$ belongs to $\mathcal{A}_{\sigma\text{-fin}}^\mu$. The equality (30) is then immediate from Proposition 4.1, applied to the measure space $(E_h, \mathcal{A}|_{E_h}, \mu|_{E_h})$, and the measure $\nu|_{E_h}$, which has density $f|_{E_h}$.

(ii) $\Rightarrow$ (i). Assume (ii) holds. If we start with some $A \in \mathcal{A}_{\sigma\text{-fin}}^\mu$, then obviously the measurable function $h = \varkappa_A$ will have $E_h = A$, so by (ii) we immediately get

$$\nu(A) = \int_X \varkappa_A d\nu = \int_X \varkappa_A f d\mu = \int_A f d\mu.$$

B. Assume now ν and f satisfy the equivalent conditions (i) and (ii). Suppose $h : X \rightarrow \mathbb{K}$ is measurable, with $E_h \in \mathcal{A}_{\sigma\text{-fin}}^\mu$, such that $h \in L_{\mathbb{K}}^1(X, \mathcal{A}, \nu)$ and $hf \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$. Then the equality (30) follows again from Proposition 4.1, applied to the measure space $(E_h, \mathcal{A}|_{E_h}, \mu|_{E_h})$, and the measure $\nu|_{E_h}$, which has density $f|_{E_h}$. $\square$

APPENDIX A

Zorn Lemma

In this Appendix we review basic set theoretical results, which are consequences of the following postulate:

AXIOM OF CHOICE. *Given any non-empty collection¹⁷ $\{X_i : i \in I\}$ of non-empty sets, the cartesian product*

$$\prod_{i \in I} X_i$$

is non-empty.

Recall that the cartesian product is defined as

$$\prod_{i \in I} = \{f : I \rightarrow \bigcup_{i \in I} X_i : f(i) \in X_i, \forall i \in I\}.$$

In order to formulate several consequences of the Axiom of Choice, we need several concepts.

DEFINITIONS. Given a set X , by a *relation on X* one means simply as subset $\mathcal{R} \subset X \times X$. The standard notation for relations is:

$$x\mathcal{R}y \iff (x, y) \in \mathcal{R}.$$

An *order relation on X* is a relation $\prec$ with the following properties:

- $x \prec x, \forall x \in X$;
- if $x, y, z \in X$ satisfy $x \prec y$ and $y \prec z$, then $x \prec z$;
- if $x, y \in X$ satisfy $x \prec y$ and $y \prec x$, then $x = y$.

In this case the pair $(X, \prec)$ is called an *ordered set*.

An ordered set $(X, \prec)$ is said to be *totally ordered*, if

- for any elements $x, y \in X$ one has either $x \prec y$ or $y \prec x$.

More generally, given an (arbitrary) ordered set $(X, \prec)$, by a *totally ordered subset of $(X, \prec)$* , one means a subset $T \subset X$, which becomes totally ordered with respect to the order relation $\prec|_T$.

EXAMPLE A.1. Fix a set M , and take $\mathcal{X}$ to be the collection of all subsets of M . Then $\mathcal{X}$ carries a natural order relation defined by inclusion:

$$A \prec B \iff A \subset B.$$

A totally ordered subset $\mathcal{C}$ of $(\mathcal{X}, \subset)$ is called a *chain of subsets of M* . Two subset $A, B \subset M$ will be said to be *comparable*, if either $A \subset B$, or $B \subset A$, i.e. the collection $\{A, B\}$ is a chain of subsets of M .

DEFINITION. Let M be a set. A collection $\mathcal{F}$ of subsets of M is said to have the *chain property*, if

¹⁷ By a “collection of sets” one simply means a set whose elements are sets themselves.

(c) whenever $\mathcal{C} \subset \mathcal{F}$ is a chain, it follows that the union $\bigcup_{C \in \mathcal{C}} C$ also belongs to $\mathcal{F}$.

LEMMA A.1. *Let M be a set, let $\mathcal{F}$ be a collection of subsets of M with the chain property. For every set $A \in \mathcal{F}$, the collection*

$$\text{COMP}(A; \mathcal{F}) = \{B \in \mathcal{F} : B \text{ comparable to } A\}$$

has the chain property.

PROOF. Let $\mathcal{C} \subset \text{COMP}(A; \mathcal{F})$ be a chain, and put $T = \bigcup_{C \in \mathcal{C}} C$. Since $\mathcal{F}$ has the chain property, we have $T \in \mathcal{F}$. To show that T is comparable with A , we consider the two possibilities:

CASE 1: $A \supset C$, for all $C \in \mathcal{C}$. In this case we have $A \supset \bigcup_{C \in \mathcal{C}} C = T$.

CASE 2: There exists $C_0 \in \mathcal{C}$, such that $A \subset C_0$. In this case we have $A \subset C_0 \subset T$. $\square$

LEMMA A.2. *Let M be some non-empty set, let $\mathcal{F}$ be a non-empty collection of subsets of M , with the chain property. Suppose one has a map*

$$\mathcal{F} \ni A \longmapsto x_A \in M,$$

with the property that

$$A \cup \{x_A\} \in \mathcal{F}, \quad \forall A \in \mathcal{F}.$$

Then there exists $A \in \mathcal{F}$ such that $x_A \in A$.

PROOF. For each $A \in \mathcal{F}$ we define $A^+ = A \cup \{x_A\}$. Call a subset $\mathcal{G} \subset \mathcal{F}$ *inductive*, if it has the chain property, and

$$(+)\quad A \in \mathcal{G} \Rightarrow A^+ \in \mathcal{G}.$$

It is quite clear that if $\mathcal{G}_i, i \in I$ is a collection of inductive subsets of $\mathcal{F}$, then the intersection $\bigcap_{i \in I} \mathcal{G}_i$ is again an inductive subset of $\mathcal{F}$.

Fix now some subset $A_0 \in \mathcal{F}$, and define

$$\mathcal{G}_0 = \bigcap_{\substack{\mathcal{G} \text{ inductive} \\ A_0 \in \mathcal{G}}} \mathcal{G}.$$

Note that the subset $\mathcal{F}_0 = \{A \in \mathcal{F} : A \supset A_0\}$ is an inductive subset of $\mathcal{F}$, so in particular, $\mathcal{G}_0$ is non-empty, and $\mathcal{G}_0 \subset \mathcal{F}_0$, i.e.

$$(1) \quad A \supset A_0, \quad \forall A \in \mathcal{G}_0.$$

Claim: The set $\mathcal{G}_0$ is a chain.

What we need to prove is the fact that $\mathcal{G}_0$ is totally ordered by inclusion. Consider the set

$$\mathcal{T} = \{T \in \mathcal{G}_0 : T \text{ is comparable with every } A \in \mathcal{G}_0\} = \bigcap_{A \in \mathcal{G}_0} \text{COMP}(A; \mathcal{G}_0),$$

and we try to prove that $\mathcal{T} = \mathcal{G}_0$. By Lemma A.1 it is clear that $\mathcal{T}$ has the chain property. Using (1), it is clear that $A_0 \in \mathcal{T}$. Finally, we need to prove property (+). We prove this indirectly as follows. Fix $T \in \mathcal{T}$, consider the collection

$$\mathcal{V}_T = \text{COMP}(T^+; \mathcal{G}_0) = \{A \in \mathcal{G}_0 : A \text{ comparable with } T^+\},$$

and let us prove that $\mathcal{V}_T = \mathcal{G}_0$, by showing that $\mathcal{V}_T$ is an inductive set, and contains A_0 . First of all, by Lemma A.1, it follows that $\mathcal{V}_T$ has the chain property. Secondly, using (1) we have $A_0 \subset T \subset T^+$, so $A_0 \in \mathcal{V}_T$. Finally, to check property (+), we

start with some $V \in \mathcal{V}_T$, and we show that $V^+ \in \mathcal{V}_T$. In the case when $T^+ \subset V$, we are done, because we have $T^+ \subset V \subset V^+$. Assume $T^+ \not\subset V$, so that we have $V \subset T$. Since T is comparable with V^+ , we either have $V^+ \subset T$, in which case we are done, or we have $T \subset V^+$. In the latter case, we have

$$V \subset T \subset V^+.$$

Since $V^+ = V \cup \{x_V\}$, the above inclusions forces either $T = V$, which gives $T^+ = V^+$, or $T = V^+$. Clearly, either case gives $V^+ \in \mathcal{V}_T$. Having shown that $\mathcal{V}_T$ is inductive, the inclusion $\mathcal{V}_T \subset \mathcal{G}_0$ will force the equality $\mathcal{V}_T = \mathcal{G}_0$. In turn, the definition of $\mathcal{V}_T$ proves that $T^+ \in \mathcal{T}$, so $\mathcal{T}$ is indeed inductive. Finally, the inclusion $\mathcal{T} \subset \mathcal{G}_0$ then forces $\mathcal{T} = \mathcal{G}_0$, and by the definition of $\mathcal{T}$, it follows that $\mathcal{G}_0$ is indeed a chain.

Having proven the Claim, we now take $A = \bigcup_{G \in \mathcal{G}_0} G$. Since $\mathcal{G}_0$ has the chain property, it follows that $A \in \mathcal{G}_0$. By construction we have

$$A \supset G, \quad \forall G \in \mathcal{G}_0.$$

In particular we have $A \supset A^+$, which clearly forces $x_A \in A$. $\square$

DEFINITIONS. Let $(X, \prec)$ be an ordered set. By a *maximal element* for X one means an element $x \in X$ with the property:

$$\{y \in X : x \prec y\} = \{x\}.$$

In other words, this means that *there is no element $y \in X$, with $x \prec y$ and $y \neq x$.*

Given a subset $S \subset X$, an element $x \in X$ is said to be an *upper bound* for S , if

$$s \prec x, \quad \forall s \in S.$$

If such an x exists, we say that S *has an upper bound*. (It is not assumed that x belongs to S !)

LEMMA A.3 (“Easy” Zorn Lemma). *Let M be a set, and let $\mathcal{F}$ be a collection of subsets of M . Assume*

- *the Axiom of Choice is true;*
- *$\mathcal{F}$ has the chain property;*
- *$\mathcal{F}$ and is hereditary, in the sense that, whenever $A \in \mathcal{F}$, it follows that all subsets of A belong to $\mathcal{F}$.*

Then, when equipped with the inclusion relation, $(\mathcal{F}, \subset)$ has at least one maximal element.

PROOF. The proof will be carried on by contradiction. Assume no $A \in \mathcal{F}$ is maximal. For each $A \in \mathcal{F}$, define

$$X_A = \{x \in M \setminus A : A \cup \{x\} \in \mathcal{F}\}.$$

Claim: For every $A \in \mathcal{F}$, the set X_A is non-empty.

Indeed, since A is not maximal, there exists some $B \in \mathcal{F}$, with $A \subsetneq B$. In particular, there exists some $x \in B \setminus A$, and since $A \cup \{x\} \subset B$, by the hereditary property, it follows that $x \in X_A$.

Use now the Axiom of Choice, to find a map

$$\mathcal{F} \ni A \longmapsto x_A \in M,$$

such that $x_A \in X_A$, $\forall A \in \mathcal{F}$. This means that $A \cup \{x_A\} \in \mathcal{F}$, and $x_A \notin A$, for all $A \in \mathcal{F}$. By Lemma A.2 this is however impossible. $\square$

THEOREM A.1 (Zorn Lemma). *Assume the Axiom of Choice is true. Let $(X, \prec)$ be a non-empty ordered set, with the following property*

(Z) *every totally ordered subset $A \subset X$ has an upper bound.*

Then X has at least one maximal element.

PROOF. Define the collection

$$\mathcal{F} = \{A \subset X : A \text{ totally ordered subset}\}.$$

Clearly $\mathcal{F}$ is non-empty (it contains, for instance, all singletons).

It is quite clear that $\mathcal{F}$ satisfies the hypothesis of Lemma A.3. So $(\mathcal{F}, \subset)$ has a maximal element A . Take now x to be an upper bound for A , i.e. $a \prec x, \forall a \in A$.

Now we prove that x is maximal for $(X, \prec)$. Suppose $y \in X$ satisfies $x \prec y$. Then clearly $A \cup \{y\}$ will still be a totally ordered subset of X , i.e. $A \cup \{y\} \in \mathcal{F}$. The maximality of A in $(\mathcal{F}, \subset)$ will force $A \cup \{y\} = A$, so we get $y \in A$, hence $y \prec x$. Since we also have $x \prec y$, this forces $y = x$. $\square$

APPENDIX B

Cardinal Arithmetic

In this Appendix we discuss cardinal arithmetic. *We assume the Axiom of Choice is true.*

DEFINITIONS. Two sets A and B are said to have the *same cardinality*, if there exists a bijective map $A \rightarrow B$. It is clear that this defines an equivalence relation on the class¹⁸ of all sets.

A *cardinal number* is thought as an equivalence class of sets. In other words, if we write a cardinal number as $\mathfrak{a}$, it is understood that $\mathfrak{a}$ consists of all sets of a given cardinality. So when we write $\text{card } A = \mathfrak{a}$ we understand that A belongs to this class, and for another set B we write $\text{card } B = \mathfrak{a}$, exactly when B has the same cardinality as A . In this case we write $\text{card } B = \text{card } A$.

NOTATIONS. The cardinality of the empty set $\emptyset$ is zero. More generally the cardinality of a *finite* set is equal to its number of elements. The cardinality of the set $\mathbb{N}$, of all natural numbers, is denoted by $\aleph_0$.

DEFINITION. Let $\mathfrak{a}$ and $\mathfrak{b}$ be cardinal numbers. We write $\mathfrak{a} \leq \mathfrak{b}$ if there exist sets $A \subset B$ with $\text{card } A = \mathfrak{a}$ and $\text{card } B = \mathfrak{b}$.

This is equivalent to the fact that, for *any* sets A and B , with $\text{card } A = \mathfrak{a}$ and $\text{card } B = \mathfrak{b}$, one of the following equivalent conditions holds:

- there exists an *injective* function $f : A \rightarrow B$;
- there exists a *surjective* function $g : B \rightarrow A$.

For two cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$, we use the notation $\mathfrak{a} < \mathfrak{b}$ to indicate that $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{a} \neq \mathfrak{b}$.

THEOREM B.1 (Cantor-Bernstein). *Suppose two cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$ satisfy $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{b} \leq \mathfrak{a}$. Then $\mathfrak{a} = \mathfrak{b}$.*

PROOF. Fix two sets A and B with $\text{card } A = \mathfrak{a}$ and $\text{card } B = \mathfrak{b}$, so there exist injective functions $f : A \rightarrow B$ and $g : B \rightarrow A$. We shall construct a bijective function $h : A \rightarrow B$. Define the sets

$$A_0 = A \setminus g(B) \text{ and } B_0 = A \setminus f(A).$$

Then define recursively the sequences $(A_n)_{n \geq 0}$ and $(B_n)_{n \geq 0}$ by

$$A_n = g(B_{n-1}) \text{ and } B_n = f(A_{n-1}), \quad \forall n \geq 1.$$

Claim 1: One has $A_m \cap A_n = B_m \cap B_n$, $\forall m > n \geq 0$.

Let us first observe that the case when $n = 0$ is trivial, since we have the inclusions $A_m = g(B_{m-1}) \subset g(B) = A \setminus A_0$ and $B_m = f(A_{m-1}) \subset f(A) = B \setminus B_0$. Next we prove the desired property by induction on m . The case $m = 1$ is clear (this forces

¹⁸ The term *class* is used, because there is no such thing as the “set of all sets.”

$n = 0$). Suppose the statement is true for $m = k$, and let us prove it for $m = k + 1$. Start with some $n < k + 1$. If $n = 0$, we are done, by the above discussion. Assume first $n \geq 1$. Since f and g are injective we have

$$\begin{aligned} A_{k+1} \cap A_n &= g(B_k) \cap g(B_{n-1}) = g(B_k \cap B_{n-1}) = \emptyset, \\ B_{k+1} \cap B_n &= f(A_k) \cap f(A_{n-1}) = f(A_k \cap A_{n-1}) = \emptyset, \end{aligned}$$

and we are done.

Put $C = A \setminus \bigcup_{n \geq 0} A_n$ and $D = B \setminus \bigcup_{n \geq 0} B_n$.

Claim 2: One has the equality $f(C) = D$.

First we prove the inclusion $f(C) \subset D$. Start with some point $c \in C$, but assume $f(c) \notin D$. This means that there exists some $n \geq 0$ such that $f(c) \in B_n$. Since $f(c) \in f(A) = B \setminus B_0$, we must have $n \geq 1$. But then we get $f(c) \in B_n = f(A_{n-1})$, and the injectivity of f will force $c \in A_{n-1}$, which is impossible.

Second, we prove that $D \subset f(C)$. Start with some $d \in D$. First of all, since $D \subset B \setminus B_0 = f(A)$, there exists some $c \in A$ with $d = f(c)$. If $c \notin C$, then there exists some $n \geq 0$, such that $c \in A_n$, and then we would get $d = f(c) \in f(A_n) = B_{n+1}$, which is impossible.

We now begin constructing the desired bijection. First we define $\phi : \bigcup_{n \geq 0} B_n \rightarrow B$ by

$$\phi(b) = \begin{cases} b & \text{if } b \in B_n \text{ and } n \text{ is odd} \\ (f \circ g)(b) & \text{if } b \in B_n \text{ and } n \text{ is even} \end{cases}$$

Claim 3: The map ϕ defines a bijection

$$\phi : \bigcup_{n \geq 0} B_n \rightarrow \bigcup_{n \geq 1} B_n.$$

It is clear that, since $\phi|_{B_n}$ is injective, the map ϕ is injective. Notice also that, if $n \geq 0$ is even, then $\phi(B_n) = f(g(B_n)) = f(A_{n+1}) = B_{n+2}$. When $n \geq 0$ is odd we have $\phi(B_n) = B_n$, so we have indeed the equality

$$\phi\left(\bigcup_{n \geq 0} B_n\right) = \bigcup_{n \geq 1} B_n.$$

Now we define $\psi : \bigcup_{n \geq 0} A_n \rightarrow B$ by $\psi = \phi^{-1} \circ f$. Clearly ψ is injective, and

$$\psi\left(\bigcup_{n \geq 0} A_n\right) = \phi^{-1}\left(\bigcup_{n \geq 0} f(A_n)\right) = \phi^{-1}\left(\bigcup_{n \geq 0} B_{n+1}\right) = \phi^{-1}\left(\bigcup_{n \geq 1} B_n\right) = \bigcup_{n \geq 0} B_n,$$

so ψ defines a bijection

$$\psi : \bigcup_{n \geq 0} A_n \rightarrow \bigcup_{n \geq 0} B_n.$$

We then combine ψ with the bijection $f : C \rightarrow D$, i.e. we define the map $h : A \rightarrow B$ by

$$h(x) = \begin{cases} \psi(x) & \text{if } x \in \bigcup_{n \geq 0} A_n \\ f(x) & \text{if } x \in A \setminus \bigcup_{n \geq 0} A_n = C. \end{cases}$$

Clearly h is injective, and

$$h(B) = \psi\left(\bigcup_{n \geq 0} A_n\right) \cup f(C) = \left(\bigcup_{n \geq 0} B_n\right) \cup D = B,$$

so h is indeed bijective. $\square$

THEOREM B.2 (Total ordering for cardinal numbers). *Let $\mathfrak{a}$ and $\mathfrak{b}$ be cardinal numbers. Then one has either $\mathfrak{a} \leq \mathfrak{b}$, or $\mathfrak{b} \leq \mathfrak{a}$.*

PROOF. Choose two sets A and B with $\text{card } A = \mathfrak{a}$ and $\text{card } B = \mathfrak{b}$. In order to prove the theorem, it suffices to construct either an injective function $f : A \rightarrow B$, or an injective function $f : B \rightarrow A$.

We define the set

$$\mathcal{X} = \{(C, D, g) : C \subset A, D \subset B, g : C \rightarrow D \text{ bijection}\}.$$

We equip $\mathcal{X}$ with the following order relation:

$$(C, D, g) \prec (C', D', g') \iff \begin{cases} C \subset C' \\ D \subset D' \\ g = g'|_C \end{cases}$$

We now check that $(\mathcal{X}, \prec)$ satisfies the hypothesis of Zorn Lemma. Let $\mathcal{A} \subset \mathcal{X}$ be a totally ordered subset, say $\mathcal{A} = \{(C_i, D_i, g_i) : i \in I\}$. Define $C = \bigcup_{i \in I} C_i$, $D = \bigcup_{i \in I} D_i$, and $g : C \rightarrow D$ to be the unique function with the property that $g|_{C_i} = g_i$, $\forall i \in I$. (We use here the fact that for $i, j \in I$ we either have $C_i \subset C_j$ and $g_j|_{C_i} = g_i$, or $C_j \subset C_i$ and $g_i|_{C_j} = g_j$. In either case, this proves that $g_i|_{C_i \cap C_j} = g_j|_{C_i \cap C_j}$, $\forall i, j \in I$, so such a g exists.) It is then pretty clear that $(C, D, g) \in \mathcal{X}$ and $(C_i, D_i, g_i) \prec (C, D, g)$, $\forall i \in I$, i.e. (C, D, g) is an upper bound for $\mathcal{A}$. Use now Zorn Lemma, to find a maximal element (A_0, B_0, f) in $\mathcal{X}$.

Claim: Either $A_0 = A$ or $B_0 = B$.

We prove this by contradiction. If we have strict inclusions $A_0 \subsetneq A$ and $B_0 \subsetneq B$, then if we choose $a \in A \setminus A_0$ and $b \in B \setminus B_0$, we can define a bijection $g : A_0 \cup \{a\} \rightarrow B_0 \cup \{b\}$ by $g(a) = b$ and $g|_{A_0} = f$. This would then produce a new element $(A_0 \cup \{a\}, B_0 \cup \{b\}, g) \in \mathcal{X}$, which would contradict the maximality of (A_0, B_0, f) .

The theorem now follows immediately from the Claim. If $A_0 = A$, then $f : A \rightarrow B$ is injective, and if $B_0 = B$, then $f : B \rightarrow A$ is injective. $\square$

We now define the operations with cardinal numbers.

DEFINITIONS. Let $\mathfrak{a}$ and $\mathfrak{b}$ be cardinal numbers.

- We define $\mathfrak{a} + \mathfrak{b} = \text{card } S$, where S is any set which is of the form $S = A \cup B$ with $\text{card } A = \mathfrak{a}$, $\text{card } B = \mathfrak{b}$, and $A \cap B = \emptyset$.
- We define $\mathfrak{a} \cdot \mathfrak{b} = \text{card } P$, where P is any set which is of the form $P = A \times B$ with $\text{card } A = \mathfrak{a}$ and $\text{card } B = \mathfrak{b}$.
- We define $\mathfrak{a}^{\mathfrak{b}} = \text{card } X$, where X is any set of the form X which is of the form

$$X = \prod_{i \in I} A_i,$$

with $\text{card } I = \mathfrak{b}$ and $\text{card } A_i = \mathfrak{a}$, $\forall i \in I$. Equivalently, if we take two sets A and B with $\text{card } A = \mathfrak{a}$, and $\text{card } B = \mathfrak{b}$, and if we define

$$A^B = \prod_B A = \{f : f \text{ function from } B \text{ to } A\},$$

then $\mathfrak{a}^{\mathfrak{b}} = \text{card}(A^B)$.

It is pretty easy to show that these definitions are correct, in the sense that they do not depend on the particular choices of the sets involved. Moreover, these operations are consistent with the usual operations with natural numbers.

REMARK B.1. The operations with cardinal numbers, defined above, satisfy:

- $\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}$,
- $(\mathfrak{a} + \mathfrak{b}) + \mathfrak{d} = \mathfrak{a} + (\mathfrak{b} + \mathfrak{d})$,
- $\mathfrak{a} + 0 = \mathfrak{a}$,
- $\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{b} \cdot \mathfrak{a}$,
- $(\mathfrak{a} \cdot \mathfrak{b}) \cdot \mathfrak{d} = \mathfrak{a} \cdot (\mathfrak{b} \cdot \mathfrak{d})$,
- $\mathfrak{a} \cdot 1 = \mathfrak{a}$,
- $\mathfrak{a} \cdot (\mathfrak{b} + \mathfrak{d}) = (\mathfrak{a} \cdot \mathfrak{b}) + (\mathfrak{a} \cdot \mathfrak{d})$,
- $(\mathfrak{a} \cdot \mathfrak{b})^\mathfrak{d} = (\mathfrak{a}^\mathfrak{d}) \cdot (\mathfrak{b}^\mathfrak{d})$,
- $\mathfrak{a}^{\mathfrak{b}+\mathfrak{d}} = (\mathfrak{a}^\mathfrak{b}) \cdot (\mathfrak{a}^\mathfrak{d})$,
- $(\mathfrak{a}^\mathfrak{b})^\mathfrak{d} = \mathfrak{a}^{\mathfrak{b} \cdot \mathfrak{d}}$,

for all cardinal numbers $\mathfrak{a}, \mathfrak{b}, \mathfrak{d} \geq 1$.

REMARK B.2. The order relation $\leq$ is compatible with all the operations, in the sense that, if $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1$, and $\mathfrak{b}_2$ are cardinal numbers with $\mathfrak{a}_1 \leq \mathfrak{a}_2$ and $\mathfrak{b}_1 \leq \mathfrak{b}_2$, then

- $\mathfrak{a}_1 + \mathfrak{b}_1 \leq \mathfrak{a}_2 + \mathfrak{b}_2$,
- $\mathfrak{a}_1 \cdot \mathfrak{b}_1 \leq \mathfrak{a}_2 \cdot \mathfrak{b}_2$,
- $\mathfrak{a}_1^{\mathfrak{b}_1} \leq \mathfrak{a}_2^{\mathfrak{b}_2}$.

PROPOSITION B.1. Let $\mathfrak{a} \geq 1$ be a cardinal number.

(i) If A is a set with $\text{card } A = \mathfrak{a}$, and if we define

$$\mathcal{P}(A) = \{B : B \text{ subset of } A\},$$

then $2^\mathfrak{a} = \text{card } \mathcal{P}(A)$.

(ii) $\mathfrak{a} < 2^\mathfrak{a}$.

PROOF. (i). Put

$$P = \{0, 1\}^A = \{f : f \text{ function from } A \text{ to } \{0, 1\}\},$$

so that $2^\mathfrak{a} = \text{card } P$. We need to define a bijection $\phi : P \rightarrow \mathcal{P}(A)$. We take

$$\phi(f) = \{a \in A : f(a) = 1\}, \quad \forall f \in P.$$

It is clear that, since a function $f : A \rightarrow \{0, 1\}$ is completely determined by the set $\{a \in A : f(a) = 1\}$, the map ϕ is indeed bijective.

(ii). The map $A \ni a \mapsto \{a\} \in \mathcal{P}(A)$ is clearly injective. This prove the inequality $\mathfrak{a} \leq 2^\mathfrak{a}$. We now prove that $\mathfrak{a} \neq 2^\mathfrak{a}$, by contradiction. Assume there is a bijection $\theta : A \rightarrow \mathcal{P}(A)$. Define the set

$$B = \{a \in A : a \notin \theta(a)\},$$

and choose $b \in A$ such that $B = \theta(b)$. If $b \in B$, then by construction we get $b \notin \theta(b) = B$, which is impossible. If $b \notin B$, we have $b \notin \theta(b)$, which forces $b \in B$, again an impossibility. $\square$

We now discuss the properties of these operations, when infinite cardinal numbers are used.

LEMMA B.1 (Properties of $\aleph_0$).

(i) For any infinite cardinal number $\mathfrak{a}$, one has the inequality $\aleph_0 \leq \mathfrak{a}$.

$$(ii) \aleph_0 + \aleph_0 = \aleph_0;$$

$$(iii) \aleph_0 \cdot \aleph_0 = \aleph_0;$$

PROOF. (i). Let $\mathfrak{a}$ be an infinite cardinal number, and let A be an infinite set A , with $\text{card } A = \mathfrak{a}$. Since for every finite subset $F \subset A$, there exists some $x \in A \setminus F$, one can construct a sequence $(x_n)_{n \in \mathbb{N}} \subset A$, with $x_m \neq x_n, \forall m > n \geq 1$. Then the subset $B = \{x_n : n \in \mathbb{N}\}$ has $\text{card } B = \aleph_0$, so the inclusion $B \subset A$ gives the desired inequality.

(ii). Consider the sets

$$A_0 = \{n \in \mathbb{N} : n, \text{ even}\} \text{ and } A_1 = \{n \in \mathbb{N} : n, \text{ odd}\}.$$

Then clearly $\text{card } A_0 = \text{card } A_1 = \aleph_0$, and the equality $A_0 \cup A_1 = \mathbb{N}$ gives

$$\aleph_0 + \aleph_0 = \text{card } A_0 + \text{card } A_1 = \text{card}(A_0 \cup A_1) = \text{card } \mathbb{N} = \aleph_0.$$

(iii). Take the set $P = \mathbb{N} \times \mathbb{N}$, so that $\aleph_0 \cdot \aleph_0 = \text{card } P$. It is obvious that $\text{card } P \geq \aleph_0$. To prove the other inequality, we define a surjection $\phi : \mathbb{N} \rightarrow P$ as follows. For each $n \geq 1$ we take $s_n = n(n-1)/2$, we set

$$B_n = \{m \in \mathbb{N} : s_n < m \leq s_{n+1}\},$$

and we define $\phi_n : B_n \rightarrow P$ by

$$\phi(m) = (n + s_n - m, m - s_n + 1), \quad \forall m \in B_n.$$

Notice that

$$(1) \quad \phi_n(B_n) = \{(p, q) \in \mathbb{N} \times \mathbb{N} : p + q = n + 1\}.$$

Notice also that $\bigcup_{n \geq 1} B_n = \mathbb{N}$, and $B_j \cap B_k = \emptyset, \forall j > k \geq 1$, so there exists a (unique) function $\phi : \mathbb{N} \rightarrow P$, such that $\phi|_{B_n} = \phi_n$, for all $n \geq 1$. By (1) it is clear that ϕ is surjective. □

THEOREM B.3. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be cardinal numbers, with $1 \leq \mathfrak{b} \leq \mathfrak{a}$, and $\mathfrak{a}$ infinite. Then:*

$$(i) \quad \mathfrak{a} + \mathfrak{b} = \mathfrak{a};$$

$$(ii) \quad \mathfrak{a} \cdot \mathfrak{b} = \mathfrak{a}.$$

PROOF. It is clear that

$$\mathfrak{a} \leq \mathfrak{a} + \mathfrak{b} \leq \mathfrak{a} + \mathfrak{a},$$

$$\mathfrak{a} \leq \mathfrak{a} \cdot \mathfrak{b} \leq \mathfrak{a} \cdot \mathfrak{a},$$

so in order to prove the theorem, we can assume that $\mathfrak{a} = \mathfrak{b}$.

(i). Fix some set A with $\text{card } A = \mathfrak{a}$. Use Zorn Lemma, to find a maximal non-empty family $\{A_i : i \in I\}$ of subsets of A with

$$(a) \quad \text{card } A_i = \aleph_0, \text{ for all } i, j \in I;$$

$$(b) \quad A_i \cap A_j = \emptyset, \text{ for all } i, j \in I \text{ with } i \neq j.$$

If we put $B = A \setminus (\bigcup_{i \in I} A_i)$, then by maximality it follows that B is finite. In particular, if we take $i_0 \in I$ then obviously $\text{card}(A_{i_0} \cup B) = \aleph_0$, so if we replace A_{i_0} with $A_{i_0} \cup B$, we will still have the above properties (a) and (b), but also $A = \bigcup_{i \in I} A_i$. This proves that $\mathfrak{a} = \text{card } A = \aleph_0 \cdot \mathfrak{d}$, where $\mathfrak{d} = \text{card } I$. In other words, we have $\mathfrak{a} = \text{card}(\mathbb{N} \times I)$. Consider then the sets

$$C_0 = \{n \in \mathbb{N} : n \text{ even}\} \text{ and } C_1 = \{n \in \mathbb{N} : n \text{ odd}\},$$

so that $(C_0 \times I) \cup (C_1 \times I) = I \times \mathbb{N}$, and $(C_0 \times I) \cap (C_1 \times I) = \emptyset$. In particular, we get

$$\begin{aligned} \mathfrak{a} &= \text{card}(C_0 \times I) + \text{card}(C_1 \times I) = \\ &= (\text{card } C_0) \cdot (\text{card } I) + (\text{card } C_1) \cdot (\text{card } I) = \\ &= \aleph_0 \cdot \mathfrak{d} + \aleph_0 \cdot \mathfrak{d} = \mathfrak{a} + \mathfrak{a}. \end{aligned}$$

(ii). Fix A a set with $\text{card } A = \mathfrak{a}$. We are going to employ Zorn Lemma to find a bijection $A \rightarrow A \times A$. Define

$$\mathcal{X} = \{(D, f) : D \subset A, f : D \rightarrow D \times D \text{ bijective}\}.$$

Equip $\mathcal{X}$ with the following order

$$(D, f) \prec (D', f') \iff \begin{cases} D \subset D' \\ f = f'|_D \end{cases}$$

Notice that $\mathcal{X}$ is non-empty, since we can find at least one set $D \subset A$ with $\text{card } D = \aleph_0$. We now check that $\mathcal{X}$ satisfies the hypothesis of Zorn Lemma. Let $\mathcal{T} = \{(D_i, f_i) : i \in I\}$ be a totally ordered subset of $\mathcal{X}$. It is fairly clear that if one takes $D = \bigcup_{i \in I} D_i$ and one defines $f : D \rightarrow D \times D$ as the unique function with $f|_{D_i} = f_i$, $\forall i \in I$, then f is injective, and

$$f(D) = \bigcup_{i \in I} f(D_i) = \bigcup_{i \in I} f_i(D_i) = \bigcup_{i \in I} (D_i \times D_i) = D \times D,$$

so the pair (D, f) indeed belongs to $\mathcal{X}$, and is an upper bound for $\mathcal{T}$.

Use Zorn Lemma to produce a maximal element $(D, f) \in \mathcal{X}$. Notice that, if we take $\mathfrak{d} = \text{card } D$, then by construction we have

$$(2) \quad \mathfrak{d} \cdot \mathfrak{d} = \mathfrak{d}.$$

We would like to prove that $D = A$. In general this is not the case (for example, when $A = \mathbb{N}$, every $(D, f) \in \mathcal{X}$, with $\mathbb{N} \setminus D$ finite, is automatically maximal). We notice however that all we need to show is the equality

$$(3) \quad \mathfrak{d} = \mathfrak{a}.$$

We prove this equality by contradiction. We know that we already have $\mathfrak{d} \leq \mathfrak{a}$. Suppose $\mathfrak{d} < \mathfrak{a}$. Put $G = A \setminus D$ notice that $\mathfrak{d} + \text{card } G = \mathfrak{a}$. Since $\mathfrak{d} < \mathfrak{a}$, by (i) we see that we must have the equality $\text{card } G = \mathfrak{a}$. Then there exists a subset $E \subset G$ with $\text{card } E = \mathfrak{d}$. Consider the set

$$P = (E \times E) \cup (E \times D) \cup (D \times E).$$

Since $E \cap D = \emptyset$, the three sets above are pairwise disjoint, so using (2) combined again with part (i), we get

$$\begin{aligned} \text{card } P &= \text{card}(E \times E) + \text{card}(E \times D) + \text{card}(D \times E) = \\ &= \mathfrak{d} \cdot \mathfrak{d} + \mathfrak{d} \cdot \mathfrak{d} + \mathfrak{d} \cdot \mathfrak{d} = \mathfrak{d} + \mathfrak{d} + \mathfrak{d} = \mathfrak{d} = \text{card } E. \end{aligned}$$

This means that there exists a bijection $g : E \times P$, which combined with the fact that $E \cap D = P \cap (D \times D) = \emptyset$, will produce a bijection $h : D \cup E \rightarrow P \cup (D \times D)$, such that $h|_D = f$ and $h|_E = g$. Since we have $P \cup (D \times D) = (D \cup E) \times (D \cup E)$, the pair $(D \cup E, h) \in \mathcal{X}$ will contradict the maximality of (D, f) . $\square$

COROLLARY B.1. *If $\mathfrak{a}$ is an infinite cardinal number, and if $\mathfrak{b}$ is a cardinal number with $2 \leq \mathfrak{b} \leq 2^{\mathfrak{a}}$, then*

$$\mathfrak{b}^{\mathfrak{a}} = 2^{\mathfrak{a}}.$$

PROOF. We have

$$2^{\mathfrak{a}} \leq \mathfrak{b}^{\mathfrak{a}} \leq (2^{\mathfrak{a}})^{\mathfrak{a}} = 2^{\mathfrak{a} \cdot \mathfrak{a}} = 2^{\mathfrak{a}},$$

and the desired equality follows from the Cantor-Bernstein Theorem. $\square$

COROLLARY B.2. *Let $\mathfrak{a}$ be an infinite cardinal number, let A be a set with $\text{card } A = \mathfrak{a}$, and define*

$$\mathcal{P}_{fin}(A) = \{F \in \mathcal{P}(A) : F \text{ finite}\}.$$

Then $\text{card } \mathcal{P}_{fin}(A) = \mathfrak{a}$.

PROOF. First of all, the map $A \ni a \mapsto \{a\} \in \mathcal{P}_{fin}(A)$ is injective, so $\mathfrak{a} \leq \text{card } \mathcal{P}_{fin}(A)$.

We now prove the other inequality. For every integer $n \geq 1$, let A^n denote the n -fold cartesian product. We treat the sequence $A^1, A^2, \dots$ as pairwise disjoint. For every $n \geq 1$ we define the map

$$\phi_n : A^n \rightarrow \mathcal{P}_{fin}(A),$$

by

$$\phi(a_1, \dots, a_n) = \{a_1, \dots, a_n\},$$

and we define the map $\phi : \bigcup_{n=1}^{\infty} A^n \rightarrow \mathcal{P}_{fin}(A)$ as the unique map such that $\phi|_{A^n} = \phi_n, \forall n \geq 1$. Notice now that, since

$$\text{card } A^n = \mathfrak{a}^n = \mathfrak{a}, \quad \forall n \geq 1,$$

it follows that

$$\text{card} \left(\bigcup_{n=1}^{\infty} A^n \right) = \aleph_0 \cdot \mathfrak{a} = \mathfrak{a},$$

which gives

$$\text{card}(\text{Range } \phi) \leq \mathfrak{a}.$$

But it is clear that

$$\{\emptyset\} \cup \text{Range } \phi = \mathcal{P}_{fin}(A),$$

and the fact that $\mathcal{P}_{fin}(A)$ is infinite, proves that

$$\text{card } \mathcal{P}_{fin}(A) = \text{card}(\text{Range } \phi) \leq \mathfrak{a}.$$

$\square$

We conclude with a result on the cardinal number $\mathfrak{c} = \text{card } \mathbb{R}$.

PROPOSITION B.2.

(i) *For two real numbers $a < b$, one has*

$$\text{card}(a, b) = \text{card}[a, b) = \text{card}(a, b] = \text{card}[a, b] = \mathfrak{c}.$$

(ii) $\mathfrak{c} = 2^{\aleph_0}$.

PROOF. (i). It is clear that, since (a, b) is infinite, we have

$$\text{card}[a, b] = 2 + \text{card}(a, b) = \text{card}(a, b).$$

The inclusions $(a, b) \subset [a, b] \subset [a, b]$ and $(a, b) \subset (a, b] \subset [a, b]$, combined with the Cantor-Bernstein Theorem, immediately give

$$\text{card}[a, b] = \text{card}(a, b] = \text{card}(a, b).$$

Finally, the bijection

$$(a, b) \ni t \longmapsto \tan\left(\frac{\pi(2t - a - b)}{2(b - a)}\right) \in \mathbb{R}$$

shows that $\text{card}(a, b) = \mathfrak{c}$.

(ii). The proof of this result uses a certain construction, which is useful for many other purposes. Therefore we choose to work in full generality. Consider the set

$$T = \{0, 1\}^{\mathbb{N}_0} = \{a = (\alpha_n)_{n \in \mathbb{N}} : \alpha_n \in \{0, 1\}, \forall n \in \mathbb{N}\},$$

so $2^{\aleph_0} = \text{card } P$. For any real number $r \geq 2$, we define the map $\phi_r : T \rightarrow [0, 1]$ by

$$\phi(a) = (r - 1) \sum_{n=1}^{\infty} \frac{\alpha_n}{r^n}, \quad \forall a = (\alpha_n)_{n \in \mathbb{N}} \in T.$$

The maps ϕ_r , $r \geq 2$ are “almost” injective. To clarify this, we define the set

$$T_0 = \{a = (\alpha_n)_{n \in \mathbb{N}} \in T : \text{the set } \{n \in \mathbb{N} : \alpha_n = 0\} \text{ is infinite}\}.$$

Note that

$$T \setminus T_0 = \{(\alpha_n)_{n \in \mathbb{N}} \in T : \text{there exists } N \in \mathbb{N}, \text{ such that } \alpha_n = 1, \forall n \geq N\}.$$

Clearly ϕ is surjective. In fact ϕ is “almost” bijective.

Claim 1: Fix $r \geq 2$. For elements $a = (\alpha_n)_{n \in \mathbb{N}}, b = (\beta_n)_{n \in \mathbb{N}} \in T_0$, the following are equivalent

- (*) $\phi_r(a) > \phi_r(b)$;
- (**) *there exists $k \in \mathbb{N}$, such that $\alpha_k > \beta_k$, and $\alpha_j = \beta_j$, for all $j \in \mathbb{N}$ with $j < k$.*

We first prove the implication $(**) \Rightarrow (*)$. If $a, b \in T_0$ satisfy $(**)$, then

$$(4) \quad \phi_r(a) - \phi_r(b) = \frac{r-1}{r^k} + (r-1) \sum_{n=k+1}^{\infty} \frac{\alpha_n - \beta_n}{r^n} \geq \frac{r-1}{r^k} - (r-1) \sum_{n=k+1}^{\infty} \frac{\beta_n}{r^n}.$$

Notice now that there are infinitely many indices $n \geq k+1$ such that $\beta_n = 0$. This gives the fact that

$$\sum_{n=k+1}^{\infty} \frac{\beta_n}{r^n} < \sum_{n=k+1}^{\infty} \frac{1}{r^n} = \frac{1}{(r-1)r^k},$$

so if we go back to (4) we get

$$\phi_r(a) - \phi_r(b) \geq \frac{r-1}{r^k} - (r-1) \sum_{n=k+1}^{\infty} \frac{\beta_n}{r^n} > \frac{r-1}{r^k} - \frac{1}{r^k} = \frac{r-2}{r^k} \geq 0,$$

so in particular we get $\phi_r(a) > \phi_r(b)$.

Conversely, if $\phi_r(a) > \phi_r(b)$, we choose

$$k = \min\{n \in \mathbb{N} : \alpha_n \neq \beta_n\}.$$

Using the implication $(**) \Rightarrow (*)$ we see that we cannot have $\beta_k > \alpha_k$, because this would force $\phi(b) > \phi(a)$. Therefore we must have $\alpha_k > \beta_k$, and we are done.

Using Claim 1, we now see that $\phi_r|_{T_0} : T_0 \rightarrow [0, 1]$ is *injective*

Claim 2: $\text{card}(T \setminus T_0) = \aleph_0$.

This is pretty clear, since we can write

$$T \setminus T_0 = \bigcup_{k=1}^{\infty} R_k,$$

where

$$R_n = \{a = (\alpha_n)_{n \in \mathbb{N}} \in T : \alpha_n = 1, \forall n \geq 1\}.$$

Since each R_n is finite, the desired result follows.

Using Claim 2, we have

$$2^{\aleph_0} = \text{card } T = \text{card}(T \setminus T_0) + \text{card } T_0 = \aleph_0 + \text{card } T_0.$$

Since $\aleph_0 < 2^{\aleph_0}$, the above equality forces

$$2^{\aleph_0} = \text{card } T_0.$$

For every $r \geq 2$, we also have $\text{card } \phi_r(T \setminus T_0) \leq \aleph_0$, which then gives $\text{card}[\phi_r(T) \setminus \phi_r(T_0)] \leq \aleph_0$, hence using the injectivity of $\phi_r|_{T_0}$, we have $\text{card } \phi_r(T_0) = \text{card } T_0 = 2^{\aleph_0}$, so we get

$$2^{\aleph_0} = \text{card } \phi_r(T_0) \leq \text{card } \phi_r(T) = \text{card } \phi_r(T_0) + \text{card}[\phi_r(T) \setminus \phi_r(T_0)] \leq \text{card } \phi_r(T_0) + \aleph_0 = 2^{\aleph_0} + \aleph_0 = 2^{\aleph_0}.$$

By the Cantor-Bernstein Theorem this forces $\text{card } \phi_r(T) = 2^{\aleph_0}$.

Now we are done, since for $r = 2$ we clearly have $\phi_2(T) = [0, 1]$. $\square$

COROLLARY B.3.

- (i) $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$.
- (ii) If we define the set

$$\mathcal{P}_{\text{count}} = \{C \subset \mathbb{R} : \text{card } C \leq \aleph_0\},$$

then $\text{card } \mathcal{P}_{\text{count}}(\mathbb{R}) = \mathfrak{c}$.

PROOF. (i). This is immediate from the equality $2^{\aleph_0} = \mathfrak{c}$ and from Corollary B.1.

(ii). Using the inclusion $\mathcal{P}_{\text{fin}}(\mathbb{R}) \subset \mathcal{P}_{\text{count}}(\mathbb{R})$, combined with Corollary B.2, we see that we have the inequality

$$\mathfrak{c} \leq \text{card } \mathcal{P}_{\text{count}}(\mathbb{R}).$$

To prove the other inequality, we define a map $\phi : \mathbb{R}^{\aleph_0} \rightarrow \mathcal{P}_{\text{count}}(\mathbb{R})$, as follows. If $a \in \mathbb{R}^{\aleph_0}$ is a sequence, say $a = (\alpha_n)_{n \in \mathbb{N}}$, we put

$$\phi(a) = \{\alpha_n : n \in \mathbb{N}\}.$$

Since ϕ is clearly surjective, using part (i) we get

$$\text{card } \mathcal{P}_{\text{count}}(\mathbb{R}) \leq \text{card } \mathbb{R}^{\aleph_0} = \mathfrak{c}^{\aleph_0} = \mathfrak{c}.$$

$\square$

APPENDIX C

Ordinal numbers

In this Appendix we discuss ordinal number arithmetic. *The Axiom of Choice is assumed to be true.*

DEFINITION. Let X be a non-empty set. A *well ordering on X* is an total order relation $\prec$ on X with the following property:

- (W) *every non-empty subset $A \subset X$ has a smallest element, i.e. there exists $a \in A$, such that $a \prec x, \forall x \in A$.*

In this case the pair $(X, \prec)$ is called a *well ordered set*.

NOTATIONS. Let $(W, \prec)$ be a well-ordered set. For any $a \in W$, we define

$$W(a) = \{x \in W : x \prec a \text{ and } x \neq a\}.$$

Remark that $(W(a), \prec)$ is well-ordered.

LEMMA C.1. *Let $(W, \prec)$ be a well ordered set. For a subset $S \subset W$, the following are equivalent:*

- (i) *for every $s \in S$, one has the inclusion $W(s) \subset S$;*
- (ii) *either $S = W$, or there exists some $a \in W$, such that $S = W(a)$.*

PROOF. $(i) \Rightarrow (ii)$. Assume $S \subsetneq W$. Take a to be the smallest element of the set $W \setminus S$. If $s \in S$, then $a \neq s$, and by (i) we cannot have $a \prec s$, since this would force $a \in W(s) \subset S$. Therefore we must have $s \prec a$, i.e. $s \in W(a)$. This prove the inclusion $S \subset W(a)$. Conversely, if $s \in W(a)$, then s must belong to S . Otherwise $s \in W \setminus S$ would contradict the minimality of a .

$(ii) \Rightarrow (i)$. This is trivial. □

DEFINITION. A subset S , as above, is called a *full* subset.

The key feature of well-ordered sets is the following.

LEMMA C.2 (Transfinite Induction Principle). *Let $(W, \prec)$ be a well-ordered set. Let $w_1 \in W$ be the smallest element of W . Assume $A \subset W$ is a set with the property*

- (1) *If $w \in W$ has the property that, $W(w) \subset A$, then $w \in A$.*

Then $A = W$.

PROOF. Consider the set

$$S = \{s \in A : W(s) \subset A\}.$$

It is obvious that S is full, and $S \subset A$. By Lemma C.1, either $S = W$, in which case we clearly get $A = W$, or there exists $w \in W$, such that $S = W(w)$. In this case we have $W(w) \subset A$. By (1) this forces $w \in A$, so we get $w \in S$, which is impossible. □

Another useful feature is

LEMMA C.3 (Recursion Principle). *Let $(W, \prec)$ be a well-ordered set, and let w_1 be the smallest element in W . Let X be a set, and assume one has a family of maps $\Phi_a : \prod_{W(a)} X \rightarrow X$, $a \in W \setminus \{w_1\}$. Then for any element $x_1 \in X$, there exists a unique function $f : W \rightarrow X$, such that*

$$(1) \quad f(w_1) = x_1 \text{ and } f(a) = \Phi_a(f|_{W(a)}), \quad \forall a \in W \setminus \{w_1\}.$$

PROOF. For every $a \in W$ let us denote the set $W(a) \cup \{a\}$ simply by W_a , and let us define the set

$$\mathcal{F}_a = \{g : W_a \rightarrow X : g(w_1) = x_1 \text{ and } g(b) = \Phi_b(g|_{W(b)}), \quad \forall b \in W_a \setminus \{w_1\}\}.$$

Remark that, for any $a, b \in W$, with $a \prec b$, one has

$$(2) \quad f|_{W_a} \in \mathcal{F}_a, \quad \forall f \in \mathcal{F}_b.$$

Claim: For every $a \in W$, the set $\mathcal{F}_a$ is a singleton.

We prove this statement using transfinite induction. Define

$$A = \{a \in W : \mathcal{F}_a \text{ is a singleton}\}.$$

Suppose $a \in W$ has the property $W(a) \subset A$, which means that $\mathcal{F}_b$ is a singleton, for all $b \in W(a)$. For each $b \in W(a)$, let $f_b : W_b \rightarrow X$ be the unique element in $\mathcal{F}_b$. We notice that, for any $b, c \in W(a)$, with $b \prec c$, using (2), we have

$$(3) \quad f_c|_{W_b} = f_b.$$

This follows immediately from the fact that $f_c|_{W_b}$ belongs to $\mathcal{F}_b$. Using the obvious equality

$$W(a) = \bigcup_{b \in W(a)} W_b,$$

we define $g : W(a) \rightarrow X$ as the unique function with the property that $g|_{W_b} = f_b$, $\forall b \in W(a)$. Finally, we define $f_a : W_a \rightarrow X$ by $f_a|_{W(a)} = g$, and $f_a(a) = \Phi_a(g)$. It is clear that $f_a \in \mathcal{F}_a$, so $\mathcal{F}_a$ has at least one element. If $h \in \mathcal{F}_a$ is another function, then for every $b \in W(a)$ we have $h|_{W_b} \in \mathcal{F}_b$, which forces $h|_{W_b} = f_b$, in particular giving $h|_{W(a)} = g = f_a|_{W(a)}$. Then $h(a) = \Phi_a(g)$, which means that we also have $h(a) = f_a(a)$, so we must have $h = f_a$.

Having proven the Claim, we now have a family of functions $f_a : W_a \rightarrow X$, $a \in W$, with $f_b|_{W_a} = f_a$, for all $a, b \in W$ with $a \prec b$. Using the equality

$$W = \bigcup_{a \in W} W_a,$$

we then define $f : W \rightarrow X$ to be the unique function such that $f|_{W_a} = f_a$, $\forall a \in W$.

Notice that, for each $a \in W \setminus \{w_1\}$, we have $f(a) = f_a(a)$, and since $f_a \in \mathcal{F}_a$, we immediately get (1). The uniqueness of f with property (1) is also clear, since any such f will automatically satisfy $f|_{W_a} \in \mathcal{F}_a$, for all $a \in W$. $\square$

COMMENT. The system of maps $\Phi_a : \prod_{W(a)} X \rightarrow X$, $a \in W$ is to be thought as a “recurrence relation,” in the sense that it is used to define the value $f(a)$ in terms of all “preceding” values $f(w)$, $w \prec a$, $w \neq a$.

DEFINITIONS. Given two well ordered sets $(W_1, \prec_1)$ and $(W_2, \prec_2)$, a map $f : (W_1, \prec_1) \rightarrow (W_2, \prec_2)$ is called an *full embedding*, if

- f is injective.
- For any two elements $x, y \in W_1$, one has

$$x \prec_1 y \Rightarrow f(x) \prec_2 f(y).$$

- $f(W_1)$ is a full subset of W_2 .

If f is a full embedding, with $f(W_1) = W_2$, then f is called an *order isomorphism*.

The properties of these types of maps are contained in the following

PROPOSITION C.1. A. Suppose $(W_1, \prec_1)$ and $(W_2, \prec_2)$, are well-ordered sets.

- (i) If $f : (W_1, \prec_1) \rightarrow (W_2, \prec_2)$ is a full embedding, then

$$f(W_1(a)) = W_2(f(a)), \quad \forall a \in W_1.$$

In particular, if w_1 is the smallest element in W_1 , and w_2 is the smallest element in W_2 , then $f(w_1) = w_2$.

- (ii) If $f : (W_1, \prec_1) \rightarrow (W_2, \prec_2)$ is an order isomorphism, then $f^{-1} : (W_2, \prec_2) \rightarrow (W_1, \prec_1)$ is again an order isomorphism.
- (iii) There exists at most one full embedding $f : (W_1, \prec_1) \rightarrow (W_2, \prec_2)$.

B. Suppose $(W_1, \prec_1)$, $(W_2, \prec_2)$, $(W_3, \prec_3)$ are well-ordered sets, and

$$(W_1, \prec_1) \xrightarrow{f} (W_2, \prec_2) \xrightarrow{g} (W_3, \prec_3)$$

are full embeddings.

- (i) The composition $g \circ f : (W_1, \prec_1) \rightarrow (W_3, \prec_3)$ is again a full embedding.
- (ii) The composition $g \circ f$ is an order isomorphism, if and only if both f and g are order isomorphisms.

PROOF. A. (i). Start first with some element $x \in W(a)$. Since $x \prec_1 a$, we have $f(x) \prec_2 f(a)$. Since f is injective, and $x \neq a$, we must have $f(x) \neq f(a)$, hence $x \in W_2(f(a))$. Conversely, if $y \in W_2(f(a))$, then using the fact that $f(W_2)$ is full in W_2 , it follows that $y \in f(W_2)$, so there exists some $x \in W_1$, with $y = f(x)$. If $a \prec_1 x$, then we would get $f(a) \prec_2 f(x)$, which is impossible. Therefore we must have $x \prec_1 a$ and $x \neq a$, i.e. $x \in W_1(a)$, so y indeed belongs to $f(W_1(a))$. The second assertion is now clear since we have

$$W_2(f(w_1)) = f(W_1(w_1)) = f(\emptyset) = \emptyset,$$

which clearly forces $f(w_1) = w_2$.

- (ii). This is obvious.

(iii). Suppose $f, g : (W_1, \prec_1) \rightarrow (W_2, \prec_2)$ are full embeddings, and let us show that we must have $f = g$. We use transfinite induction. Define the set

$$A = \{w \in W_1 : f(w) = g(w)\}.$$

Let $w \in W_1$ be some element such that $W_1(w) \subset A$, and let us prove that $w \in A$, i.e. $f(w) = g(w)$. Denote $f(w)$ by a , and $g(w)$ by b . Using the fact that $f|_{W_1(w)} = g|_{W_1(w)}$, combined with (i), we have

$$W_2(a) = W_2(f(w)) = f(W_1(w)) = g(W_1(w)) = W_2(g(w)) = W_2(b).$$

This clearly forces $a = b$. Indeed, if $a \neq b$, then either $a \prec b$, in which case $a \in W_2(b) \setminus W_2(a)$, or $b \prec a$, in which case $b \in W_2(a) \setminus W_2(b)$. In either case, we will get $W_2(a) \neq W_2(b)$.

B.(i). It is clear that $g \circ f$ is injective, and satisfies the second condition in the definition, so the only thing we need to prove is the fact that $(g \circ f)(W_1)$ is full. If $f(W_1) = W_2$, there is nothing to prove, since we would get $(g \circ f)(W_1) = g(W_2)$, which is full.

Assume $f(W_1) = W_2(a)$, for some $a \in W_2$. Then by (i) we have

$$(g \circ f)(W_1) = g(f(W_1)) = g(W_2(a)) = W_3(g(a)),$$

so again $(g \circ f)(W_1)$ is full.

(ii). Assume first that both f and g are order isomorphisms. Then $g \circ f : (W_1, \prec_1) \rightarrow (W_3, \prec_3)$ is a full embedding, by (i), and it is clearly surjective, hence $g \circ f$ is indeed an order isomorphism.

Conversely, assume $g \circ f : (W_1, \prec_1) \rightarrow (W_3, \prec_3)$ is an order isomorphism. This clearly forces g to be surjective, hence an order isomorphism. But then g^{-1} is an order isomorphism, and so will be $g^{-1} \circ (g \circ f) = f$. $\square$

COROLLARY C.1. *If $(W, \prec)$ is a well-ordered set, and $a \in W$, then there is no full embedding $(W, \prec) \rightarrow (W(a), \prec)$.*

PROOF. Suppose there exists a full embedding $f : (W, \prec) \rightarrow (W(a), \prec)$. Since the inclusion $\iota : (W(a), \prec) \hookrightarrow (W, \prec)$ is obviously a full embedding, the composition $\iota \circ f : (W, \prec) \rightarrow (W, \prec)$ is a full embedding. Since we also have $\text{Id}_W : (W, \prec) \rightarrow (W, \prec)$ as a full embedding, this would force $\iota \circ f = \text{Id}_W$, which would force ι to be surjective. But this is obviously impossible. $\square$

DEFINITIONS. Two well-ordered sets $(W_1, \prec_1)$ and $(W_2, \prec_2)$ are said to have the *same order type*, if there exists an order isomorphism $(W_1, \prec_1) \rightarrow (W_2, \prec_2)$. By the above considerations, this defines an equivalence relation on the class of all well-ordered sets.

An *ordinal number* is thought as an equivalence class of well-ordered sets. In other words, if we write a cardinal number as α , it is understood that α consists of all well-ordered sets of a given order type. So when we write $\text{ord}(W, \prec) = \alpha$ we understand that $(W, \prec)$ belongs to this class, and for another well-ordered set $(W', \prec')$ we write $\text{ord}(W', \prec') = \alpha$, exactly when $(W', \prec')$ has the same order type as $(W, \prec)$. In this case we write $\text{ord}(W', \prec') = \text{ord}(W, \prec)$.

We regard the empty set $\emptyset$ as a well-ordered set, with the empty relation. We write $\text{ord}(\emptyset) = 0$.

COMMENTS. If $(W_1, \prec_1)$ and $(W_2, \prec_2)$ are well-ordered sets, then one has the obvious implication

$$\text{ord}(W_1, \prec_1) = \text{ord}(W_2, \prec_2) \implies \text{card } W_1 = \text{card } W_2.$$

Conversely, if the well-ordered sets $(W_1, \prec_1)$ and $(W_2, \prec_2)$ are *finite*, and $\text{card } W_1 = \text{card } W_2$, then $\text{ord}(W_1, \prec_1) = \text{ord}(W_2, \prec_2)$. Indeed, if we take $n = \text{card } W_1$, then one can define recursively a finite sequence $(w_k)_{k=1}^n \subset W_1$, by taking w_1 to be the smallest element of W_1 , and defining, for each $k \in \{2, 3, \dots, n\}$ the element w_k to be the smallest element of the set $W_1 \setminus \{w_1, w_2, \dots, w_{k-1}\}$. The obvious bijection

$$\{1, 2, \dots, n\} \ni k \mapsto w_k \in W_1$$

will then define an order isomorphism

$$(\{1, \dots, n\}, \leq) \rightarrow (W_1, \prec_1).$$

Likewise $(W_2, \prec_2)$ has same order type as $(\{1, \dots, n\}, \leq)$.

Using the above notations, we can then regard all non-negative integers as ordinal numbers, by identifying $\text{ord}(W, \prec) = \text{card}(W)$, for all finite well-ordered sets $(W, \prec)$.

NOTATION. If α is an ordinal number, say $\alpha = \text{ord}(W, \prec)$, for some well-ordered set $(W, \prec)$, then the cardinal number $\text{card } W$ does not depend on the particular choice of $(W, \prec)$. We will denote it by $\text{card } \alpha$. As discussed above, if

$$\text{card } \alpha = \text{card } \beta = \text{finite cardinal},$$

then $\alpha = \beta$. As we shall see later, this implication holds only for finite ordinal numbers.

DEFINITIONS. Let α_1 and α_2 be ordinal numbers, say $\alpha_1 = \text{ord}(W_1, \prec_1)$ and $\alpha_2 = \text{ord}(W_2, \prec_2)$, where $(W_1, \prec_1)$ and $(W_2, \prec_2)$ are two well-ordered sets. We write $\alpha_1 \leq \alpha_2$, if there exists a full embedding $f : (W_1, \prec_1) \rightarrow (W_2, \prec_2)$. By Proposition C.1, this definition is independent of the choices of $(W_1, \prec_1)$ and $(W_2, \prec_2)$.

We write $\alpha_1 < \alpha_2$ if $\alpha_1 \leq \alpha_2$ and $\alpha_1 \neq \alpha_2$.

REMARK C.1. If α_1 and α_2 are ordinal numbers, with $\alpha_1 \leq \alpha_2$, then $\text{card } \alpha_1 \leq \text{card } \alpha_2$.

PROPOSITION C.2. *The relation $\leq$ is an order relation, on any set of ordinal numbers.*

PROOF. It is obvious that $\alpha \leq \alpha$, for any ordinal number α .

Assume α_1 and α_2 are ordinal numbers with $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_1$, and let us show that this forces $\alpha_1 = \alpha_2$. Let $(W_1, \prec_1)$ and $(W_2, \prec_2)$ be well-ordered sets with $\alpha_1 = \text{ord}(W_1, \prec_1)$ and $\alpha_2 = \text{ord}(W_2, \prec_2)$. Since $\alpha_1 \leq \alpha_2$, there exists a full embedding $f : (W_1, \prec_1) \rightarrow (W_2, \prec_2)$. Since $\alpha_2 \leq \alpha_1$, either there exists a full embedding $g : (W_2, \prec_2) \rightarrow (W_1, \prec_1)$. By Proposition C.1.B, the composition $g \circ f : (W_1, \prec_1) \rightarrow (W_1, \prec_1)$ is a full embedding. Since we already have a full embedding $\text{Id}_{W_1} : (W_1, \prec_1) \rightarrow (W_1, \prec_1)$, by Proposition C.1.A, we must have $g \circ f = \text{Id}_{W_1}$. Using Proposition C.1.B this forces f (and g) to be order isomorphisms, so we indeed have $\alpha_1 = \alpha_2$.

Finally, suppose α_1 , α_2 and α_3 are ordinal numbers such that $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_3$. The fact that $\alpha_1 \leq \alpha_3$ follows immediately from Proposition C.1.B. $\square$

THEOREM C.1 (Ordinal Comparability Theorem). *Let α_1 and α_2 be ordinal numbers. Then either $\alpha_1 \leq \alpha_2$, or $\alpha_2 \leq \alpha_1$.*

PROOF. Let $(W_1, \prec_1)$ and $(W_2, \prec_2)$ be well-ordered sets with $\alpha_1 = \text{ord}(W_1, \prec_1)$ and $\alpha_2 = \text{ord}(W_2, \prec_2)$. For every $a \in W_1$ we denote the set $W_1(a) \cup \{a\}$ simply by W_1^a . It is clear that $(W_1^a, \prec_1)$ is well-ordered. Consider the set

$$A = \{a \in W_1 : \text{there exists a full embedding } (W_1^a, \prec_1) \rightarrow (W_2, \prec_2)\}.$$

By Proposition C.1.A, we know that for any $a \in A$, there exists a *unique* full embedding $(W_1^a, \prec_1) \rightarrow (W_2, \prec_2)$. We denote this full embedding by f_a .

Claim 1: The set A is full. Moreover, for any $a, b \in A$, with $b \prec a$, we have

$$f_b = f_a|_{W_1^b}.$$

Start with some $a \in A$, and let us prove that $W_1(a) \subset A$. Fix some arbitrary $b \in W(a)$. Then the inclusion $\iota : (W_1^b, \prec_1) \hookrightarrow (W_1^a, \prec_1)$ is obviously a full embedding, since we can write

$$W_1^b = W_1(c),$$

where c is the smallest element of the set

$$D_b = \{x \in W_1 : b \prec_1 x \text{ and } b \neq x\}.$$

(The fact that $a \in D_b$ shows that $D_b \neq \emptyset$.) Then the composition

$$f_a \circ \iota : (W_1^b, \prec_1) \rightarrow (W_2, \prec_2)$$

is a full embedding, so b indeed belongs to A . Moreover, we will have $f_b = f_a \circ \iota = f_a|_{W_1^b}$.

Define the map $\phi : A \rightarrow W_2$ by

$$\phi(a) = f_a(a), \quad \forall a \in A.$$

Remark that

$$(4) \quad \phi|_{W_1^a} = f_a, \quad \forall a \in A.$$

Indeed, if we take some $b \in W_1(a)$, then by Claim 1, we have $\phi(b) = f_b(b) = f_a(b)$, so we get $\phi|_{W_1(a)} = f_a|_{W_1(a)}$.

Claim 2: $\phi : (A, \prec_1) \rightarrow (W_2, \prec_2)$ is a full embedding.

We start by proving the first two conditions. Let $a, b \in A$ be such that $b \prec_1 a$ and $b \neq a$, and let us show that $\phi(b) \prec_2 \phi(a)$ and $\phi(b) \neq \phi(a)$. We have $b \in W_1^a$ and $\phi|_{W_1^a} = f_a$, so using the fact that $f_a : (W_1^a, \prec_1) \rightarrow (W_2, \prec_2)$ is a full embedding, we indeed get $\phi(b) = f_a(b) \prec_2 f_a(a) = \phi(a)$, and $\phi(b) \neq \phi(a)$.

We now show that $\phi(A)$ is full in $(W_2, \prec_2)$. Start with some $y \in \phi(A)$, and let us show that $W_2(y) \subset \phi(A)$. On the one hand, since we obviously have

$$A = \bigcup_{a \in A} W_1^a,$$

we also have

$$\phi(A) = \phi\left(\bigcup_{a \in A} W_1^a\right) = \bigcup_{a \in A} \phi(W_1^a),$$

so there exists some $a \in A$, such that $y \in \phi(W_1^a) = f_a(W_1^a)$. On the other hand, since $f_a : (W_1^a, \prec_1) \rightarrow (W_2, \prec_2)$ is a full embedding, it follows that $f_a(W_1^a)$ is full, so we get $W_2(y) \subset f_a(W_1^a) = \phi(W_1^a) \subset \phi(A)$.

We now finish the proof. Since both A and $\phi(A)$ are full, there are three cases to examine

CASE 1: $A = W_1$. In this case $\phi : (W_1, \prec_1) \rightarrow (W_2, \prec_2)$ is a full embedding, so we get $\alpha_1 \leq \alpha_2$.

CASE 2: $\phi(A) = W_2$. In this case $\phi : (A, \prec_1) \rightarrow (W_2, \prec_2)$ is an order isomorphism, so $\phi^{-1} : (W_2, \prec_2) \rightarrow (W_1, \prec_1)$ is a full embedding, and we get $\alpha_1 \leq \alpha_2$.

CASE 3: $A \subsetneq W_1$ and $\phi(A) \subsetneq W_2$. This means there exist $a_1 \in W_1$ and $a_2 \in W_2$ such that $A = W_1(a_1)$ and $\phi(A) = W_2(a_2)$. This case turns out to be impossible. To see this, we define $\psi : W_1^{a_1} \rightarrow W_2$ by $\psi|_{W_1(a)} = \phi$ and $\psi(a_1) = a_2$, then $\psi : (W_1^{a_1}, \prec_1) \rightarrow (W_2, \prec_2)$ will still be an order isomorphism. Indeed, the first two conditions in the definition are clear, while the equality

$$\psi(W_1^{a_1}) = W_2^{a_2} = \{y \in W_2 : y \prec_2 a_2\},$$

proves that $\psi(W_1^{a_1})$ is full. The existence of ψ then forces $a_1 \in A$, which contradicts the equality $A = W_1(a_1)$. $\square$

THEOREM C.2. *Let α be an ordinal number. Then the class P_α of all ordinal numbers β with $\beta < \alpha$ is a set. More explicitly, if $(W, \prec)$ is a well-ordered set with $\text{ord}(W, \prec) = \alpha$, then the map*

$$\phi : W \ni a \mapsto \text{ord}(W(a), \prec) \in P_\alpha$$

is a bijection. Moreover, $(P_\alpha, \leq)$ is well-ordered, and $\phi : (W, \prec) \rightarrow (P_\alpha, \leq)$ is an order isomorphism.

PROOF. Let β be an ordinal number with $\beta < \alpha$. Then there exists a well-ordered set $(W_1, \prec_1)$, and a full embedding $\phi : (W_1, \prec_1) \rightarrow (W, \prec)$, such that

- $\beta = \text{ord}(W_1, \prec_1)$,
- $\phi(W_1) = W(a_1)$,

for some $a_1 \in W$. This fact already proves that P_α is a set.

Claim: *The element $a_1 \in W$ does not depend on the particular choice of $(W_1, \prec_1)$.*

Indeed, if $(W_2, \prec_2)$ is another well-ordered set, and $\psi : (W_2, \prec_2) \rightarrow (W, \prec)$ is another full embedding with

- $\beta = \text{ord}(W_2, \prec_2)$,
- $\psi(W_2) = W(a_2)$,

for some $a_2 \in W$, then we would get the existence of an order isomorphism $\gamma : (W(a_1), \prec) \rightarrow (W(a_2), \prec)$. We can assume (otherwise we replace γ with γ^{-1}) that $a_1 \prec a_2$. If $a_1 \neq a_2$, we would have $a_1 \in W(a_2)$, so if we work with the well-ordered set $Z = W(a_2)$ we would have an order isomorphism $(Z, \prec) \rightarrow (Z(a_1), \prec)$. By Corollary C.1 this is impossible. Therefore, we must have $a_1 = a_2$.

Using the Claim, we then define a_β as the unique element in W , such that $\text{ord}(W(a_\beta), \prec) = \beta$. Define the map $\psi : P_\alpha \ni \beta \mapsto a_\beta \in W$. It is clear that $\phi \circ \psi = \text{Id}_{P_\alpha}$.

Let us prove now that $\psi \circ \phi = \text{Id}_W$. Start with some arbitrary $a \in W$, and put $\beta = \phi(a) = \text{ord}(W(a), \prec)$. Since $\text{ord}(W(a), \prec) = \beta$, by the Claim, we must have $a_\beta = a$, i.e. $\psi(\beta) = a$, which means that $(\psi \circ \phi)(a) = a$.

Finally, we note that, if $a, b \in W$ are elements with $a \prec b$, then the obvious full embedding $(W(a), \prec) \hookrightarrow (W(b), \prec)$ proves that $\text{ord}(W(a), \prec) \leq \text{ord}(W(b), \prec)$, i.e. $\phi(a) \leq \phi(b)$.

Since ϕ is bijective, it is clear that, for $a, b \in W$, we have in fact the equivalence

$$a \prec b \iff \phi(a) \leq \phi(b).$$

This proves that $(P_\alpha, \leq)$ is well-ordered, and $\phi : (W, \prec) \rightarrow (P_\alpha, \leq)$ is an order isomorphism. $\square$

COROLLARY C.2. *If $\mathcal{S}$ is a set of ordinal numbers, then $(\mathcal{S}, \leq)$ is well-ordered.*

PROOF. By Theorem C.1, $(\mathcal{S}, \leq)$ is totally ordered. Fix some non-empty subset $\mathcal{A} \subset \mathcal{S}$, and let us show that $\mathcal{A}$ has a smallest element. Start with some arbitrary $\alpha \in \mathcal{A}$. If $\alpha \leq \beta, \forall \beta \in \mathcal{A}$, we are done. Otherwise, the intersection $\mathcal{A} \cap P_\alpha$ is non-empty. We then use the fact that $(P_\alpha, \leq)$ is well-ordered, to choose α_1 to be its smallest element. If we start with some arbitrary $\beta \in \mathcal{A}$, then either $\alpha \leq \beta$, in which case we immediately get $\alpha_1 < \beta$, or $\beta < \alpha$, in which case $\beta \in \mathcal{A} \cap P_\alpha$, and we again get $\alpha_1 \leq \beta$. So α_1 is in fact the smallest element of $\mathcal{A}$. $\square$

THEOREM C.3 (Well ordering Theorem). *Every non-empty set has a well ordering.*

PROOF. Let

$$\mathcal{W} = \{(W, \prec) : (W, \prec) \text{ well-ordered, and } W \subset X\}.$$

For two elements $(W_1, \prec_1)$ and $(W_2, \prec_2)$, we define $(W_1, \prec_1) \sqsubset (W_2, \prec_2)$, if and only if $W_1 \subset W_2$, and the inclusion map $(W_1, \prec_1) \hookrightarrow (W_2, \prec_2)$ is a full embedding. (This is equivalent to the fact that W_1 is a full subset of $(W_2, \prec_2)$, and $\prec_1 = \prec_2 \upharpoonright_{W_1}$.)

It is obvious that $(\mathcal{W}, \sqsubset)$ is an ordered set. We want to apply Zorn Lemma to this set. We need to check the hypothesis. Start with a totally ordered subset $\mathcal{T} = \{(W_i, \prec_i) : i \in I\} \subset \mathcal{W}$, and let us show that $\mathcal{T}$ has an upper bound in $\mathcal{W}$. Define $W = \bigcup_{i \in I} W_i$. For $a, b \in W$, we define $a \prec b$, if and only if there exists $i \in I$, such that $a, b \in W_i$, and $a \prec_i b$. Let us check that $(W, \prec)$ is a well-ordered set. First of all, we need to show that $\prec$ is an order relation on W . It is clear that $a \prec a$, $\forall a \in W$. Suppose $a, b \in W$ satisfy $a \prec b$ and $b \prec a$, and let us show that $a = b$. We know there exists $i, j \in I$ such that $a, b \in W_i$ and $a \prec_i b$, and $a, b \in W_j$ and $b \prec_j a$. Now there are two possibilities: either $(W_i, \prec_i) \sqsubset (W_j, \prec_j)$, or $(W_j, \prec_j) \sqsubset (W_i, \prec_i)$. In the first case we get $a \prec_i b$ and $b \prec_i a$, so we would get $a = b$. In the other case, by symmetry, we again get $a = b$. Let us show now transitivity. Suppose $a, b, c \in W$ satisfy $a \prec b$ and $b \prec c$, and let us show that $a \prec c$. We know there exist $i, j \in I$, such that $a, b \in W_i$ and $a \prec_i b$, and $b, c \in W_j$ and $b \prec_j c$. As above, we have two possibilities: either $(W_i, \prec_i) \sqsubset (W_j, \prec_j)$, or $(W_j, \prec_j) \sqsubset (W_i, \prec_i)$. In the first case we get $a, b, c \in W_j$ and $a \prec_j b \prec_j c$, so we get $a \prec_j c$. In the second case, we get $a, b, c \in W_i$ and $a \prec_i b \prec_i c$, so we get $a \prec_i c$. In either case we get $a \prec c$.

Next we show that $(W, \prec)$ is totally ordered. Start with arbitrary $a, b \in W$, and let us prove that either $a \prec b$ or $b \prec a$. If we choose $i, j \in I$ such that $a \in W_i$ and $b \in W_j$, then using the two possibilities $(W_i, \prec_i) \sqsubset (W_j, \prec_j)$ or $(W_j, \prec_j) \sqsubset (W_i, \prec_i)$ we immediately see that we can find $k \in I$ (k is either i or j), such that $a, b \in W_k$. Then using the fact that $(W_k, \prec_k)$ is totally ordered, we either have $a \prec_k b$, or $b \prec_k a$. This gives either $a \prec b$, or $b \prec a$.

In order to prove that $(W, \prec)$ is well-ordered, and $(W_i, \prec_i) \sqsubset (W, \prec)$, $\forall i \in I$, we shall use the following

Claim: For any $i \in I$, one has the implication:

$$a \in W_i \implies W(a) \subset W_i.$$

Indeed, if there exists some $b \in W(a)$, but $b \notin W_i$, this would mean that there exists some $j \in I$, with $b \in W_j$, $b \prec_j a$, and $b \neq a$. This would then force $(W_i, \prec_i) \sqsubset (W_j, \prec_j)$, and $b \in W_j(a)$. But this is impossible, since the fact that W_i is full in $(W_j, \prec_j)$ would force $b \in W_j(a) \subset W_i$.

Let us show now that $(W, \prec)$ is well-ordered. Start with some arbitrary non-empty subset $A \subset W$. Choose $i \in I$, such that $A \cap W_i \neq \emptyset$, and take a to be the smallest element in $A \cap W_i$, in the well-ordered set $(W_i, \prec_i)$, i.e.

$$(5) \quad a \in A \cap W_i, \text{ and } a \prec_i x, \quad \forall x \in A \cap W_i.$$

Let us prove that a is in fact the smallest element of A , in $(W, \prec)$. Start with some arbitrary element $b \in A$, and let us prove that $a \prec b$. Assume the opposite, which using the fact that $(W, \prec)$ is totally ordered, this means that $b \prec a$, and $b \neq a$,

i.e. $b \in W(a)$. By the Claim however, this will force $b \in W_i$, so we would get $b \in A \cap W_i$, and the choice of a would give $a \prec_i b$, which would then give $a \prec b$, thus contradicting the assumption on b .

We now prove $(W_i, \prec_i) \sqsubset (W, \prec)$, $\forall i \in I$. It is clear that the inclusion map $\iota : (W_i, \prec_i) \hookrightarrow (W, \prec)$ satisfies the first two conditions in the definition of full embeddings, so the only thing we need is the fact that W_i is full in $(W, \prec)$. But this is precisely the content of the above Claim.

Having shown that every totally ordered subset $\mathcal{T} \subset \mathcal{W}$ has an upper bound, we now invoke Zorn Lemma, to get the existence of a maximal element $(W, \prec) \in \mathcal{W}$. The proof of the Theorem will be finished once we prove that $W = X$. We prove this equality by contardiction. Assume $W \subsetneq X$. Pick an element $x \in X \setminus W$, and define the set $W_1 = W \cup \{x\}$. Equip W_1 with the order relation $\prec_1$ defined by

$$a \prec b \iff \begin{cases} a, b \in W \text{ and } a \prec b, \\ \text{or } b = x \end{cases}$$

It is pretty obvious that $W = W_1(x)$ and $\prec = \prec_1 \upharpoonright_W$, so $(W_1, \prec_1)$ is well-ordered and $(W, \prec) \sqsubset (W_1, \prec_1)$. Since $W \subsetneq W_1$, this would contardict the maximality. $\square$

COMMENT. An interesting consequence of the Well-Ordering Theorem is the following: *For any cardinal number $\mathfrak{a}$, there exists an ordinal number α , such that $\text{card } \alpha = \mathfrak{a}$.*

Another interesting application is the following:

COROLLARY C.3. *If $\mathcal{C}$ is a set of cardinal numbers, then $(\mathcal{C}, \leq)$ is well-ordered.*

PROOF. For any $\mathfrak{a} \in \mathcal{C}$ we choose a well-ordered set $(W_{\mathfrak{a}}, \prec_{\mathfrak{a}})$ with $\text{card } W_{\mathfrak{a}} = \mathfrak{a}$. Choose any set X with

$$\mathfrak{a} < \text{card } X, \quad \forall \mathfrak{a} \in \mathcal{C}.$$

(For example, we can take $Y = \bigcup_{\mathfrak{a} \in \mathcal{C}} W_{\mathfrak{a}}$, so that $\mathfrak{a} \leq \text{card } Y$, $\forall \mathfrak{a} \in \mathcal{C}$, and then we define $X = \{0, 1\}^Y$.) Choose a well-ordering $\prec$ on the set X . Define $\alpha = \text{ord}(X, \prec)$ and $\alpha_{\mathfrak{a}} = \text{ord}(W_{\mathfrak{a}}, \prec_{\mathfrak{a}})$, $\forall \mathfrak{a} \in \mathcal{C}$. Since

$$\text{card } \alpha_{\mathfrak{a}} = \mathfrak{a} < \text{card } X = \text{card}(X, \prec), \quad \forall \mathfrak{a} \in \mathcal{C},$$

it follows that we have $\alpha_{\mathfrak{a}} < \alpha$, i.e. $\alpha_{\mathfrak{a}} \in P_{\alpha}$, $\forall \mathfrak{a} \in \mathcal{C}$.

Apply now the fact that the ordinal set $(P_{\alpha}, \leq)$ is a well-ordered, to find some $\mathfrak{a}_0 \in \mathcal{C}$, such that

$$\alpha_{\mathfrak{a}_0} \leq \alpha_{\mathfrak{a}}, \quad \forall \mathfrak{a} \in \mathcal{C}.$$

This will clearly imply

$$\mathfrak{a}_0 = \text{card } \alpha_{\mathfrak{a}_0} \leq \text{card } \alpha_{\mathfrak{a}} = \mathfrak{a}, \quad \forall \mathfrak{a} \in \mathcal{C}. \quad \square$$

EXAMPLES C.1. As previously discussed, for every *finite* cardinal number $n \geq 0$, there exists exactly one ordinal number with n as its cardinality.

The next interesting case is the class

$$A = \{\alpha : \alpha \text{ ordinal number with } \text{card } \alpha \leq \aleph_0\},$$

then A is a set. Indeed if we choose an ordinal number γ_1 with $\text{card } \gamma_1 = \mathfrak{c}$, then A is a subset of P_{γ_1} . Moreover, if we choose an ordinal number γ_2 with $\text{card } \gamma_2 = 2^{\mathfrak{c}}$, then we see that $\gamma_1 \in P_{\gamma_2} \setminus A$. We can then take Ω to be the smallest element of the non-empty set $P_{\gamma_2} \setminus A$, and we have

$$A = P_{\Omega}.$$

The inclusion $A \subset P_\Omega$ is clear. To prove the other inclusion, we start with some ordinal number $\alpha < \Omega$ and we see that this forces $\alpha \in P_{\gamma_2}$, so it will be impossible to have $\aleph_0 < \text{card } \alpha$, because this would give $\alpha \in P_{\gamma_2} \setminus A$, contradicting the minimality of Ω .

The ordinal number Ω is called the *smallest uncountable ordinal number*.

Fact 1: The set P_Ω is uncountable.

This follows from the fact that

$$\Omega = \text{ord}(P_\Omega, \leq),$$

which gives $\aleph_0 < \text{card } \Omega = \text{card } P_\Omega$.

Fact 2: The cardinal number $\aleph_1 = \text{card } \Omega = \text{card } P_\Omega$ is the smallest uncountable cardinal number.

Indeed, if one starts with some cardinal number $\mathfrak{a} < \aleph_1$, then if we choose a well-ordered set $(W, <)$ with $\text{card } W = \mathfrak{a}$, then, since we have $\text{card } W < \text{card } \Omega$ we must have $\text{ord}(W, <) < \Omega$, which then forces $\mathfrak{a} \leq \aleph_0$.

Fact 3: Any countable subset $A \subset P_\Omega$ has a strict upper bound in P_Ω , that is, there exists $\beta \in P_\Omega$, such that $\alpha < \beta, \forall \alpha \in A$.

We prove this by contradiction. Assume A has no strict upper bound in P_Ω , which means that for every $\beta \in P_\Omega$, there exists some $\alpha \in A$ such that $\beta \leq \alpha$. This gives

$$(6) \quad P_\Omega = \bigcup_{\alpha \in A} (P_\alpha \cup \{\alpha\}).$$

But for every $\alpha \in P_\Omega$ we have $\text{ord } P_\alpha = \alpha$, which forces $\text{card } P_\alpha \leq \aleph_0$. Then the fact that A is countable, combined with (6) will force P_Ω to be countable, which is impossible.

The above construction can be generalized to arbitrary cardinal numbers, giving the following

Fact 4: Given any cardinal number $\mathfrak{a}$, there exist a smallest ordinal number $\Omega_{\mathfrak{a}}$ with $\mathfrak{a} < \text{card } \Omega_{\mathfrak{a}}$, and the cardinal number $\mathfrak{a}' = \text{card } \Omega_{\mathfrak{a}}$ is the smallest cardinal number with $\mathfrak{a} < \mathfrak{a}'$. Any set $A \subset P_{\Omega_{\mathfrak{a}}}$, with $\text{card } A \leq \mathfrak{a}$, has a strict upper bound in $P_{\Omega_{\mathfrak{a}}}$.